Locality for singular stochastic PDEs

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Non-translation invariant SPDEs

Denote by $(t, x) \in \mathbb{R} \times \mathbf{T}$ a typical spacetime point. We consider the operator

$$\mathcal{L}_x^i := a^i(\cdot)\partial_x^2 + b^i(\cdot)\partial_x, \qquad (1 \le i \le k_0),$$

with smooth coefficients a^i and b^i. We consider systems of parabolic equations of the form

$$\big(\partial_t - \mathcal{L}_x^i\big) u_i = f^i(u)\xi + g^i(u, \partial_x u), \qquad (1 \le i \le k_0),$$

with $\xi = (\xi_1, \ldots, \xi_{n_0})$ an n_0-dimensional spacetime 'noise'.

Context

- When the a^i and b^i are constants, Regularity Structures can be used for solving this system (see [Hai14],[BHZ19],[CH16] and [BCCH21]).
- If the coefficients are non constants then it has been considered via paraconrolled calculus in some specific setting see ([GIP15], works from Ismael Bailleul).
- Motivation: intermediate step for defining SPDEs on a manifold in a systematic way (Work in progress of Hairer and Singh).

A local perturbative expansion

$$\partial_t u = \partial_x^2 u + f(u)\,(\partial_x u)^2 + g(u)\xi, \quad \partial_t v = \partial_x^2 v + \xi.$$

Then u cannot be described by $u = v + w$ but through a Taylor type expansion:

$$u = \sum_{\tau \in T} c_{\tau,x} u_{\tau,x} + R_{T,x}$$

where

- T is a finite set of decorated trees
- $u_{\tau,x}$ are recentered (Gaussian) stochastic processes
- $c_{\tau,x}$ are coefficients of the Taylor expansion
- $R_{T,x}$ is a remainder nicer than the $u_{\tau,x}$.

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type …

Recentered stochastic processes

u$_{,x} = K * \xi - (K * \xi)(x), \qquad u_{X^k,x} = (\cdot - x)^k$

u$_{,x} = K * (\partial_x K * \xi)^2 - (K * (\partial_x K * \xi)^2)(x)$

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Main Steps

- Analytical step (no radial kernel $K(x, y)$) use the formalism given in *Tourist's guide for regularity structures* of Bailleul and Hoshino.
- Define the model (recentered iterated integrals), identical to the original construction.
- Renormalised Model (replace constants by functions).
- Convergence of the renormalised model is open.
- Renormalised equation (main result obtained).

Main Result

Theorem (Bailleul, B. 2021)

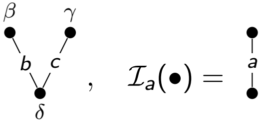
There exists a finite vector space T and for a map $R : (\mathbb{R} \times \mathbf{T}) \times T \to T$ called strong preparation map, the renormalised equation is given by

$$(\partial_t - \mathcal{L}_x^i)u_i = f^i(u)\xi + g^i(u, \partial_x u) + \sum_{l=0}^{n_0} F_i\Big(\big(R(\cdot)^* - Id\big)\zeta_l\Big)(u, \partial_x u)\,\xi_l, \quad (1 \leq i \leq k_0),$$

for some explicit functions $F_i(\tau)(u, \partial_x u)$ indexed by $\tau \in T$ and where ζ_l is associated to ξ_l.

Decorated trees

- Rooted trees.
- Decorations on the edges (kernels, noises and derivatives).
- Decorations on the nodes (polynomials).

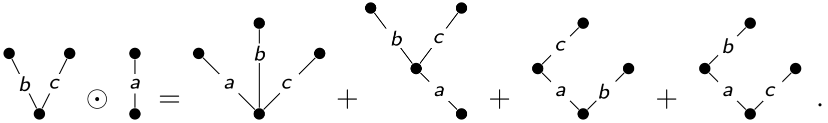


We consider a finite dimensional space of decorated tree T generated by the equation.

- For $\tau \in T$, $|\tau|_\zeta$ is the number of noise symbols.
- $\deg : T \to \mathbb{R}$ is a degree map.

Connes-Kreimer coproduct

We consider a product $\odot$ given by:

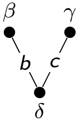


In fact, one has

$$\langle \tau_1 \odot \tau_2, \tau_3 \rangle = \langle \tau_1 \otimes \tau_2, \Delta_{CK} \tau_3 \rangle$$

where Δ_{CK} is a variant of the Connes-Kreimer coproduct.

Deformed Connes-Kreimer coproduct (B., Manchon 2020)



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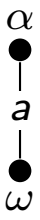
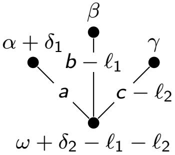


$= \sum_{\delta=\delta_1+\delta_2} \Big(\sum_{\ell} \binom{\omega}{\ell}$


$+ \dots \Big).$

One gets:

$$\tau \star \sigma = \tau \odot \sigma + \text{terms of lower order.}$$

In fact, one has

$$\langle \tau_1 \star \tau_2, \tau_3 \rangle = \langle \tau_1 \otimes \tau_2, \tilde{\Delta}_{CK} \tau_3 \rangle$$

where $\tilde{\Delta}_{CK}$ is a deformation of the Connes-Kreimer coproduct.

(Strong) Preparation maps

A **preparation map** is a linear map $R : T \to T$ such that for each $\tau \in T$ there exist finitely many $\tau_i \in T$ and constants λ_i such that

$$R\tau = \tau + \sum_i \lambda_i \tau_i, \quad \text{with} \quad \deg(\tau_i) \geq \deg(\tau) \quad \text{and} \quad |\tau_i|_\zeta < |\tau|_\zeta$$

and one has

$$R^* \left(\sigma \star \tau\right) = \sigma \star \left(R^* \tau\right) \tag{1}$$

for all $\sigma \in T^+$ and $\tau \in T$.

A **strong preparation map** will have (1) for all $\sigma, \tau \in T$.

Example of preparation maps

The main examples are preparation maps used for the BPHZ renormalisation:

$$R_\ell^*(\tau) := \sum_{\sigma \in T^-} \frac{\ell(\sigma)}{S(\sigma)} (\tau \star \sigma).$$

where

- T^- are trees of negative degree.
- $\ell : T^- \to \mathbb{R}$ is a character:

  $$\ell(\tau) = -\mathbb{E}\big[(\Pi^{R_\ell}\tau)(0)\big]$$

  with Π^{R_ℓ} the renormalised smooth model associated with R_ℓ.
- $S(\sigma)$ is the symmetry factor of $\sigma \in T$.

Renormalisation maps

A **good multi-pre-Lie morphism on** T is a map $M : T \to T$ such that:

$$M^* \left(\mathcal{I}_a(\tau) \star \sigma\right) = \mathcal{I}_a(M^* \tau) \star M^* \sigma, \quad M^*(\bullet_k \star \sigma) = \bullet_k \star M^* \sigma.$$

Example: BPHZ renormalisation map, Translation maps for Branched Rough Paths,...

Given a strong preparation map R, we define $M_R^\times : T \to T$ as multiplicative and by the induction relation

$$M_R^\times \left(\mathcal{I}_a(\tau)\right) = \mathcal{I}_a\left(M_R^\times (R\tau)\right).$$

Then M_R is defined as

$$M_R = M_R^\times R$$

Decorated trees expansion

Abstract solution u in $\mathcal{T}$ is given by:

$$\mathrm{u}_i = \sum_{\tau \in \mathcal{T}} \frac{F_i(\tau)(u, \partial_{x_1} u)}{S(\tau)} \tau.$$

where there exists an explicit expression for $F_i(\tau)$. The lift G of a smooth enough function G is given for any $\mathrm{a} = a_{\mathbf{1}}\mathbf{1} + \mathrm{a}' \in \mathcal{T}$ by

$$\mathrm{G}(\mathrm{a}) = \sum_k \frac{D^k G(a_{\mathbf{1}})}{k!} (\mathrm{a}')^k.$$

The lift of $F_i(\tau)$ is denoted by $\mathrm{F}_i(\tau)$.

"A one page proof of [BCCH21]"

One has the following identity:

$$(\partial_t - \mathcal{L}_x^i) u_i = \mathrm{R}^{\mathrm{M}^R}(\mathrm{v}_i), \quad \mathrm{v}_i = \sum_{\deg(\tau) < \gamma - 2} \frac{F_i(\tau)(u, \partial_x u)}{S(\tau)} \tau,$$

where

$$\left(\mathrm{R}^{\mathrm{M}^R} \mathrm{v}_i\right)(x) = \left(\hat{\Pi}_x^R \left(R \mathrm{v}_i(x)\right)\right)(x), \quad \hat{\Pi}_x^R \text{ multiplicative}$$

Then

$$R\mathrm{v}_i(x) = \sum_{l=0}^{n_0} \mathrm{F}_i\left(R^* \zeta_l\right)\left(\mathrm{u}(x), D\mathrm{u}(x)\right) \zeta_l.$$

where the last identity uses the right morphism property of R.

"A one page proof of [BCCH21]"

Using the (crucial) multiplicativity property of $\widehat{\Pi}_x^R$ we see that

$$\begin{aligned}
((\partial_t - \mathcal{L}_x^i)u_i)(x) &= (\mathsf{R}^{\mathsf{M}^R}\mathsf{v}_i)(x) = \widehat{\Pi}_x^R(R(x)\mathsf{v}_i(x))(x) \\
&= \sum_{l=0}^{n_0} \widehat{\Pi}_x^R\Big(\mathsf{F}_i(R^*(x)\zeta_l)(\mathsf{u}(x), D\mathsf{u}(x))\zeta_l\Big)(x) \\
&= \sum_{l=0}^{n_0} F_i(R^*(x)\zeta_l)\Big((\widehat{\Pi}_x^R\mathsf{u}(x))(x), \partial_x(\widehat{\Pi}_x^R\mathsf{u}(x))(x)\Big)\widehat{\Pi}_x^R\zeta_l \\
&= \sum_{l=0}^{n_0} F_i(R^*(x)\zeta_l)(u(x), \partial_x u(x))\xi_l(x).
\end{aligned}$$

Conclusion

- A short proof of the renormalised equation that works in the non-translation invariant context.
- Equivalence between preparation maps and BPHZ renormalisations maps for Branched Rough Paths. Equivalence open for Regularity Structures.
- Convergence of the renormalised model could use the recursive definition with R.