Classical identification of emergent geometries in AdS spacetimes and quantum algorithms for algebras in dual CFTs.

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Large-N matrix models and emergent geometry

Based on

J. Ben Geloun and S. Ramgoolam, "The quantum detection of projectors in finite-dimensional algebras and holography, " arXiv:2303.12154 [quant-ph]
J. Ben Geloun and S. Ramgoolam, “Quantum mechanics of bipartite ribbon graphs: Integrality, Lattices and Kronecker coefficients” arXiv:2010.04054v1 [hep-th]

Basic example of emergence from large N : AdS5/CFT4

String theory on $AdS_5 \times S^5$ geometry and its fluctuations are captured by $\mathcal{N} = 4$ SYM with $U(N)$ gauge group.

CFT operators : Half-BPS

General half-BPS gauge invariant operators of dimension n can be parameterised by using permutations $\sigma \in S_n$:

$$\mathcal{O}_\sigma(Z) = \sum_{i_1,\cdots,i_n} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}}$$

Different choices of permutation σ give rise to different trace structures.

For example, for $n = 3$,

$$\sigma = (1,2,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$\mathcal{O}_\sigma(Z) = \sum_{i_1,i_2,i_3} Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} = \mathrm{tr}(Z^3)$$

$$\sigma = (1,2)(3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$\mathcal{O}_\sigma(Z) = \sum_{i_1,i_2,i_3} Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} = \mathrm{tr}(Z^2)\mathrm{tr}(Z)$$

$$\sigma = (1)(2)(3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$
$$\mathcal{O}_\sigma(Z) = \sum_{i_1,i_2,i_3} Z^{i_1}_{i_1} Z^{i_2}_{i_2} Z^{i_3}_{i_3} = \mathrm{tr}(Z)\mathrm{tr}(Z)\mathrm{tr}(Z) = (\mathrm{tr}Z)^3$$

CFT operators :

For fixed $n < N$, the number of trace structures is equal to the number of partitions of n (more on general n shortly).

From the permutation parameterisation,

$$\mathcal{O}_\sigma(Z) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(Z)$$

Operators associated to conjugate permutations are the same. Conjugacy classes in S_n are in 1-1 correspondence with cycle structures. For a multi-trace operator of dimension n of the form

$$(\mathrm{tr}(Z))^{p_1}(\mathrm{tr}(Z^2))^{p_2}\cdots(\mathrm{tr}(Z^n))^{p_n}$$

with $p_1 + 2p_2 + \cdots + np_n = n$, we can pick a permutation $\sigma^{(p)}$ with cycle structure $[1^{p_1}, 2^{p_2}, \cdots]$ and write

$$(\mathrm{tr}(Z))^{p_1}(\mathrm{tr}(Z^2))^{p_2}\cdots(\mathrm{tr}(Z^n))^{p_n} = \mathcal{O}_{\sigma^{(p)}}(Z) = \frac{1}{|\mathcal{C}_p|}\sum_{i=1}^{|\mathcal{C}_p|}\mathcal{O}_{\sigma_i^{(p)}}(Z)$$

CFT operators

Formal linear combinations of permutations with complex coefficients live in the group algebra $\mathbb{C}(S_n)$. Combinations of this form

$$T_p = \frac{1}{|\mathcal{C}_p|} \sum_i \sigma_i^{(p)}$$

live in the centre of the group algebra, denoted $\mathcal{Z}(\mathbb{C}(S_n))$.

As p ranges over all partitions of n, the T_p form a basis for $\mathcal{Z}(\mathbb{C}(S_n))$.

Projector basis for $\mathcal{Z}(\mathbb{C}(S_n))$

There is a basis for $\mathcal{Z}(\mathbb{C}(S_n))$ consisting of orthogonal projectors. The elements of the basis are labelled by Young diagrams R

$$P_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi^R(\sigma) \ \sigma = \frac{d_R}{n!} \sum_{p \vdash n} \chi^R(\sigma^{(p)}) \ \left(T_p |\mathcal{C}_p|\right)$$

$$P_R P_S = \delta_{RS} P_S$$

Projectors and Schur-basis of CFT operators

The projectors are directly related to the Schur-basis operators

$$\mathcal{O}_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^R(\sigma) \ \mathcal{O}_\sigma(Z)$$

which were defined and shown to have orthogonal 2-point functions in free-field $\mathcal{N} = 4$ SYM :

$$\langle \mathcal{O}_R(Z(x_1)) \mathcal{O}_R(Z^\dagger(x_2)) \rangle = \frac{1}{|x_1 - x_2|^{2n}} f_R(N)$$

Corley, Jevicki, Ramgoolam, "Exact Correlators of Giant Gravitons from dual N=4 SYM", ATMP-2001,

https://arxiv.org/abs/hep-th/0111222

The orthogonality of projectors is directly related (diagrammatic) to the orthogonality of the two-point functions.

Corley, Ramgoolam, "Finite Factorization equations and Sum Rules for BPS correlators in N=4 SYM theory"

JHEP-2002, https://arxiv.org/abs/2201.12917

Complex matrix model and background fields

The combinatorial computations are essentially computations in a complex matrix model with partition function

$$\mathcal{Z} = \int [dZ] \; e^{-\mathrm{tr} ZZ^{\dagger}}$$

The combinatorial computations (neatly simplified using algebras and diagrams) are reviewed and extended to actions with background fields of the form

$$\mathcal{Z} = \int [dZ] \; e^{-\mathrm{tr} ZAZ^{\dagger}}$$

A is hermitian matrix background field.

Ramgoolam and Sword, "Matrix and tensor witnesses of hidden symmetry algebras" , JHEP-2023,

https://arxiv.org/abs/2302.01206

Complex matrix model and background fields

In these cases the normalisation factors are replaced by Schur-functions of the background field

$$f_R(N) \to \mathcal{O}_R(A)$$

This makes contact with the super-integrability program of Morozov-Mironov

A. Mironov, A. Morozov, "Superintegrability summary" https://arxiv.org/abs/2201.12917

Young diagrams and giants.

It was argued that the Young diagram basis allows the identification of operators dual to semi-classical giants which are large in the AdS as well as the sphere directions (CJR-2001).

There is an underlying free-fermion picture for this half-BPS sector which sheds further light on the dictionary between Young diagram operators and giants (CJR-2001, Berenstein-2004).

Berenstein “A toy model for the AdS/CFT correspondence” , https://arxiv.org/abs/hep-th/0403110

Extensive evidence in subsequent papers : DB, Robert de Mello Koch, SR, many others ... multi-matrix operators related to strings attached to giants ...

Many giants and LLM geometries

Giant gravitons are described in space-time in terms of solutions to 3-brane actions with embedding in $AdS_5 \times S^5$ bulk space-time. They correspond to operators of dimension $n \sim N$.

For $n \sim N^2$, the giant gravitons back-react and produce large deformations of space-time, with $AdS_5 \times S^5$ asymptotics. The general half-BPS super-gravity solutions were characterised by Lin, Lunin, Maldacena (2004) - LLM-2004.

A distinguished role is played by a 2-dimensional plane in space-time, which can be identified with a free-fermion phase space. Colourings of the plane by concentric black rings in a white background can be mapped to Young diagram row lengths and equivalently excitations of N free fermions in a 1D harmonic oscilliator potential, making contact with the free-fermion/Young-diagram connection from the matrix model of a complex scalar.

Outline of Talk :

- Detecting the R label of projectors P_R (using the Hilbert space structure of $\mathcal{Z}(\mathbb{C}(S_n))$ and a system of eigenvalue equations).
- Detecting the LLM geometry corresponding to R using measurements of the LLM metric
- Quantum detection of projectors (the eigenvalue system and obtain complexity estimates using standard results in quantum information and group theory)
- Holographic Classical detection (estimates comparable to above (modulo assumptions) ; Classical detection by randomised algorithms .. exponentially harder.).

Part 1 : Detecting P_R in $\mathcal{Z}(\mathbb{C}(S_n))$

Number of P_R = Number of T_p = Number of partitions of n $\sim e^{\sqrt{n}}$.

R is a Young diagram (an irrep of S_n) ; p is a partition of n associated with a cycle structure.

P_R satisfy eigenvalue equations, with known simple eigenvalues.

$$T_p P_R = \frac{\chi^R(T_p)}{d_R} P_R$$

Part 1 : Detecting P_R in $\mathcal{Z}(\mathbb{C}(S_n))$

A subset of the T_p are inretesting: when p corresponds to a partition associated with a single cycle of length k and remaining cycles of length 1, i.e. $p = [k, 1^{n-k}]$:

$$T_{p=[k,1^{n-k}]} \equiv T_k$$

The eigenvalues for T_k have nice expressions in terms of power sums of contents of boxes in Young diagrams.

Math papers by Lasalle and by Corteel, Goupille, Schaeffer

NUmber of $T_k = n$.

The eigenvalues of the set of T_k, for $k = 1 \cdots n$, uniquely specify the P_R.

Kemp, Ramgoolam, "BPS states in $N = 4$ SYM theory and centres of symmetric group algebras" JHEP

Part 1 : Detecting P_R in $\mathcal{Z}(\mathbb{C}(S_n))$

In fact, for any given finite n, we only need a small number of T_k to distinguish the P_R

E.g. for $n \in \{2, 3, 4, 5, 7\}$ it suffices to know T_2.

With $\{T_2, T_3\}$ we can distinguish all P_R for n up to 14.

For general n, there is some $k_*(n)$, such that

$$\{T_2, T_3, \cdots, T_{k_*(n)}\}$$

uniquely specify the R. Such subsets non-linearly generate the centre $\mathcal{Z}(\mathbb{C}(S_n))$.
We computed this for n up to 80 .. e.g. $k_*(n = 80) = 6$. !

Kemp, Ramgoolam, "BPS states in $N = 4$ SYM theory and centres of symmetric group algebras" JHEP-2020

Part 2 : Detecting LLM geometries

Lin-Lunin-Maldacena (2004) classified the half-BPS super-gravity solutions with $AdS_5 \times S^5$ asymptotics, which take the form:

$$ds^2 = -h^{-2}(dt + \sum_{i=1}^{2} V_i dx_i)^2 + h^2(dy^2 + \sum_{i=1}^{2} dx_i dx_i) + R^2 d\Omega^2 + R^2 d\tilde{\Omega}^2$$

The functions V_1, V_2, h, R appearing above are all functions of (x_1, x_2, y), and are all determined by one function $u(x_1, x_2, y)$. The function obeys a harmonic equation in y and is determined by its value on the $y = 0$ plane.

The function $u(x_1, x_2)$ on the LLM plane is determined by using a Wigner phase space distribution associated to the quantum many-body fermion state (associated with Young diagram R).

Vijay Balasubramanian, Bartlomiej Czech, Klaus Larjo, Joan Simon, "Integrability vs. Information Loss: A Simple Example," -2006, https://arxiv.org/abs/hep-th/0602263 (BGLS-2006)

Part 2 : Detecting LLM geometries

The upshot of this discussion gives an expression for u that allows the determination of the conserved charges from the semi-classical geometry :

$$u(\rho, \theta) = 2\cos^2\theta \sum_{l=0}^{\infty} \frac{\sum_{f\in\mathcal{F}} A^l(f)}{\rho^{2l+2}} (-1)^l (l+1)\ {}_2F_1(-l, l+1; 1; \sin^2\theta)$$

Here $\rho \in [0, \infty]$, $\theta \in [0, \frac{\pi}{2}]$, $\mathcal{F} = \{f_1, f_2, \ldots, f_N\}$ a set of increasing integers related to the eigenvalues of individual Fermion $E_i = \hbar(f_i + \frac{1}{2})$, $i = 1, 2, \ldots, N$. In such an expansion, $A^l(f)$ is a polynomial of order l in f (its explicit form can be found in BGLS-2006).

Part 2 : Detecting LLM geometries

These polynomials

$$\sum_{f\in\mathcal{F}} A^l(f) = C_l(R)$$

are Casimirs of $U(N)$.

These are in turn related to the eigenvalues of $\frac{\chi_R(T_p)}{d_R}$ - for example there are relations of the form

$$C_2(R) = Nn + \frac{\chi_R(T_2)}{d_R}$$

Such relations follow from Schur-Weyl duality which relates $U(N)$ rep theory to rep theory of

$$\bigoplus_{n=0}^{\infty} \mathbb{C}(S_n)$$

Knowledge of Casimirs $C_2(R), \cdots, C_k(R)$ is equivalent to knowing the normalised characters $\frac{\chi_R(T_2)}{d_R}, \cdots, \frac{\chi_R(T_k)}{d_R}$.

Part 2 : Detecting LLM geometries

The point made in BGLS-2006 was that determining a general Young diagram and corresponding LLM geometry requires knowing N Casimirs, while the Planck scale cutoff means we have access to far fewer Casimirs/multi-pole moments – order $N^{1/4}$ - an interpretation of information loss as a toy model for black holes.

We can give a more careful discussion of this argument by defining a $k_*(n, N)$ - number of cycle central elements (Casimirs) needed to distinguish all Young diagrams with n boxes and no more than N rows.

When $n < N$, $k_*(n, N) = k_*(n)$ which is the case we stick with for now.

Part 3: Quantum detection of projectors P_R

The task of identifying the projector P_R using the T_k eigenvalue equations lends itself to standard quantum algorithms - quantum phase estimation, which come with associated complexity estimates (query and gate complexity).

JBG-SR-2023 : J. Ben Geloun and S. Ramgoolam, "The quantum detection of projectors in finite-dimensional algebras and holography," JHEP-2023

This allows the exploitation of exponential improvements provided by quantum algorithms (compared to classical algorithms) for certain computational tasks in linear algebra, along the lines of

Harrow, Hassidim, Lloyd, "Quantum algorithm for solving linear systems of equations," PRL-2009, https://arxiv.org/abs/0811.3171

Part 3: Quantum detection of projectors P_R

The algorithm uses black box unitaries $U_k = e^{\frac{2\pi i}{\chi_k^{\max}} T_k}$ and

$$U_k P_R = e^{\frac{2\pi i}{\chi_k^{\max}} \widehat{\chi}^R(T_k)} P_R$$

$\chi_k^{\max}$ is the maximum of the eigenvalues $\frac{\chi^R(T_k)}{d_R}$ as R ranges over Young diagrams with n boxes.

Quantum phase estimation involves applications of powers of U_k - assumed to be available as black boxes - to the initial state P_R, which is assumed to be given as a quantum state in the Hilbert space $\mathcal{Z}(\mathbb{C}(S_n))$.

Query complexity counts the number of uses of these black boxes in the quantum circuit. Known results for QPE are used from standard texts e.g. Nielsen and Chuang. Gate complexity counts the total number of other boxes in the circuit.

Part 3: Quantum detection of projectors P_R

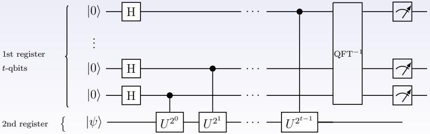

Figure 1: Quantum phase estimation by a quantum circuit acting on the initial state $|0\rangle^{\otimes t} \otimes |\psi\rangle$: H-boxes are Hadamard gates, U^{2^i}-boxes stand for CU-operators, $i = 0, \ldots, t-1$, QFT^{-1} for the inverse quantum Fourier transform, and the last stage involves a measurement on the first register.

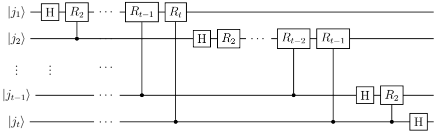

Figure 2: A circuit for quantum Fourier transform $|j_1 j_2 \ldots j_t\rangle$: H-boxes are Hadamard gates, R_k-boxes stand for C-R_k-operators, $k = 2, \ldots, t$.

Part 3: Quantum detection of projectors P_R

Standard result in QPE - for one unitary - gives query complexity $\mathcal{O}(t)$ and gate complexity $\mathcal{O}(t^2)$, where t is the number of bits needed to code the eigenvalues of interest.

S_n group theory input – about values of the normalised characters $\frac{\chi^R(T_k)}{d_R}$ - relates this to n.

In our application we have a range of unitaries $\{U_2, \cdots, U_{k_*(n)}\}$. We argued for a heuristic approximate lower bound $k_*(n) \gtrapprox n^{1/4}$ in the large n limit (JBG-SR-2023), and we assumed $k_*(n) \sim n^{\alpha}$ with $1/4 \leq \alpha < 1/2$.

$k_*(n) \leq n^{1/2}$ has been argued more recently in

Kemp, " A generalized dominance ordering for 1/2-BPS states" - https://arxiv.org/abs/2305.06768

Part 3: Quantum detection of projectors P_R

Based on the input from Quantum information, and the S_n rep theory estimates, we arrived at the complexity estimate for the detection of the projectors P_R :

Query complexity : $\mathcal{O}(n^{2\alpha} \log n)$

Gate complexity : $\mathcal{O}(n^{3\alpha} (\log n)^2)$

These are both bounded by $n^{3/2+\epsilon}$ with $\epsilon > 0$.

Hence the quantum projector detection is polynomially bounded in n - although $p(n) \sim e^{\sqrt{n}}$ and $n! \sim e^{n \log n - n}$.

Part 4: Holographic classical detection of projectors P_R

Remarkably - Because of AdS/CFT, the same task of detecting P_R has a classical counterpart. Use the long-distance behaviour of the metric/form-fields, which are determined by one function $u(\rho, \theta)$, to identify the Casimirs C_l up to a cut-off $l \sim k_*(n) \sim n^\alpha$ and estimate the complexity of the task.

In JBG-SR-2023, we described how to use the standard "Fast Fourier Transform algorithm" - along with known analytic properties of $u(\rho, \theta)$ from LLM and BGLS-2006, to reconstruct the Casimirs from the values of $u(\rho, \theta)$ at different values of θ.

And we used standard results on the complexity of FFT, along with the $k_*(n) \sim n^\alpha$ with $1/4 \leq \alpha < 1/2$, to obtain complexity estimates.

Part 4: Holographic classical detection of projectors P_R

Formula for $u(\rho,\theta)$:

$$u(\rho,\theta) = 2\cos^2\theta \sum_{l=0}^{\infty} \frac{\sum_{f\in\mathcal{F}} A^l(f)}{\rho^{2l+2}} (-1)^l (l+1)\ {}_2F_1(-l, l+1; 1; \sin^2\theta)$$

Introduce a cut-off $\Lambda \sim n^\alpha$ and expressing the $2F1$ in terms of Jacobi polynomials

$$u(\rho,\theta) = 2\cos^2\theta \sum_{l=0}^{\Lambda} \frac{\sum_{f\in\mathcal{F}} A^l(f)}{\rho^{2l+2}} (-1)^l (l+1) P_l^{0,0}(\cos 2\theta)$$

A rescales form of u :

$$\widetilde{u}(\rho, X, \Lambda) = \frac{u(\rho, X, \Lambda)}{(1+X)} = \sum_{l=0}^{\Lambda} U(l,\rho) P_l^{0,0}(X)$$

where the Casimirs of interest are given by

$$\sum_{f\in\mathcal{F}} A^l(f) = (-1)^l \rho^{2l+2} \frac{U(l,\rho)}{l+1}$$

Part 4: Holographic classical detection of projectors P_R

With a little re-organisation

$$\widetilde{u}(\rho, \theta, \Lambda) = \sum_{l=0}^{\Lambda} U(l, \rho) \sum_{m=-l}^{l} \widetilde{p}_{l,m} \, e^{2i\theta m}$$

$$= \sum_{m=-\Lambda}^{\Lambda} \left[\sum_{l=|m|}^{\Lambda} U(l, \rho) \, \widetilde{p}_{l,m} \right] e^{2i\theta m} = \sum_{m=-\Lambda}^{\Lambda} \widetilde{C}_m(\rho, \Lambda) \, e^{2i\theta m}$$

$$\widetilde{C}_m(\rho, \Lambda) := \sum_{l=|m|}^{\Lambda} U(l, \rho) \, \widetilde{p}_{l,m}$$

We have a Fourier expansion, with coefficients that know about the Casimirs- and needs solution of a linear system involving Jacobi polynomial coeffs. - to go from Fourier coeffs. to Casimirs.

Part 4: Holographic classical detection of projectors P_R

The physical input into the algorithm (FFT and linear-system inversion) is a set of values of $u(\rho, \theta)$ at Λ discrete values of θ.

We assume that the computational complexity of measuring the $\widetilde{u}(\rho, \theta_l = \frac{\pi l}{\Lambda+1}, \Lambda)$, $l = 0, \ldots, \Lambda$, is bounded from above by a certain function $c_{\widetilde{u}}(\Lambda)$, the complexity of measuring $\widetilde{u}$ at separations of $2\pi/(\Lambda + 1)$. The estimation of $c_{\widetilde{u}}(\Lambda)$ will require a complexity analysis of measurements in classical gravity, which we leave for future discussion and calculation.

Putting everything together, we arrive at the complexity of detecting the P_R/corresponding-LLM-geometry :

$$f(\Lambda) \leq c_0 \, \Lambda \, c_{\widetilde{u}}(\Lambda) + c_1 \Lambda \log \Lambda + c_2 \Lambda^2$$

where c_0, c_1 and c_2 are n-independent constants.

Part 4: Holographic classical detection of projectors P_R

Recalling $\Lambda \sim k_*(n) \sim n^\alpha$ with $1/4 \leq \alpha < 1/2$, and assuming $c_{\tilde{u}}(\Lambda)$ grows at most polynomially with Λ, we find that this holographic classical detection of R has a complexity which is polynomial in n - like the quantum phase estimation algorithm we described before.

Quantum Linear algebra algorithms (of HHL type) have been compared to randomised classical algorithms. In concrete real world-applications (recommendation systems) where the data is classical - and has to be converted to a quantum state vector - the (quantum-inspired) randomised classical algorithms were found to be competitive with the quantum algorithm.

Ewin Tang "A quantum-inspired classical algorithm for recommendation systems"

For the projector detection task at hand - intrinsically a more quantum problem - the corresponding “quantum-inspired classical algorithms” we came up with in JBG-SR-2023 were exponentially worse than the quantum algorithm.

Summary and Outlook

We defined a "quantum detection of projectors in algebras" task inspired by AdS/CFT.

In the simplest case considered, motivated by half-BPS states, the algebra is $\mathcal{Z}(\mathbb{C}(S_n))$.

Other algebras considered in JBG-SR-2023 are related to Littlewood-Richardson coefficients and Kronecker coefficients.

Our discussion on the classical gravity side is a "proof of concept" complexity claculation – important and interesting conceptual questions remain on the intrinsic compolexities of the gravitational merasurements. Are there classical gravitational analogs of the query v/s gate complexities in the quantum side.

Explore similar classical/quantum comparisons in other instances of quantum-state/classical-geometry correspondences in string theory, e.g. Mathur programme, AdS3 etc.