THE INTEGRAL FORM OF SUPERGRAVITY

P.A. Grassi, Università del Piemonte Orientale, Italy

in collaboration with
Roberto Catenacci (Univ. Piemonte Orientale),
Carlo Alberto Cremonini and Simone Noja (Univ. Insubria)
Leonardo Castellani (UPO) and Pietro Fré (UNITO)

References and History

- Picture Changing Operators Friedan-Martinec-Shenker
- Some math work by Schwartz, Voronov, Zorich, Bernstein, Leites, Manin
- Two interesting papers (unpublished) by Belopolsky
- Multiloop amplitudes in Pure Spinor String Theory by Berkovits
- Geometry of supermanifolds and target space PCO PAG-Policastro
- Integration of superforms and super-Thom class PAG-Marescotti
- Balanced super varieties and Integral form cohomology Catenacci-Debernardi-Matessi-PAG
- Superstring perturbation theory revised Witten
- Poincaré dual, Hodge operator, Hodge theory and dualities Castellani-Catenacci-PAG
- D=3 N=1 Super Chern-Simons on Supermanifolds Maccaferri-PAG
- D=3 N=1 Supergravity Castellani-Catenacci-PAG
- D=1 N=1,2 Super Quantum Mechanics Castellani-Catenacci-PAG
- D=3 N=2 Super Chern-Simons with Matter (ABJM) Fré-PAG
- D=4 N=1,2 Wess-Zumino and Super-Yang-Mills Castellani-Catenacci-PAG
- D=10 N=1 Super-Yang-Mills Fré-PAG

Plan of the talk

- The main idea
- A preliminary warm-up
- Integral forms and integration on supermanifolds
- Definition of PCO
- Cohomology
- Hodge operator
- Wess-Zumino model D=4 N=1
- Super-Yang-Mills D=10 N=1
- Conclusions

Integration on supermanifolds

Integration of Forms on Supermanifolds

Let us begin with a conventional manifold $\mathcal{M}$ with dimension = n, given a generic differential form

$$\omega \in \Omega^\bullet(\mathcal{M})$$

This is a section of the exterior bundle and it can be decomposed as

$$\omega = \omega^0 + \omega^1 + \omega^2 + \cdots + \omega^n$$

where the last term is the top form. Locally, a generic form can be written as

$$\omega(x, dx) = \sum_{p=0}^{n} \omega_{[\mu_1 \dots \mu_p]}(x) dx^{\mu_1} \dots dx^{\mu_p}$$

and its integral on the manifold is

$$\int_{\mathcal{M}} \omega = \int f(x)[d^n x]\,, \quad f(x) = \sqrt{g}\, \omega_{[\mu_1 \dots \mu_n]}(x) \epsilon^{\mu_1 \dots \mu_n}$$

where the second member is a Lebesgue/Riemann integral of the function built in terms of the differential form.

Differential forms on a supermanifold

Let us now move to supermanifolds. We denote by $\mathcal{M}$ a (n|m)-dimensional supermanifold parametrised by the local coordinates (x^μ, θ^α)

We introduce also the corresponding 1-forms $(dx^\mu, d\theta^\alpha)$ with the properties

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \quad dx^\mu \wedge d\theta^\alpha = d\theta^\alpha \wedge dx^\mu \quad d\theta^\alpha \wedge d\theta^\beta = d\theta^\beta \wedge d\theta^\alpha$$

Then a generic (super) form looks like

$$\omega = \sum_{k=1,l=1}^{k=p,l=q} \omega_{[\mu_1 \dots \mu_k](\alpha_1 \dots \alpha_l)} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} d\theta^{\alpha_1} \wedge \cdots \wedge d\theta^{\alpha_l}$$

where the components $\omega_{[\mu_1 \dots \mu_k](\alpha_1 \dots \alpha_l)}(x, \theta)$ are functions of the manifold coordinates. The indices $[\mu_1 \dots \mu_k]$ are anti-symmetrised while $(\alpha_1 \dots \alpha_l)$ are symmetrised. The total form degree is fixed by the p + q, summing the form degree of the bosonic coordinates and the form degree of the fermionic ones. This implies that there is no upper bound to the form degree and there is no top form.

For geometric integration theory on supermanilfolds see the work Berstein, Leites, Manin, Zorich, Voronov, Khudaverdian, Belopoloski, Witten

The integrals over the fermionic coordinates (dx, θ) are Berezin integrals, over the x-coordinates are the usual Lebesgue/Riemann integrals, but the integral over dθ is not well defined on the superforms.

We define the integration over dθ by introducing a special type of form: the Dirac delta of dθ

$$\delta(d\theta^\alpha)$$

such that $$\int f(d\theta^\alpha)\delta(d\theta^\alpha) = f(0)$$

with the usual properties $$d\delta(d\theta^\alpha) = \delta'(d\theta)d^2\theta = 0$$

They formally share all distributional properties of the usual Dirac delta functions. In addition, they are forms and therefore we can apply the usual geometric differential operators.

For the Dirac delta functions we assume the following properties (distributional properties)

$$\delta(d\theta^\alpha) \wedge \delta(d\theta^\beta) = -\delta(d\theta^\beta) \wedge \delta(d\theta^\alpha)$$

this follows by assuming an oriented integration measure.
In this way, we see that there is an upper bound to the number of delta's: the number of fermionic coordinates.

A fundamental property is the distributional equation

$$d\theta^\alpha \delta(d\theta^\alpha) = 0$$

In the same way, using the distributional properties of delta's, we have that

$$d\theta^\alpha \delta^{(n)}(d\theta^\alpha) = -n\delta^{(n-1)}(d\theta^\alpha)$$

That equation tells us that the derivatives of delta's carry negative form degree.
In this way, multiplying by dθ, it reduces the negative power. The Dirac delta has no form degree.

Now a generic **(pseudo)-form** can be written as

$$\omega = \sum_{p,r,s} \omega_{[\mu_1 \ldots \mu_p](\alpha_1 \ldots \alpha_r)[\alpha_{r+1} \ldots \alpha_s]}(x,\theta) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge d\theta^{\alpha_1} \wedge \cdots \wedge d\theta^{\alpha_r} \wedge \delta(d\theta^{\alpha_{r+1}}) \wedge \cdots \wedge \delta(d\theta^{\alpha_s})$$

each pieces are differential forms with fixed **form degree = p + r** and **picture number = s - r**

A generic (p|q) form is written in terms of

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} d\theta^{\alpha_1} \wedge \cdots \wedge d\theta^{\alpha_r} \delta^{(s_1)}(d\theta^{\beta_1}) \wedge \ldots \delta^{(s_q)}(d\theta^{\beta_q})$$

and we denote by $\Omega^{(p|q)}(\mathcal{M})$ the space of pseudo-forms. For q=0, we have the well-known **superforms**, for q=m we have the **integral forms** and for 0< q<m, we have the space of **pseudo-forms**.

We can apply the complete Cartan calculus (Lie derivatives, contractions, inner products….)

Form complexes

Now we have the following complexes

$$0 \to \Omega^{(0|0)} \to \Omega^{(1|0)} \to \dots \to \Omega^{(n|0)} \to \Omega^{(n+1|0)} \dots$$

where all spaces are finite dimensional. The complex is not bounded from above.
The differential **d** acts along the arrows.

$$\dots \to \Omega^{(-2|m)} \to \Omega^{(-1|m)} \to \dots \to \boxed{\Omega^{(n|m)}} \to 0$$

this is the complex of integral forms. It is unbounded from below, but it is bounded from above.
The last space is the space of top forms. Notice that when we have the maximum number of delta's, there is no room for $d\theta' s$

There are additional complexes of the form:

$$\dots \to \Omega^{(-2|q)} \to \Omega^{(-1|q)} \to \dots \to \Omega^{(n|q)} \to \dots$$

which is not bounded from above nor from below. In addition, each single space is **infinite dimensional** space and their geometry is completely unknown .

In summary, we have

$$\begin{array}{cccccccccccc}
 & 0 & \xrightarrow{\mathrm{d}} & \Omega^{(0|0)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|0)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|0)} & \xrightarrow{\mathrm{d}} & \Omega^{(m+1|0)} \ \dots \\
 & z\Big\uparrow & & z\Big\uparrow\Big\downarrow Y & & & z\Big\uparrow\Big\downarrow Y & & & z\Big\uparrow\Big\downarrow Y & & \Big\downarrow Y \\
 & \vdots & & \vdots & & & \vdots & & & \vdots & & \vdots \\
\dots & \Omega^{(-1|s)} & \xrightarrow{\mathrm{d}} & \Omega^{(0|s)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|s)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|s)} & \xrightarrow{\mathrm{d}} & \Omega^{(m+1|s)} \ \dots \\
 & \vdots & & \vdots & & & \vdots & & & \vdots & & \vdots \\
 & z\Big\uparrow & & z\Big\uparrow\Big\downarrow Y & & & z\Big\uparrow\Big\downarrow Y & & & z\Big\uparrow\Big\downarrow Y & & \Big\downarrow Y \\
\dots & \Omega^{(-1|n)} & \xrightarrow{\mathrm{d}} & \Omega^{(0|n)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|n)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|n)} & \xrightarrow{\mathrm{d}} & 0
\end{array}$$

The operators **Y** and **Z** are known as Picture Changing Operators and act vertically in the complexes.

The **Y** operators are elements of the cohomology

$$H^{(0|m)}(\mathcal{M})$$

This implies that given a pseudo form (p|q) and multiplying it by a PCO $Y_i = \theta_i \delta(d\theta_i)$
we have

$$Y_i : H^{(p|q)}(\mathcal{M}) \to H^{(p|q+1)}(\mathcal{M})$$

This observation implies that if there were cohomology in a given space, this can be mapped into a space with another picture. Since the two complexes $\Omega^{(p|0)}(\mathcal{M})$ and $\Omega^{(p|m)}(\mathcal{M})$ are either bounded from below or from above, this means that there is no cohomology below and above.

So, the cohomology is entirely contained into the square bounded by the 0-forms with 0 pictures and from the integral forms with n-form degree and m-picture.

Definition of PCO’s

Definition of the PCO's

Suppose immersing a bosonic surface into a supermanifold

$$\iota : \mathcal{M}^{(n)} \longrightarrow \mathcal{M}^{(n|m)}$$

in the trivial way: by setting the fermionic coordinates to zero.
Then, its Poincaré dual is

$$\mathbb{Y}^{(0|m)}_{spacetime} = \prod_{\alpha=1}^{m} \theta^\alpha \delta(d\theta^\alpha)$$

1. It is closed
2. It is not exact (so it belongs to a cohomology space)
3. Any variation of the immersion is d-exact

$$\delta \mathbb{Y}^{(0|m)}_{spacetime} = d\Omega^{(-1|m)}$$

Cartan calculus on supermanifolds

Differential

$$d = d\theta^\alpha D_\alpha + (dx^m + \theta\gamma^m d\theta)\partial_m$$

Even/Odd Vector fields:

$$v = v^\alpha D_\alpha + v^m \partial_m$$ with v^α odd/even, v^m even/odd

Contraction and Lie derivatives

Even ι_v , $\quad \iota_v^2 = 0\,, \quad \mathcal{L}_v = d\iota_v + \iota_v d$

Odd $\iota_{\tilde{v}}$, $\quad \iota_{\tilde{v}}^2 \neq 0\,, \quad \mathcal{L}_{\tilde{v}} = d\iota_{\tilde{v}} - \iota_{\tilde{v}} d$

New differential operators (distribution-like operators acting on the space of forms)

$$\delta(\iota_{\tilde{v}}) = \int_{-\infty}^{\infty} dt\, e^{it\iota_{\tilde{v}}} \qquad \Theta(\iota_{\tilde{v}}) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \frac{e^{it\iota_{\tilde{v}}}}{t+i\epsilon} dt$$

Finally, following string theory suggestion, we can define our PCO Z. According to our notations, it decreases the picture by removing delta functions.

$$Z_D = \Big[d, \Theta(\iota_D)\Big]$$

- it is closed
- it is not exact (Heaviside Theta function is not a distribution with compact support)
- it depends upon an odd vector field D. But any variation of D implies that Z it is exact
- it is not a derivation with respect to the wedge product of forms
- it acts vertically along the complexes of forms, from integral form to diff. forms
- it can be combined with other PCO's Z as follows

$$Z_{Max} = \prod_{p=0}^{m} Z_{D_p}$$

where the odd vector fields D_p are linearly independent.

How to compute with these PCO's

Given an odd vector field $$D = D^1 \frac{\partial}{\partial\theta^1} + D^2 \frac{\partial}{\partial\theta^2}$$

We define the map (contractions)

$$\iota_D : \Omega^{(p|q)}_{\mathbb{P}^{1|2}} \longrightarrow \Omega^{(p-1|q)}_{\mathbb{P}^{1|2}}\,,$$
$$\omega \longmapsto \iota_D(\omega)$$

$$\iota_D(d\theta^\alpha) = d\theta^\alpha(D) = D^\alpha .$$

It satisfies the Cartan algebra

$$\mathcal{L}_D = [d, \iota_D]\,, \qquad [\mathcal{L}_D, \iota_{D'}] = \iota_{\{D,D'\}}\,, \qquad \{\iota_D, \iota_{D'}\} = 0\,,$$

Acting on integral forms

$$\begin{aligned}\Theta(\iota_D)\delta(d\theta^\alpha) &= -i\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt \frac{e^{it\iota_D}}{t+i\epsilon}\delta(d\theta^\alpha) = -i\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt \frac{\delta(d\theta^\alpha + iD^\alpha t)}{t+i\epsilon} \\ &= -\frac{1}{D^\alpha}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt \frac{\delta(t - i\frac{d\theta^\alpha}{D^\alpha})}{t+i\epsilon} = \boxed{\frac{i}{d\theta^\alpha} \in \Omega^{(-1|0)}_{\mathbb{P}^{1|2}}}\end{aligned}$$

LARGE
HILBERT
SPACE

For more than one fermionic coordinate we have for example

$$\Theta(\iota_{D'})\Theta(\iota_D)\Big(\delta(d\theta_1)\delta(d\theta_2)\Big) = \frac{\det(D',D)}{(D'\cdot d\theta)(D\cdot d\theta)} \in \Omega^{(-2|0)}_{\mathbb{P}^{1|2}}$$

Let us compute for the top form, the action of 2 PCO's

$$\omega^{(1|2)} = A(z,\theta)dz\delta(d\theta^1)\delta(d\theta^2)$$

Acting with the first Z

$$\begin{aligned} Z_D(\omega^{(1|2)}) &= d\left[-i\Theta(\iota_D)Adz\delta(d\theta^1)\delta(d\theta^2)\right] - i\Theta(\iota_D)\left[d\left(Adz\delta(d\theta^1)\delta(d\theta^2)\right)\right] \\ &= d\Big[A\Big(\frac{D^1}{d\theta^1}+\frac{D^2}{d\theta^2}\Big)dz\,\delta(D\cdot d\theta)\Big] \\ &= 2\Big((D^1\partial_1 A + D^2\partial_2 A)\,dz\,\delta(D\cdot d\theta)\Big) \\ &= 2D^\alpha\partial_\alpha A\,dz\,\delta(D\cdot d\theta)\Big) \in \Omega^{(1|1)}_{\mathbb{P}^{1|2}}, \end{aligned}$$

acting with the second Z

$$Z_{D'}\Big[2D^\alpha \partial_\alpha A\, dz\, \delta(D\cdot d\theta)\Big)\Big] \quad = \quad 2\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta A\, dz \in \Omega^{(1|0)}_{\mathbb{P}^{1|2}}$$

- The dependence on D and D' disappeared.
- Since the original form was a top form, then it was closed, the result is closed
- Even if we have passed through the Large Hilbert Space (admitting inverses) LHS, the result is in Small Hilbert Space (SHS).

As in string theory, there exists a differential operator which selects the SHS inside the LHS.

$$\eta = -2\Pi \lim_{\epsilon\to 0}\sin(i\epsilon\iota_D) : \Omega^{(p|q)}_{\mathbb{P}^{1|2}} \to \Omega^{(p+1|q+1)}_{\mathbb{P}^{1|2}}$$

Acting as

$$\begin{cases} \eta\Big(\frac{1}{(d\theta^\alpha)^p}\Big) = \frac{(-1)^{p-1}}{(p-1)!}\delta^{(p-1)}(d\theta^\alpha) & p > 1, \\ \eta\Big((d\theta^\alpha)^p\Big) = 0 & p \geq 0, \\ \eta\Big(\delta^{(p)}(d\theta^\alpha)\Big) = 0 & p \geq 0, \end{cases}$$

Actions on Supermanifolds

Actions on supermanifolds

$$S = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)}(\Phi, d\Phi; V, \psi) \wedge \mathbb{Y}^{(0|m)}(V, \psi)$$

$\mathcal{M}^{(n|m)}$ **Supermanifold**, which locally is described by a superspace with n bosonic coordinates and m fermionic coordinates

(V^a, ψ^α) Supervielbein of the supermanifold a=1,…,n, α=1,…,m

For flat superspace: $V^a = dx^a + \theta\gamma^a d\theta\,, \quad \psi^\alpha = d\theta^\alpha$

$\mathcal{L}^{(n|0)}(\Phi, d\Phi; V, \psi)$ Geometric Lagrangian. It is a function of fields, their differentials, and of the supervielbein. It is a **n-superform** (differential superform)

$\mathbb{Y}^{(0|m)}(V, \psi)$ Poincaré dual to the immersion of a bosonic submanifold into the supermanifold, are view as **Picture Changing Operator**.

n: form number m: picture number

$$S = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)} \wedge \mathbb{Y}^{(0|m)}$$

Choosing a suitable PCO, the geometric action reduces to the component action

$$\mathbb{Y}^{(0|m)}_{space-time}(V, \psi)$$

Choosing a manifest supersymmetric PCO, the geometric action reduces to the superspace action

$$\mathbb{Y}^{(0|m)}_{susy}(V, \psi)$$

$$S = \int_{\mathcal{M}^n_{bos}} [d^n x] \mathcal{L}^{(n)}(\phi, \partial\phi)$$

$$\mathcal{L}^{(n)}(\phi, \partial\phi) = \mathcal{L}^{(n)}(\Phi, \partial\Phi; V, \psi)\Big|_{\theta=0,\psi=0}$$

$$S = \int_{\mathbb{R}^{(n|m)}} [d^n x d^m \theta] \widehat{\mathcal{L}}^{(n)}(\phi, \partial\phi, \theta)$$

Equivalence

$$S = \int_{\mathcal{M}^n_{bos}} [d^n x] \mathcal{L}^{(n)}(\phi, \partial\phi) \quad \longleftrightarrow \quad S = \int_{\mathbb{R}^{(n|m)}} [d^n x d^m \theta] \widehat{\mathcal{L}}^{(n)}(\phi, \partial\phi, \theta)$$

The two actions are equivalent iff

$$\boxed{d\mathcal{L}^{(n|0)}(\Phi, d\Phi; V, \psi) = 0}$$

The action is closed under some conditions (superspace constraints).
Note that it is **n-superform**, so its differential is not trivial.

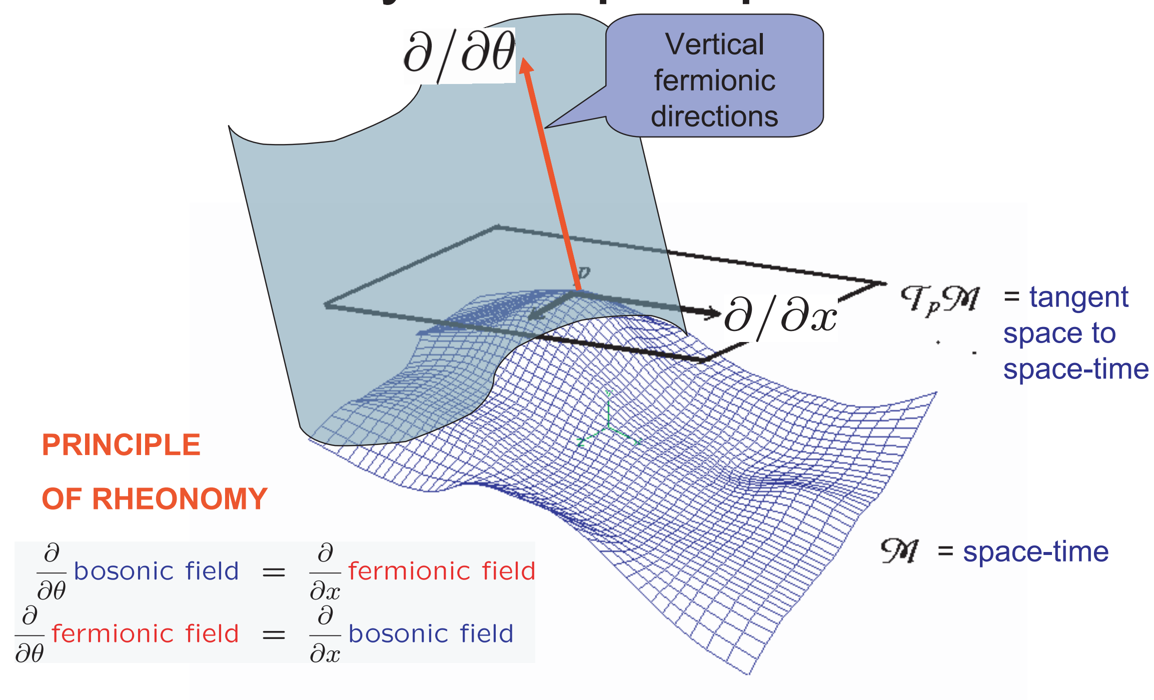
and two different PCO's differ by exact terms

$$\boxed{\mathbb{Y}^{(0|m)}_{susy}(V, \psi) = \mathbb{Y}^{(0|m)}_{spacetime}(V, \psi) + d\Omega^{(-1|m)}}$$

Supergravity Action: from Rheonomy to Integral Forms

The basic observation is the implementation of the idea represented by the following picture

Rheonomy of superspace



which implies that all the fields are promoted to superfields (differently to what happen to usual superspace approach where the different components of a superfield describe different representatives of the supermultiplet),

In that context, to get an action, reproducing the parametrizations and the equations of motion, one constructs a Lagrangian which is a n-form (depending upon all coordinates and 1-forms)

$$\mathcal{L}_{Sugra}(x, dx, \theta, d\theta) \longrightarrow S_{\rm rheo} = \int_{\mathcal{M}} \mathcal{L}_{Sugra}(x, dx, 0, 0)$$

Wess-Zumino model

Wess-Zumino model D=4 N=1

It is described by a superfield Φ, whose differential is

$$
\begin{aligned}
d\Phi &= V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Phi + \psi^\alpha D_\alpha \Phi + \bar\psi^{\dot\alpha}\bar D_{\dot\alpha}\Phi \\
&= V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Phi + \psi^\alpha W_\alpha\,,
\end{aligned}
$$

W is a new superfield whose first component is the super partner of the scalar.
Its differential is

$$
dW_\alpha = V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}W_\alpha - 2i\bar\psi^{\dot\alpha}\partial_{\alpha\dot\alpha}\Phi + \psi_\alpha F\,,
$$

F is a superfield whose first component is the auxiliary field and its differential is

$$
dF = V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}F + 2i\bar\psi^{\dot\alpha}\partial_{\dot\alpha\alpha}W^\alpha\,,
$$

$$
\Phi = \phi + \mathcal{O}(\theta)\,, \qquad W_\alpha = \lambda_\alpha + \mathcal{O}(\theta)\,, \qquad F = f + \mathcal{O}(\theta)\,.
$$

$$S = \int_{\mathcal{SM}^{(4|4)}} \mathcal{L}^{(4|0)}(\Phi, W, F) \wedge \mathbb{Y}^{(0|4)} \,,$$

The action is a (4|0) -superform, built with the superfields, their differentials, and the supervielbeins. It is known as **Rheonomic action** for Wess-Zumino theory (the fields ξ and its conjugate are needed for a first order formalism).

$$\begin{aligned}
\mathcal{L}^{(4|0)}_{kin} &= (V^4)\,(\bar{\xi}^{\alpha\dot{\alpha}}\xi_{\alpha\dot{\alpha}} + \bar{F}F) \\
&+ (V^3)^{\alpha\dot{\alpha}}\Big[(d\Phi - \psi^\beta W_\beta)\bar{\xi}_{\alpha\dot{\alpha}} + (d\bar{\Phi} - \bar{\psi}^{\dot{\beta}}\bar{W}_{\dot{\beta}})\xi_{\alpha\dot{\alpha}} + x(\bar{W}_{\dot{\alpha}} dW_\alpha + d\bar{W}_{\dot{\alpha}} W_\alpha)\Big] \\
&+ (V^2_+)^{\alpha\beta}\left[y(W_\alpha \psi_\beta d\bar{\Phi}) + z(W_\alpha \psi_\beta \bar{W}^{\dot{\gamma}} \bar{\psi}_{\dot{\gamma}})\right] \\
&+ (V^2_-)^{\dot{\alpha}\dot{\beta}}\Big[y(\bar{W}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}} d\Phi) + z(\bar{W}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}} W^\gamma \psi_\gamma)\Big] \\
&+ V^{\alpha\dot{\alpha}}\left[t(\bar{\Phi} d\Phi - d\bar{\Phi}\Phi)\psi_\alpha \bar{\psi}_{\dot{\alpha}}\right] \,.
\end{aligned}$$

the parameters (x,t,y,z) are fixed to (1/2, - i/2, 1,1) and the super potential

$$\begin{aligned}
\mathcal{L}^{(4|0)}_{sup} &= \Big(\mathcal{W}'(\Phi)F - \frac{1}{2}\mathcal{W}''(\Phi) W_\alpha W^\alpha\Big)(V^4) + \mathcal{W}'(\Phi) W^\alpha \bar{\psi}^{\dot{\alpha}} (V^3)_{\alpha\dot{\alpha}} \\
&+ \mathcal{W}(\Phi)\bar{\psi}^{\dot{\alpha}}\bar{\psi}^{\dot{\beta}}(V^2_-)_{\dot{\alpha}\dot{\beta}} + \text{h.c.}
\end{aligned}$$

Let us choose the following PCO

$$\boxed{\mathbb{Y}^{(0|4)}_{s.t.} = \theta^2 \delta^2(\psi) \wedge \bar{\theta}^2 \delta(\bar{\psi}) \,,}$$

applying the formula we get

$$\begin{aligned}\mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)}_{s.t.} = & \Big[(\bar{\xi}^{\alpha\dot{\alpha}} \xi_{\alpha\dot{\alpha}} + \bar{f} f) d^4x \\ & + \Big(d\phi \bar{\xi}^{\alpha\dot{\alpha}} + d\bar{\phi} \xi^{\alpha\dot{\alpha}} + \frac{i}{2}(\bar{\lambda}^{\dot{\alpha}} d\lambda^\alpha + d\bar{\lambda}^{\dot{\alpha}} \lambda^\alpha)\Big)(d^3x)_{\alpha\dot{\alpha}} \\ & + \Big(\mathcal{W}'(\phi) f - \frac{1}{2}\mathcal{W}''(\phi)\lambda^\alpha \epsilon_{\alpha\beta}\lambda^\beta\Big) d^4x + \text{h.c.}\Big] \theta^2 \bar{\theta}^2 \delta^2(\psi) \delta^2(\bar{\psi}) \,,\end{aligned}$$

which is the component action (in the first order formalism) with the super potential W(Φ).

The action is supersymmetric up to total derivatives.

This is due to the fact that the PCO is not manifestly supersymmetric, but its supersymmetry variation is d-exact. Since the rheonomic action is d-closed, any variation of the PCO is compensated by an integration-by-part.

On the other side, we can use another PCO given by

$$\boxed{\mathbb{Y}^{(0|4)}_{s.s.} = \Big(- 4(\theta V \bar\iota) \wedge (\bar\theta V \iota) + \theta^2 (\iota V \wedge V \iota) + \bar\theta^2 (\bar\iota V \wedge V \bar\iota) \Big) \delta^4(\psi)}$$

It is an element of the cohomology.

It shows three pieces: the first one is non-chiral and hermitian.
The second and the third terms are chiral and conjugated to each other.

and finally we have

$$\begin{aligned}
S &= \int_{\mathcal{M}^{(4|4)}} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)}_{s.s.} \\
&= \int_{\mathcal{M}^{(4|4)}} \Big(\overline{W} V \psi)(\bar\psi V W) + \mathcal{W}(\Phi)(\bar\psi V \wedge V \bar\psi) + \overline{\mathcal{W}}(\bar\Phi)(\psi V \wedge V \psi) \Big) \wedge \mathbb{Y}^{(0|4)}_{s.s.} \\
&= \int_{\mathcal{M}^{(4|4)}} \Big(\overline{W}^{\dot\alpha} \bar\theta_{\dot\alpha} W_\alpha \theta^\alpha + \mathcal{W}(\Phi) \bar\theta^2 + \overline{\mathcal{W}}(\bar\Phi)\theta^2 \Big) V^4 \delta^4(\psi) \\
&\boxed{= \int [d^4 x d^2\theta d^2\bar\theta] \Big(\overline{W}^{\dot\alpha} \bar\theta_{\dot\alpha} W_\alpha \theta^\alpha + \mathcal{W}(\Phi) \bar\theta^2 + \overline{\mathcal{W}}(\bar\Phi)\theta^2 \Big) \, .}
\end{aligned}$$

The last line is the superspace W-Z action.

D=4 N=1 Supergravity

The restriction to the bosonic subspace can be implemented by observing that the choice of the submanifold immersed into the supermanifold corresponds exactly to the following (4|4)-form

$$\omega^{4|4} = \mathcal{L}_{Sugra}(x, dx, \theta, d\theta)\theta^2\bar{\theta}^2\delta^2(d\theta)\delta^2(d\bar{\theta}) = \mathcal{L}_{Sugra}(x, dx, 0, 0)\theta^2\bar{\theta}^2\delta^2(d\theta)\delta^2(d\bar{\theta}) \,,$$

which can be integrated on the full supermanifold

$$S_{\rm rheo} = \int_{\widehat{\mathcal{M}}} \mathcal{L}_{Sugra}(x, dx, 0, 0)\theta^2\bar{\theta}^2\delta^2(d\theta)\delta^2(d\bar{\theta})$$

where

$$\mathcal{L}_{Sugra} = \epsilon_{abcd}\,\mathfrak{R}^{ab} \wedge V^c \wedge V^d - 4\left(\bar{\psi}^\bullet \wedge \gamma_a\,\rho_\bullet + \rho^\bullet \wedge \gamma_a\,\psi_\bullet\right) \wedge V^a$$

then, if it must be a (4|4)-form, why don't we put the basic volume form?

$$\Omega^{(4|4)} = \epsilon_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d \wedge \delta^2(\psi) \wedge \delta^2(\bar{\psi})$$

but finally by passing to the basis (dx, dθ) we get

$$\boxed{\Omega^{(4|4)} = \mathrm{Sdet}(E)\, d^4x \wedge \delta^2(d\theta) \wedge \delta^2(d\bar{\theta}) \,,}$$

Super-Yang-Mills d=10

SUPER YANG-MILLS D=10 N=1

As usual the gauge field is described by a gauge potential and its field strength

$$
\begin{aligned}
F^{(2|0)} &\equiv dA^{(1|0)} + A^{(1|0)} \wedge A^{(1|0)} \\
&= F_{ab} V^a \wedge V^b + F_{a\alpha} V^a \wedge \psi^\alpha + F_{\alpha\beta} \psi^\alpha \wedge \psi^\beta \,,
\end{aligned}
$$

to remove redundant degrees of freedom, one imposes the following constraints

$$
F_{\alpha\beta} = \nabla_{(\alpha} A_{\beta)} + \gamma^a_{\alpha\beta} A_a = 0 \,,
$$

and they imply the equations of motion

$$
\nabla^a F_{ab} = 0 \,, \qquad \gamma^a_{\alpha\beta} \nabla_a W^\beta = 0 \,.
$$

The action is

$$
\begin{aligned}
\mathcal{L}^{(10|0)} \;=\;& \Big(-\frac{1}{90}\mathfrak{F}_{ab}\mathfrak{F}^{ab}\, V^{a_1}{}_\wedge \ldots {}_\wedge V^{a_{10}} + \mathfrak{F}^{a_1 a_2}\, F^{(2|0)}{}_\wedge V^{a_3} \ldots {}_\wedge V^{a_{10}} \\
& +2i\mathfrak{F}^{a_1 a_2} W \gamma_a \psi_\wedge V^a{}_\wedge V^{a_3} \ldots {}_\wedge V^{a_{10}} + \frac{4}{9} i W \gamma^{a_1} \nabla W_\wedge V^{a_2} \ldots {}_\wedge V^{a_{10}} \\
& +\frac{8}{3} i W \gamma^{a_1 \ldots a_3} \psi_\wedge F^{(2|0)}{}_\wedge V^{a_4} \ldots {}_\wedge V^{a_{10}} \\
& + \left(1+\frac{3}{8}a\right) W \gamma^{a_1 \ldots a_3} W \psi_\wedge \gamma_a \psi_\wedge V^a{}_\wedge V^{a_4}_\wedge \ldots {}_\wedge V^{a_{10}} \\
& +a\, W\, \gamma^{a_1 a_2 b}\, W\, \psi_\wedge \gamma_b \psi_\wedge V^{a_3}_\wedge \ldots {}_\wedge V^{a_{10}} \Big)\ \epsilon_{a_1 \ldots a_{10}} \\
& -84i \left(A^{(1|0)}{}_\wedge F^{(2|0)} - \frac{1}{3} A^{(1|0)}{}_\wedge A^{(1|0)}{}_\wedge A^{(1|0)}\right) {}_\wedge \psi_\wedge \gamma^{a_1 \ldots a_5} \psi_\wedge V_{a_1 \wedge} \ldots {}_\wedge V_{a_5}
\end{aligned}
$$

If projected with the spacetime PCO, we get the component action

$$
\begin{aligned}
S \;&=\; \int_{\mathcal{M}^{(10|16)}} \mathcal{L}^{(10|0)} \wedge \mathbb{Y}^{(0|16)} = \int \Big(-\frac{1}{90}\mathfrak{F}_{ab}\mathfrak{F}^{ab}\, V^{a_1}{}_\wedge \ldots {}_\wedge V^{a_{10}} \\
&+\; \mathfrak{F}^{a_1 a_2}\, F^{(2|0)}{}_\wedge V^{a_3} \ldots {}_\wedge V^{a_{10}} + \frac{4}{9} i W \gamma^{a_1} \nabla W_\wedge V^{a_2} \ldots {}_\wedge V^{a_{10}} \Big)\ \epsilon_{a_1 \ldots a_{10}} \wedge \mathbb{Y}^{(0|16)} \\
&=\; \int d^{10}x \left(-\frac{1}{4}\mathcal{F}_{ab}\mathcal{F}^{ab} + \chi \gamma^a \nabla_a \chi\right)
\end{aligned}
$$

but with a new PCO (inspired by Pure Spinor String Theory)

$$\mathbb{Y}^{(0|16)}_{p.s.} = V^{a_1}{}_\wedge V^{a_2}{}_\wedge V^{a_3}{}_\wedge V^{a_4}{}_\wedge V^{a_5} \epsilon_{\beta_1 \dots \beta_{16}} \theta^{\beta_1} \dots \theta^{\beta_{11}} (\gamma_{a_1} \iota)^{\beta_{12}} \dots (\gamma_{a_5} \iota)^{\beta_{16}} \delta^{16}(\psi)$$

we finally get the action

$$S = \int_{\mathcal{M}^{(10|16)}} \left[\left(\mathcal{A}_\wedge \mathcal{F} - \frac{1}{3} \mathcal{A}_\wedge \mathcal{A}_\wedge \mathcal{A} \right) \wedge \psi \gamma^{a_1 \dots a_5} \psi_\wedge V_{a_1 \wedge} \dots {}_\wedge V_{a_5} \right] \wedge \mathbb{Y}^{(0|16)}_{p.s.}$$

computing the integral (Berezin and form integrals), one found the component action. Without computing the Berezin integral, one finds the Berkovits formulation of Super-Yang-Mills based on pure spinor formulation.

Supergravity d=11

As is well-kwon D=11 supergravity is a very geometrical model. For that the Rheonomic construction has been given by D'Auria and Fré in a very detailed paper. The fields, the action and the equations of motion are given and they are based on Maurer-Cartan form related to gauging a given group.

The fields are the usual supervielbein E, the spin connection and the 3-form A

They curvatures are given by the equations

$$
\begin{aligned}
R^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^{b} \,, \\
T^a &= \mathcal{D}V^a - \frac{i}{2}\bar{\psi} \wedge \Gamma^a \psi \,, \\
\rho &= \mathcal{D}\psi = d\psi - \frac{1}{4}\omega_{ab} \wedge \Gamma^{ab}\psi \,, \\
F &= dA - \frac{1}{2}\bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b \,,
\end{aligned}
$$

Again the use of the differential forms has a lot of advantages. All fields are extended to be super fields (or superforms) which depend upon the anticommuting coordinates. The supermanifold has dimension (11|32).

With these definitions, we can write the Rheonomic action as a (11|0) differential form on the (11|32) supermanifold.

$$
\begin{aligned}
\mathcal{L}^{(11|0)} = & -\frac{1}{9} R^{a_1 a_2}{}_{\wedge} V^{a_3} \ldots {}_{\wedge} V^{a_{11}} \epsilon_{a_1 \ldots a_{11}} \\
& + \frac{7i}{30} T^a{}_{\wedge} V_{a \wedge} \bar{\psi}_{\wedge} \Gamma^{b_1 \ldots b_5} \psi_{\wedge} V^{b_6} \ldots {}_{\wedge} V^{b_{11}} \epsilon_{b_1 \ldots b_{11}} \\
& + 2 \bar{\rho}_{\wedge} \Gamma_{c_1 \ldots c_8} \psi_{\wedge} V^{c_1} \ldots {}_{\wedge} V^{c_8} \\
& - 84 F_{\wedge} \left(i \bar{\psi}_{\wedge} \Gamma_{a_1 \ldots a_5} \psi V^{a_1} \ldots V^{a_5} - 10 A_{\wedge} \bar{\psi}_{\wedge} \Gamma_{ab} \psi_{\wedge} V^a{}_{\wedge} V^b \right) \\
& + \frac{1}{4} \bar{\psi}_{\wedge} \Gamma_{a_1 a_2} \psi_{\wedge} \bar{\psi}_{\wedge} \Gamma_{a_3 a_4} \psi_{\wedge} V^{a_5} \ldots {}_{\wedge} V^{a_{11}} \epsilon_{a_1 \ldots a_{11}} \\
& - 210\, \bar{\psi}_{\wedge} \Gamma_{a_1 a_2} \psi_{\wedge} \bar{\psi}_{\wedge} \Gamma_{a_3 a_4} \psi_{\wedge} V_{a_1} \ldots {}_{\wedge} V_{a_4 \wedge} A \\
& - 840 F_{\wedge} F_{\wedge} A + \frac{1}{330} F_{a_1 \ldots a_4} F^{a_1 \ldots a_4} V^{c_1} \ldots {}_{\wedge} V^{c_{11}} \epsilon_{c_1 \ldots c_{11}} \\
& + 2 F_{a_1 \ldots a_4} F_{\wedge} V_{a_5} \ldots {}_{\wedge} V_{a_{11}} \epsilon^{a_1 \ldots a_{11}}
\end{aligned}
$$

Note that by using the space time PCO, this reproduces the usual component action of CJS D=11 supergravity (the 3-form field strength action is in the first order formalism).

A new PCO can be used instead (here we give only
its first term to give an idea how to reproduce the full action)

$$\mathbb{Y}^{(0|32)} = \epsilon_{\alpha_1 \ldots \alpha_{32}} \theta^{\alpha_1} \ldots \theta^{\alpha_{23}} (V_{a_1} \Gamma^{a_1} \iota)^{\alpha_{24}} \ldots (V_{a_9} \Gamma^{a_9} \iota)^{\alpha_{32}} \delta^{32}(\psi)$$

Inserting this expression in the integral form action, we finally select the following two-pieces.

$$\mathcal{L}^{(11|0)} \wedge \mathbb{Y}^{(0|32)} = 840 \Big(F \wedge A_\wedge (\bar{\psi}_\wedge \Gamma_{ab} \psi)_\wedge V^a{}_\wedge V^b - F_\wedge F_\wedge A \Big) \wedge \mathbb{Y}^{(0|32)}$$

The expression has to be computed using Berezinian integration to
see how the different kinetic terms emerge. However, we have a faster way to
check, at least the very few terms. It can be compared with the action of the form

$$S_{SFT} = \left\langle \Phi^{(1)}, Q\Phi^{(1)} \right\rangle$$

given by N. Berkovits and by M. Cederwall for pure spinor superspace formulation.

Conclusions and future directions

- We explored the integral form formalism for quantum field theories. For rigid supersymmetric models to supergravity theories. The framework unifies all superspace formulations in a geometrical picture.

- We also explored non-factorised Lagrangian (such as Chern-Simons theory with N=1 susy where the gauge fields are taken in the (1|1) form/picture and the action leads to A-infinity structures).

- The supergravity models still require a deep analysis to understand how to formulation works in the case of curved supermanifolds. The full fledged analysis has been performed only for 3D case (even for massive gravity).

- Harmonic superspace is a useful technique used for supergravity models, but in the preset case has not been yet developed.

- The quantum version is not yet studied

the end

thank you for the attention.

Super Chern-Simons and non-associative algebras

Let us now consider the supersymmetric version of CS theory. The general expression is of the following form

$$S_{SCS} = \int_{\mathcal{SM}^{(3|2)}} \mathrm{Tr}\Big(A^{(1|0)} \wedge dA^{(1|0)} + \frac{2}{3} A^{(1|0)} \wedge A^{(1|0)} \wedge A^{(1|0)} + \frac{1}{2} W^{(0|0)} \cdot W^{(0|0)} V^{(3|0)}\Big) \wedge \mathbb{Y}^{(0|2)}$$

- The gauge connection is replaced by a (1|0) form.
- The gauge group is still a bosonic group, the gauge connection is Lie-algebra valued.
- The integral is over the full supermanifold according to the discussion above.
- $\mathbb{Y}^{(0|2)}$ is a generic PCO, which transforms the action into a integral form.

The function $W^{(0|0)}$ is related to the gauge connection using the Bianchi identities

$$F^{(2|0)} = V^a \wedge V^b F_{[ab]} + V^a (\psi \gamma^b W) \qquad dW^\alpha = V^a \nabla_a W^\alpha - \frac{1}{4} (\gamma^{ab} \psi)^\alpha F_{[ab]}$$

The 3d vielbeins satisfy the following MC equations

$$d\psi = 0 \qquad dV^a = \frac{1}{2} \psi \gamma^a \psi \qquad V^{(3|0)} = \epsilon_{abc} V^a \wedge V^b \wedge V^c$$

• If we simplified to the an abelian gauge group, we drop the interaction term (and the covariant derivative from the Bianchi ids).

$$S^{abelian}_{SCS} = \int_{\mathcal{SM}^{(3|2)}} \left(A^{(1|0)} \wedge dA^{(1|0)} + \frac{1}{2} W^{(0|0)} \cdot W^{(0|0)} V^{(3|0)}\right) \wedge \mathbb{Y}^{(0|2)}$$

• Now, we observe that any (0|2) PCO can be decomposed into a product to two (0|1) PCO (with the correct properties) - up to total derivatives

$$\mathbb{Y}^{(0|2)} = \mathbb{Y}^{(0|1)} \wedge \mathbb{Y}^{(0|1)} + d\Omega^{(-1|2)}$$

• Thus, we can rewrite the action by distributing the PCO's over the fields as follows

$$S^{abelian}_{SCS} = \int_{\mathcal{SM}^{(3|2)}} \left((A^{(1|0)} \wedge \mathbb{Y}^{(0|1)}) \wedge d(A^{(1|0)} \wedge \mathbb{Y}^{(0|1)}) + \frac{1}{2}(W^{(0|0)}\mathbb{Y}^{(0|1)}) \cdot (W^{(0|0)}\mathbb{Y}^{(0|1)}) V^{(3|0)}\right)$$

• Finally, we can rewrite the action as follows (in terms of pseudoforms)

$$S^{abelian}_{SCS} = \int_{\mathcal{SM}^{(3|2)}} \left(A^{(1|1)} \wedge dA^{(1|1)} + \frac{1}{2} W^{(0|1)} \cdot W^{(0|1)} V^{(3|0)}\right)$$

To add the interactions, we need a 2-product with the following property:

$$M_2 : \Omega^{(1|1)} \otimes \Omega^{(1|1)} \longrightarrow \Omega^{(2|1)}$$

This situation has strong analogies with string theory and superstring theory (the ghost number has to be correctly compensated for meaningful actions. In the case of superstrings the picture number has the same role as here. It must be saturated for non-trivial contributions. In the case of g super Riemann surfaces q = 2-2g

Using the PCO Z, $Z_D : \omega^{(p|q)} \longrightarrow \omega^{(p|q-1)}$

Erler, Konopka and Sachs (arXiv:1312.2948) proposed the expression

$$M_2\big(\omega_1^{(1|1)}, \omega_2^{(1|1)}\big) = \frac{1}{3}\Big(Z_D(\omega_1^{(1|1)} \wedge \omega_2^{(1|1)}) + Z_D(\omega_1^{(1|1)}) \wedge \omega_2^{(1|1)} + (\omega_1^{(1|1)}) \wedge Z_D(\omega_2^{(1|1)})\Big)$$

in terms of which we have

$$\mathcal{L}^{Int} = \mathrm{Tr}\Big(A^{(1|1)} \wedge M_2(A^{(1|1)}, A^{(1|1)})\Big)$$

The 2-product of EKS is not associative

$$M_2(M_2(A,B),C) + M_2(A, M_2(B,C)) \neq 0$$

If we identify the differential d with the 1-product, with the property

$$dM_2(A,B) = M_2(dA,B) + (-1)^{|A|} M_2(A, dB))$$

it turns out that the 2-product satisfies

$$M_2(M_2(A,B),C) + M_2(A, M_2(B,C)) =$$
$$dM_3(A,B,C) + M_3(dA,B,C) + (-1)^{|A|} M_3(A,dB,C) + (-1)^{|A|+|B|} M_3(A,B,dC)$$

which is the starting relation for an A_∞ algebra.

Every algebraic structure is purely based on differential forms and on the supergeometry I discussed. The A_∞ is extended to the whole complex of forms.

The final action with the complete set of interaction terms, which is gauge invariant under the L-infinity gauge transformation is now an integral form.

$$S_{SCS} = \int_{\mathcal{SM}^{(3|2)}} Tr\Big(A^{(1|1)} \wedge \sum_{n=1}^{\infty} M_n(A^{(1|1)}, A^{(1|1)})\Big)$$

- It gives the correct equations of motion
- It is supersymmetric (it is not easy to see, it satisfies the susy constraints.
- It is gauge invariant
- It is invariant under superdiffeomorphisms
- It has exactly the same form as Open Superstring Field Theory
- The dependence by the vector v inside of Z is exact and it drops out.