Comments on heterotic/F-theory duality in 8d

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Motivation

F-theory on elliptically fibered K3 dual to Heterotic on T^2 [Vafa; Vafa,Morrison '96]

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map known at every point in moduli space in few cases, e.g.(2 moduli):

K3 $$y^2 = x^3 + a\, u^4 v^4\, x\, z^4 + (b\, u^6 v^6 + d\, u^5 v^7 + u^7 v^5)\, z^6$$

x, y, z, and u, v: homogeneous coordinates of the fiber ambient variety $\mathbb{P}_{2,3,1}$, and the base $\mathbb{P}^1$

singularities: II^* (E_8) at $v = 0$, II^* (E_8) at $u = 0$

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Wilson lines

Heterotic $$\tau = \frac{G_{12} + i\sqrt{\det G}}{G_{11}},\ \rho = B_{12} + i\sqrt{\det G},\quad a_1 = a_2 = 0 \text{ i.e. no WL}$$

group $E_8 \times E_8 \times U(1)^2$, $U(1)^2$ can enhance, e.g. to $SU(3)$ at $\tau = \rho = e^{i\pi/3}$

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$(\tau,\rho) \leftrightarrow (a,b,d)$ via $SL(2,\mathbb{Z})$ modular invariant j [Cardoso, Curio, Lüst, Mohaupt '96]

$$j(\tau)j(\rho) = -1728^2\,\frac{a^3}{27d}\,, \quad (j(\tau)-1728)(j(\rho)-1728) = 1728^2\,\frac{b^2}{4d}$$

mirror map [Lerche, Stieberger '99]

$\tau = \rho = e^{i\pi/3} \Rightarrow j(\tau) = j(\rho) = 0 \Rightarrow a = 0, b = \pm 2, d = 1$

$\Rightarrow$ extra K3 singularity at $u = \mp v$, type IV $(SU(3))$

without WL, also heterotic with gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2 \times U(1)^2$

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HE and HO K3's are birationally equivalent [McOrist, Morrison, Sethi '10]

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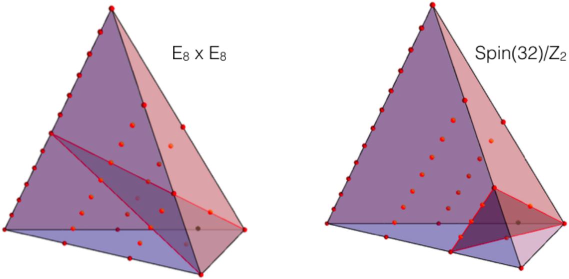
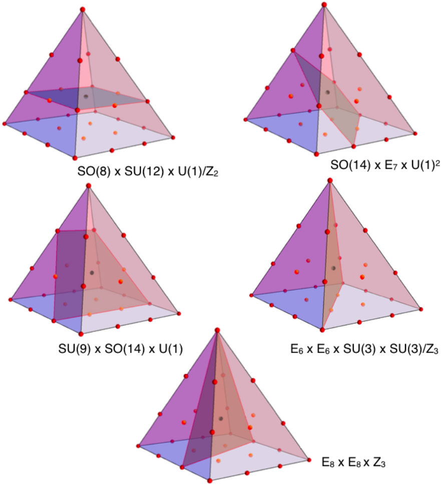
two fibrations of same 3d reflexive polyhedron [Candelas, Skarke '97]



taken from Candelas, Constantin, Damian, Larfors, Morales (CCDLM) '14

Kreuzer-Skarke list: 9 K3's with 2 moduli, each with several fibrations, e.g.

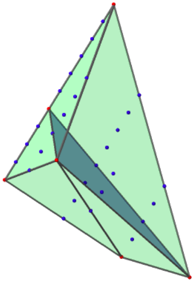
from CCDLM '14



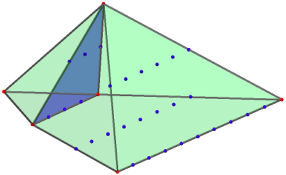
CCDLM found mirror map in 3 of the 9 K3's

heterotic duals ? matching of enhancement patterns ?

more moduli ? 24 K3's with 3 moduli in Kreuzer-Skarke catalog, e.g.



$E_8 \times E_7$



$SU(2) \times \mathrm{Spin}(28)/\mathbb{Z}_2$

map to heterotic in terms of Siegel modular forms [Malmendier, Morrison '14]

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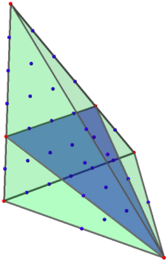
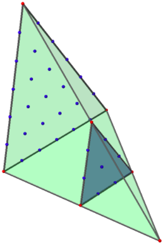
less moduli ? 2 K3's with 1 modulus in Kreuzer-Skarke list, e.g.

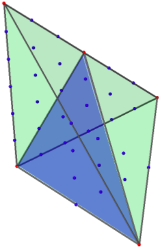
$$x_1^6 + x_2^6 + x_3^6 + x_4^2 - c_0\, x_1\, x_2\, x_3\, x_4 = 0$$



$E_7 \times SO(20)/\mathbb{Z}_2$



$SU(18)/\mathbb{Z}_3$



$E_8 \times E_8 \times SU(2)$

known mirror map [Lian, Yau '94]

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Outline

- Motivation ✓
- Review of moduli spaces
  - Heterotic on T^2
  - F-theory on K3
- Examples
  - 3 moduli
  - 2 moduli
  - 1 modulus
- Final remarks

Review of moduli spaces

Heterotic on T^2

massless states arrange in 8d, $\mathcal{N} = 1$ multiplets: gravity and gauge

gauge group has rank 18, not including graviphotons

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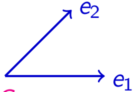

$G_{ij} = e_i \cdot e_j$

moduli: bkgd of metric and B field $\{G_{11}, G_{12}, G_{22}, B_{12}\}$

2 Wilson lines a_i^K, $K = 1, \ldots, 16$, $\quad \langle A_i \rangle = a_i^K H_K$ in Cartan

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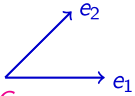

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Narain moduli space: $\underbrace{O(18,2,\mathbb{Z})}_{\text{T-duality group}} \backslash O(18,2,\mathbb{R}) / O(18,\mathbb{R}) \times O(2,\mathbb{R})$

$O(18,2,\mathbb{Z})$ generators: shift of B_{12} by integer, change of $\{e_1, e_2\}$ basis, factorized dualities (exchange of winding and momenta in one direction) traslation of a_i by vector Q in even, self-dual lattice Γ_{16}

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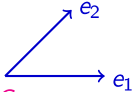

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ex. for no WL, T-duality group $O(2,2,\mathbb{Z}) \sim SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) \times \mathbb{Z}_2 \times \mathbb{Z}_2$

acting on (τ, ρ)

Heterotic on T^2, complex moduli

[Kiritsis, Obers '97]

$$\tau = \frac{G_{12} + i\sqrt{\det G}}{G_{11}}, \qquad \zeta = a_1 \tau - a_2, \qquad \rho = B_{12} + i\sqrt{\det G} + \tfrac{1}{2} a_1 \cdot \zeta$$

complex structure — Wilson lines — Kähler

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selected elements of $O(18,2,\mathbb{Z})$ acting on (τ,ρ,ζ)

$$\tau' = -\tfrac{1}{\tau}, \quad \rho' = \rho - \tfrac{\frac{1}{2}\zeta^2}{\tau}, \quad \zeta' = \tfrac{\zeta}{\tau}, \qquad \tau' = \tau+1, \quad \rho' = \rho, \quad \zeta' = \zeta$$

$$\tau' = \tau, \quad \rho' = \rho + \zeta\cdot Q + \tfrac{1}{2}Q^2\tau, \quad \zeta' = \zeta + Q\tau, \qquad \tau' = -\bar{\tau}, \quad \rho' = -\bar{\rho}, \quad \zeta' = \bar{\zeta}$$

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Narain lattice vectors $(P, p_L; p_R)$

$$p_R^2 = \frac{1}{2|\rho_2\tau_2 - \frac{1}{2}\zeta_2^2|}\left|m_2 - \tau m_1 + \rho n^1 + (\rho\tau - \tfrac{1}{2}\zeta^2)n^2 + \zeta\cdot q\right|^2$$

$$P^2 + p_L^2 - p_R^2 = q^2 + 2n^i m_i$$

m_i: quantized momenta, n^i: winding, $q \in \Gamma_{16}$

massless states have $p_R = 0$, gauge enhancement when $P^2 + p_L^2 = 2$

Heterotic on T^2, the $O(3,2,\mathbb{Z})$ case

choose WLs that break an $SU(2)$

$$a_i = (\lambda_i, \lambda_i, 0, ..., 0) \;\Rightarrow\; \zeta = (\beta, \beta, 0, ..., 0), \qquad \beta = \lambda_1 \tau - \lambda_2$$

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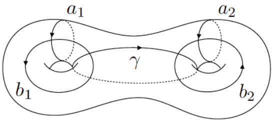
$\{\mathcal{A}_1, \mathcal{B}_1, \Gamma, \mathcal{Z}_1\}$ generate $Sp(4,\mathbb{Z})$

$\{\mathcal{A}_1, \mathcal{B}_1, \Gamma, \mathcal{A}_2, \mathcal{B}_2\}$: Dehn twists



gen.	τ'	ρ'	β'
$\mathcal{A}_1$	$\tau + 1$	ρ	β
$\mathcal{B}_1$	$\frac{\tau}{1-\tau}$	$\rho + \frac{\beta^2}{1-\tau}$	$\frac{\beta}{1-\tau}$
Γ	$\tau + 1$	$\rho + 1$	$\beta - 1$
$\mathcal{Z}_1$	ρ	τ	β
$\mathcal{Z}_2$	$-\bar{\tau}$	$-\bar{\rho}$	$\bar{\beta}$
$\mathcal{A}_2$	τ	$\rho + 1$	β
$\mathcal{B}_2$	$\tau + \frac{\beta^2}{1-\rho}$	$\frac{\rho}{1-\rho}$	$\frac{\beta}{1-\rho}$

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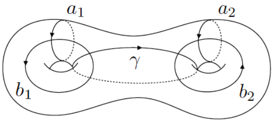

(τ, ρ, β) are moduli of genus-two surface

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$\mathcal{B}_2$	$\tau + \frac{\beta^2}{1-\rho}$	$\frac{\rho}{1-\rho}$	$\frac{\beta}{1-\rho}$

$$\Omega = \begin{pmatrix} \tau & \beta \\ \beta & \rho \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4,\mathbb{Z}), \qquad \Omega \to (A\Omega + B)(C\Omega + D)^{-1}$$

period matrix

F-theory on K3

elliptically fibered K3: $y^2 = x^3 + f(u,v)\,x\,z^4 + g(u,v)\,z^6$

f, g polynomials (sections) of degree 8 and 12 in $(u,v) \in$ base $\mathbb{P}^1$

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complex degrees of freedom$=$ 9 + 13 - 4 $=$18

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moduli space: $\mathcal{M}_{\mathrm{K3},n} = O(\Gamma_{n,2})\backslash O(n,2,\mathbb{R})/O(n,\mathbb{R}) \times O(2,\mathbb{R})$

moduli space of algebraic K3 with Picard number $(20-n)$ [Aspinwall '96]

[Braun, Fucito, Morales '13]

$O(\Gamma_{n,2})$: group of isometries of sublattice of $H^2(\mathrm{K3},\mathbb{Z})$ orthogonal to Picard lattice

$O(\Gamma_{n,2}) \sim O(n,2,\mathbb{Z})$, $\mathcal{M}_{\mathrm{K3},n} \sim \mathcal{M}_{\mathrm{het},n}$, n: number of complex moduli

F-theory on K3, gauge group

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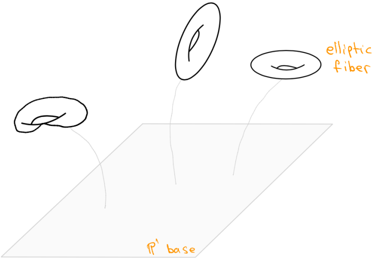

7-branes located where fiber degenerates

singularity type $\Rightarrow$ gauge group in 7-brane

depends on vanishing deg $\mu(f), \mu(g), \mu(\Delta)$

$\Delta = 4f^3 + 27g^2$

$\mu(f)$	$\mu(g)$	$\mu(\Delta)$	type	singularity
≥ 0	≥ 0	0	I_0	–
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	cusp
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
≥ 2	≥ 3	6	I_0^*	D_4
2	3	$n+6$	I_n^*	D_{4+n}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Kodaira Classification

Examples

3 moduli

3 moduli example, the F-theory story

HE: $\quad y^2 = x^3 + (a\,u^4 v^4 + c\,u^3 v^5)\,x\,z^4 + (b\,u^6 v^6 + d\,u^5 v^7 + u^7 v^5)\,z^6$

singularities: II^* (E_8) at $v = 0$, III^* (E_7) at $u = 0$ $\to \mathrm{II}^*$ for $c = 0$ $\Rightarrow$ no WL

HO: $\quad y^2 = x^3 + v(u^3 + a\,u\,v^2 + b\,v^3)\,x^2 z^2 + v^7(c\,u + d\,v)\,x\,z^4$

singularities: I_2 ($SU(2)$) at $cu + dv = 0$, I_{10}^* ($SO(28)$) at $v = 0$ $\to \mathrm{I}_{12}^*$ for $c = 0$

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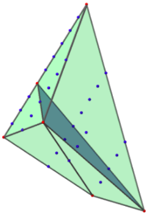

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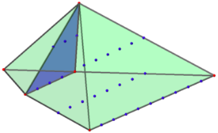

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generic gauge group: $E_8 \times E_7 \times U(1) \times U(1)^2$ HE

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for $\beta \neq 0$, $U(1)^3$ can enhance to

a) $SU(2) \times U(1)^2$ $\quad \tau = \rho$

b) $SU(2)^2 \times U(1)$ $\quad \tau = \rho, \beta = -\frac{1}{2}$

c) $SU(3) \times U(1)$ $\quad \tau = \rho, \beta = \frac{1}{2}(\tau - 1)$

d) $SU(3) \times SU(2)$ $\quad \tau = \rho = \frac{2i}{\sqrt{3}}, \beta = \frac{i}{\sqrt{3}}$ $\star$

e) $SU(4)$ $\quad \tau = \rho = \frac{1+2i\sqrt{2}}{3}, \beta = \frac{-1+i\sqrt{2}}{3}$ $\star$

$$\Omega_d = \begin{pmatrix} \frac{2i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{2i}{\sqrt{3}} \end{pmatrix}, \quad \Omega_e = \begin{pmatrix} \frac{1+2i\sqrt{2}}{3} & \frac{-1+i\sqrt{2}}{3} \\ \frac{-1+i\sqrt{2}}{3} & \frac{1+2i\sqrt{2}}{3} \end{pmatrix}$$

fixed pts of $Sp(4, \mathbb{Z})$ subgroups

[Gottschling '61]

3 moduli example, the map

heterotic $(\tau, \rho, \beta) \leftrightarrow$ K3 (a, b, c, d) [Kumar '07, Clingher, Doran '10; Malmendier, Morrison '14]

$$a = -\frac{1}{48}\psi_4(\Omega)\,,\;\; b = -\frac{1}{864}\psi_6(\Omega)\,,\;\; c = -4\chi_{10}(\Omega)\,,\;\; d = \chi_{12}(\Omega)\,,\;\; \Omega = \begin{pmatrix} \tau & \beta \\ \beta & \rho \end{pmatrix}$$

$\psi_4,\ \psi_6,\ \chi_{10},\ \chi_{12}$: $Sp(4,\mathbb{Z})$ Siegel modular forms of weight 4,6,10,12 resp.

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when $\beta = 0$ ($c = 0$) map reduces to

$$j(\tau)j(\rho) = -1728^2\frac{a^3}{27d}\,, \qquad (j(\tau)-1728)(j(\rho)-1728) = 1728^2\frac{b^2}{4d}$$

let's check enhancing at fixed points of $Sp(4,\mathbb{Z})$!

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modular forms have Fourier expansions, $F(\Omega) = \sum_{n,m,r} A(n,r,m)e^{2\pi i n\tau}e^{2\pi i m\rho}e^{2\pi i r\beta}$

evaluated numerically via sage package `degree2` [Takemori '14]

found that Igusa invariants $\frac{\psi_4^3}{\chi_{12}}, \frac{\psi_6^2}{\chi_{12}}, \frac{\psi_4^2\psi_6\chi_{10}}{\chi_{12}^2}$, are integers at fixed points

N.B. $\Omega_{\rm d}$ and $\Omega_{\rm e}$ are period matrices of genus-two $y^2 = x^6 - 1$ and $y^2 = x^5 - x$, resp.

3 moduli example, enhancing in K3 side

focus on HE, work in patch $v = 1$

$$\Delta_{\mathrm{K3}} = \text{const.}\, u^9 \left[4(au + c)^3 + 27u(u^2 + bu + d)^2\right] = u^9 \Delta_5$$

enhancing when Δ_5 develops multiple roots, depends on

$$a = -\tfrac{1}{48}\psi_4(\Omega)\,,\;\; b = -\tfrac{1}{864}\psi_6(\Omega)\,,\;\; c = -4\chi_{10}(\Omega)\,,\;\; d = \chi_{12}(\Omega)\,,$$

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at $\Omega_{\mathrm{e}} = \begin{pmatrix} \frac{1+2i\sqrt{2}}{3} & \frac{-1+i\sqrt{2}}{3} \\ \frac{-1+i\sqrt{2}}{3} & \frac{1+2i\sqrt{2}}{3} \end{pmatrix}$, Δ_5 has a root of order 4 $\Rightarrow$ I_4 singularity

agrees with $SU(4)$ observed in heterotic

analogous findings in HO

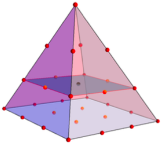
2 moduli

2 moduli examples, the F-theory story # 3

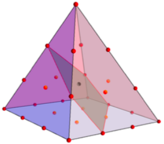
$$c_1\, z_1^3 z_2^3 + c_2\, z_1^3 z_4^3 + c_3\, z_2^3 z_3^3 + c_4\, z_2^3 z_4^3 + c_5\, z_5^3 - c_0\, z_1 z_2 z_3 z_4 z_5 = 0$$

$$\text{moduli:} \quad \frac{\xi}{27} = \frac{c_1 c_4 c_5}{c_0^3}, \qquad \frac{\eta}{27} = \frac{c_2 c_3 c_5}{c_0^3}$$

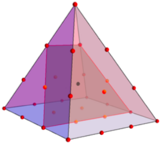
five fibration structures [CCDLM '14]



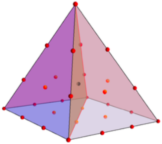
SO(8) x SU(12) x U(1)/Z_2



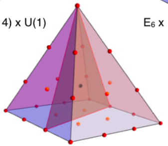
SO(14) x E_7 x U(1)^2



SU(9) x SO(14) x U(1)



E_6 x E_6 x SU(3) x SU(3)/Z_3



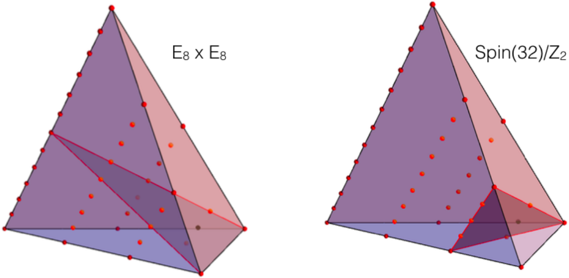
E_8 x E_8 x Z_3

2 moduli examples, the F-theory story # 2

$$c_1\, z_1^{12} + c_2\, z_1^6 z_2^6 + c_3\, z_3^{12} + c_4\, z_3^3 + c_5\, z_4^2 - c_0\, z_1 z_2 z_3 z_4 = 0$$

$$\text{moduli:} \quad \xi = \frac{c_1 c_3}{c_2^2}, \qquad \eta = \frac{c_2 c_4^2 c_5^2}{c_0^6}$$

two fibration structures [CCDLM '14]



CCDLM and CCLM maps equivalent

2 moduli examples, the F-theory story # 1

$$c_1\, z_1^3 z_2^3 + c_2\, z_3^3 z_4^3 + c_3\, z_5^3 z_6^3 + c_4\, z_2^2 z_4^2 z_6^2 + c_5\, z_1^2 z_3^2 z_5^2 - c_0\, z_1 z_2 z_3 z_4 z_5 z_6 = 0$$

$$\text{moduli:}\quad \frac{\xi}{27} = \frac{c_1 c_2 c_3}{c_0^3}, \qquad \frac{\eta}{4} = \frac{c_4 c_5}{c_0^2}$$

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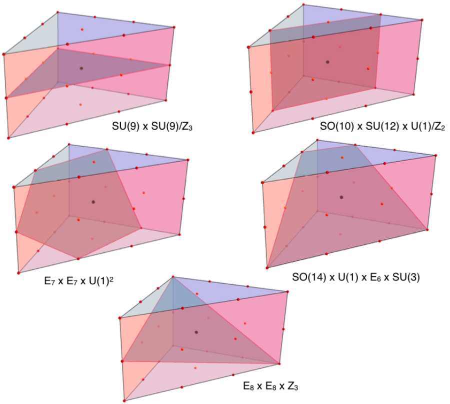

2 moduli example # 1, the mirror map [CCDLM '14]

$$p = c_1\, z_1^3 z_2^3 + c_2\, z_3^3 z_4^3 + c_3\, z_5^3 z_6^3 + c_4\, z_2^2 z_4^2 z_6^2 + c_5\, z_1^2 z_3^2 z_5^2 - c_0\, z_1 z_2 z_3 z_4 z_5 z_6$$

discriminant locus: $D = (\xi - 1 - 3\eta)^2 - \eta(\eta+3)^2 = 0, \quad \dfrac{\xi}{27} = \dfrac{c_1 c_2 c_3}{c_0^3}, \ \dfrac{\eta}{4} = \dfrac{c_4 c_5}{c_0^2}$

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instead of (ξ, η), work with ratios of periods of holomorphic 2-form

$$\tau^{(1)} = \frac{\varpi_1}{\varpi_0}, \qquad \tau^{(2)} = \frac{\varpi_2}{\varpi_0}$$

ϖ_i satisfy Picard-Fuchs equations

monodromy around $(\xi, \eta) = (0, 0) \Rightarrow \tau^{(i)} \to \tau^{(i)} + 1$

monodromy around $D = 0 \Rightarrow \tau^{(1)} \leftrightarrow \tau^{(2)}$

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mirror map given in terms of invariants $j_1 = j(\tau^{(1)})$ and $j_2 = j(\tau^{(2)})$

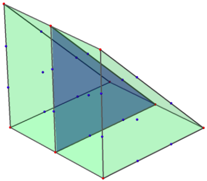
$$j_1 = \frac{216}{\xi\eta^3}\left(1 - \xi - \eta - \sqrt{D}\right)^3, \quad j_2 = \frac{216}{\xi\eta^3}\left(1 - \xi - \eta + \sqrt{D}\right)^3$$

for instance

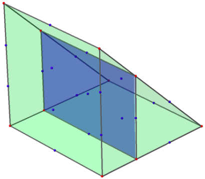
(ξ, η)	$(27, 4)$	$(\frac{27}{64}, \frac{1}{16})$
$\tau^{(1)} = \tau^{(2)}$	$\frac{1+i\sqrt{7}}{2}$	$2i$

2 moduli example # 1, fibration structures

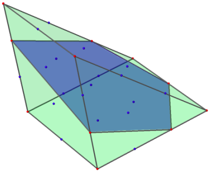
via `sage.geometry.polyhedron.ppl_lattice_polytope` [V.Braun '12]



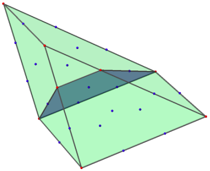
$SU(9) \times SU(9)/\mathbb{Z}_3$



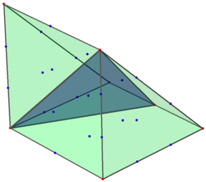
$SO(10) \times SU(12) \times U(1)/\mathbb{Z}_2$



$E_7 \times E_7 \times U(1)^2$



$SO(14) \times U(1) \times E_6 \times SU(3)$



$E_8 \times E_8 \times \mathbb{Z}_3$

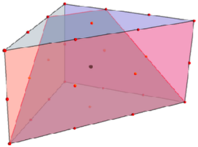
2 moduli example # 1, fibration with fiber F_{14}

defining equation

$$c_1 x^2 y^2 z_1 z_2 + c_2 x t^3 + c_3 y z^3 z_1^2 z_2^2 + c_4 z^2 t^2 z_2^2$$
$$+ c_5 x^2 y^2 z_1^2 - c_0 xyzt z_1 z_2 = 0$$

base coordinates z_1, z_2 = affine nodes of extended Dynkin diagram

[Candelas, AF '96; Bouchard, Skarke '03; Grassi, Perduca '13]



SO(14) x U(1) x E_6 x SU(3)

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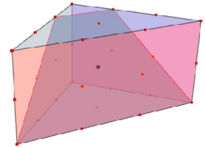


SO(14) x U(1) x E6 x SU(3)

Weierstraß form

$$f = -\frac{1}{48} z_1^2 z_2^3 (z_1^2 z_2 c_0^4 - 24 z_1^2 z_2 c_0 c_1 c_2 c_3 - 8 z_1 z_2^2 c_0^2 c_1 c_4 + 16 z_2^3 c_1^2 c_4^2 - 24 z_1^3 c_0 c_2 c_3 c_5 - 8 z_1^2 z_2 c_0^2 c_4 c_5$$
$$+ 32 z_1 z_2^2 c_1 c_4^2 c_5 + 16 z_1^2 z_2 c_4^2 c_5^2)\,,$$

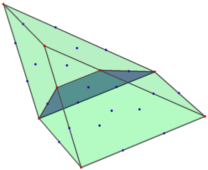
$$g = -\frac{1}{864} z_1^3 z_2^4 (-z_1^3 z_2^2 c_0^6 + 36 z_1^3 z_2^2 c_0^3 c_1 c_2 c_3 - 216 z_1^3 z_2^2 c_1^2 c_2^2 c_3^2 + 12 z_1^2 z_2^3 c_0^4 c_1 c_4 - 144 z_1^2 z_2^3 c_0 c_1^2 c_2 c_3 c_4 + 64 z_1^3 z_2^2 c_4^3 c_5^3$$
$$- 48 z_1 z_2^4 c_0^2 c_1^2 c_4^2 + 64 z_2^5 c_1^3 c_4^3 + 36 z_1^4 z_2 c_0^3 c_2 c_3 c_5 - 432 z_1^4 z_2 c_1 c_2^2 c_3^2 c_5 + 12 z_1^3 z_2^2 c_0^4 c_4 c_5 - 288 z_1^3 z_2^2 c_0 c_1 c_2 c_3 c_4 c_5$$
$$- 96 z_1^2 z_2^3 c_0^2 c_1 c_4^2 c_5 + 192 z_1 z_2^4 c_1^2 c_4^3 c_5 - 216 z_1^5 c_2^2 c_3^2 c_5^2 - 144 z_1^4 z_2 c_0 c_2 c_3 c_4 c_5^2 - 48 z_1^3 z_2^2 c_0^2 c_4^2 c_5^2 + 192 z_1^2 z_2^3 c_1 c_4^3 c_5^2)\,,$$

$$\Delta = -\frac{1}{16} c_3^2 c_2^2 z_1^9 z_2^8 (z_2 c_1 + z_1 c_5)^3 (z_1^3 z_2 c_0^3 c_2 c_3 - 27 z_1^3 z_2 c_1 c_2^2 c_3^2 + z_1^2 z_2^2 c_0^4 c_4 - 36 z_1^2 z_2^2 c_0 c_1 c_2 c_3 c_4 - 8 z_1 z_2^3 c_0^2 c_1 c_4^2$$
$$+ 16 z_2^4 c_1^2 c_4^3 - 27 z_1^4 c_2^2 c_3^2 c_5 - 36 z_1^3 z_2 c_0 c_2 c_3 c_4 c_5 - 8 z_1^2 z_2^2 c_0^2 c_4^2 c_5 + 32 z_1 z_2^3 c_1 c_4^3 c_5 + 16 z_1^2 z_2^2 c_4^3 c_5^2)\,.$$

generic group $SO(14) \times E_6 \times SU(3) \times U(1) \times U(1)^2$

enhances at points (ξ, η) where Δ develops additional multiple roots

2 moduli example # 1, enhancement patterns



fiber F_{14}

generic group $SO(14) \times E_6 \times SU(3) \times U(1) \times U(1)^2$

at $D = 0$ $U(1)^2$ enhances generically to $SU(2) \times U(1)$

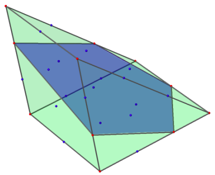
discriminant locus: $D = (\xi - 1 - 3\eta)^2 - \eta(\eta + 3)^2$

$$\frac{\xi}{27} = \frac{c_1 c_2 c_3}{c_0^3}, \quad \frac{\eta}{4} = \frac{c_4 c_5}{c_0^2}$$

at special points $SU(3) \times U(1)^2$ enhances to

$SU(5)$	$(\xi, \eta) = (27, 4)$,	$\tau^{(1)} = \tau^{(2)} = \frac{1+i\sqrt{7}}{2}$
$SU(4) \times SU(2)$	$(\xi, \eta) = (\frac{27}{64}, \frac{1}{16})$,	$\tau^{(1)} = \tau^{(2)} = 2i$
$SU(3) \times SU(2)^2$	$(\xi, \eta) = (-8, -3)$,	$\tau^{(1)} = \tau^{(2)} = i$

2 moduli example # 1, enhancement patterns



fiber F_5

generic group $E_7 \times E_7 \times U(1)^2 \times U(1)^2$

at $D = 0$ $U(1)^2$ enhances generically to $SU(2) \times U(1)$

at special points $U(1)^4$ enhances to

$$\begin{array}{ll} SU(4) \times U(1) & \tau^{(1)} = \tau^{(2)} = \frac{1+i\sqrt{7}}{2} \\ SU(3) \times U(1)^2 & \tau^{(1)} = \tau^{(2)} = \frac{1+i\sqrt{19}}{2} \\ SU(2)^3 \times U(1) & \tau^{(1)} = \tau^{(2)} = i\sqrt{3} \\ SU(2)^2 \times U(1)^2 & \tau^{(1)} = \tau^{(2)} = i\sqrt{7}, i \end{array}$$

to do: match in heterotic, i.e. find slice of heterotic moduli space (τ, ρ, ζ) where same enhancement pattern occurs

1 modulus

1 modulus example # 1, the F-theory story

$x_1^4 + x_2^4 + x_3^4 + x_4^4 - c_0\, x_1\, x_2\, x_3\, x_4 = 0,$ singular at $\xi = 0, 1, \infty,\ \xi = \dfrac{4^4}{c_0^4}$

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ratio of periods $t = \dfrac{\varpi_1}{\varpi_0}$ transforms under $\Gamma_0^+(2)$

e.g. $t \to -\dfrac{1}{2t}$ under monodromy around $\xi = 1$ [Nagura, Sugiyama '93]

mirror map: $\dfrac{4^4}{\xi} = \hat{\jmath}_2(e^{2\pi i t})$ modular invariant of $\Gamma_0^+(2)$, $\hat{\jmath}_2 = 0, t = \frac{1+i}{2}$

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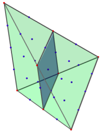
ratio of periods $t = \dfrac{\varpi_1}{\varpi_0}$ transforms under $\Gamma_0^+(2)$

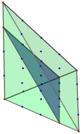
e.g. $t \to -\dfrac{1}{2t}$ under monodromy around $\xi = 1$ [Nagura, Sugiyama '93]

mirror map: $\dfrac{4^4}{\xi} = \hat{\jmath}_2(e^{2\pi i t})$ modular invariant of $\Gamma_0^+(2)$, $\hat{\jmath}_2 = 0, t = \frac{1+i}{2}$

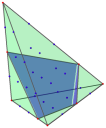
4 fibration structures



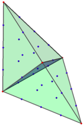
$SO(16) \times SO(16) \times U(1)/\mathbb{Z}_2$



$E_6 \times SU(12)/\mathbb{Z}_3$



$E_8 \times E_8 \times U(1)$



$E_7 \times E_7 \times SU(4)/Z_2$

enhancement at $\xi = 1, t = \frac{i}{\sqrt{2}}$,
also at $\xi \to \infty, t = \frac{1+i}{2}$

to do: relate heterotic (τ, ρ, ζ) to t
e.g. in 1c $(\tau, \rho, \zeta) = (t, 2t, 0)$

1 modulus example # 2, the F-theory story

$x_1^6 + x_2^6 + x_3^6 + x_4^2 - c_0\, x_1\, x_2\, x_3\, x_4 = 0,$ singular at $\xi = 0, 1, \infty$, $\xi = \frac{12^3}{c_0^6}$

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ratio of periods $t = \frac{\varpi_1}{\varpi_0}$ transforms under $SL(2, \mathbb{Z})$

mirror map: $\frac{12^3}{\xi} = j(e^{2\pi i t})$ modular invariant of $SL(2, \mathbb{Z})$, $j = 0, t = \frac{1+i\sqrt{3}}{2}$

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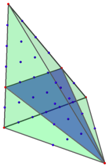
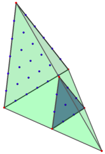
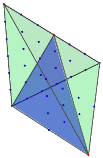
3 fibration structures [Kimura, Mizoguchi '17]



$E_7 \times SO(20)/\mathbb{Z}_2$



$SU(18)/\mathbb{Z}_3$



$E_8 \times E_8 \times SU(2)$

enhancement at $\xi = 1, t = i$,
also at $\xi \to \infty, t = \frac{1+i\sqrt{3}}{2}$

map to heterotic $\tau \sim t, \rho = \tau, \zeta = a_1 \tau$

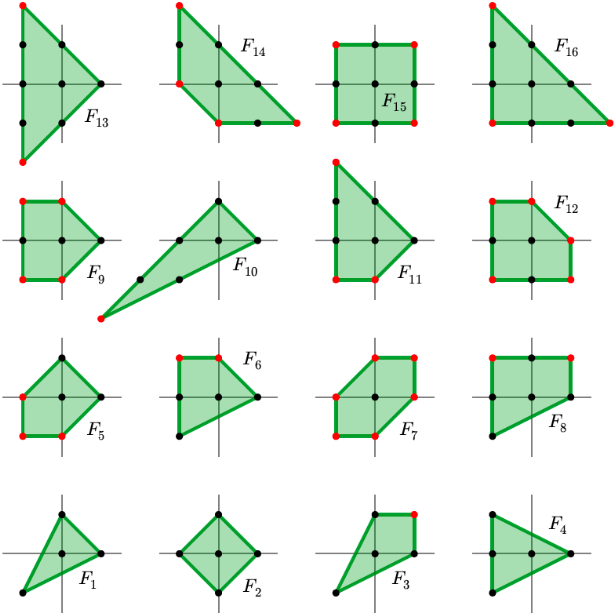
Final remarks

- studied aspects of F-theory and heterotic in 8d
- considered K3's with $n = 1, 2, 3$ moduli (Picard number 19, 18, 17)

  more $n = 3$ examples in [Chabrol '19]
- matched enhancing patterns via 'mirror map', in some examples
- to do:
  - complete analysis of enhancement and matching
  - explore examples with rank reduction (CHL) [Bouchard, Skarke '03]
  - ...

Fin

The 16 reflexive polygons



taken from Braun, Grimm, Keitel '13

Fiber polygon	Toric sections	Relations	MW_T
F_3	$(1,1) \simeq f_0$		0
F_5	$(0,-1) \simeq f_0$ $(-1,-1) \simeq f_1$ $(-1,0) \simeq f_2$		$\mathbb{Z} \oplus \mathbb{Z}$
F_6	$(0,1) \simeq f_0$ $(-1,1) \simeq f_1$		$\mathbb{Z}$
F_7	$(-1,-1) \simeq f_0$ $(0,-1) \simeq f_1$ $(1,0) \simeq f_2$ $(1,1) \simeq f_3$ $(0,1) \simeq f_4$ $(-1,0) \simeq f_5$	$\sigma_1 = \sigma_3 + \sigma_4$ $\sigma_5 = \sigma_2 + \sigma_3$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
F_8	$(-1,1) \simeq f_0$ $(1,1) \simeq f_1$		$\mathbb{Z}$
F_9	$(-1,-1) \simeq f_0$ $(0,-1) \simeq f_1$ $(0,1) \simeq f_2$ $(1,1) \simeq f_3$	$\sigma_1 = \sigma_2 + \sigma_3$	$\mathbb{Z} \oplus \mathbb{Z}$
F_{10}	$(-3,-2) \simeq f_0$		0
F_{11}	$(-1,-1) \simeq f_0$ $(0,-1) \simeq f_1$ $(-1,2) \simeq f_2$	$\sigma_1 = 2\sigma_2$	$\mathbb{Z}$
F_{12}	$(-1,-1) \simeq f_0$ $(1,-1) \simeq f_1$ $(1,0) \simeq f_2$ $(0,1) \simeq f_3$ $(-1,1) \simeq f_4$	$\sigma_1 = \sigma_3 + \sigma_4$ $\sigma_4 = \sigma_1 + \sigma_2$	$\mathbb{Z} \oplus \mathbb{Z}$
F_{13}	$(-1,-2) \simeq f_0$ $(-1,2) \simeq f_1$	$2\sigma_1 = 0$	$\mathbb{Z}_2$
F_{14}	$(-1,0) \simeq f_0$ $(-1,2) \simeq f_1$ $(2,-1) \simeq f_2$ $(0,-1) \simeq f_3$	$\sigma_1 = 2\sigma_2$ $\sigma_3 = \sigma_1 + \sigma_2$	$\mathbb{Z}$
F_{15}	$(-1,-1) \simeq f_0$ $(1,-1) \simeq f_1$ $(1,1) \simeq f_2$ $(-1,1) \simeq f_3$	$2\sigma_2 = 0$ $\sigma_3 = \sigma_1 + \sigma_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
F_{16}	$(-1,-1) \simeq f_0$ $(2,-1) \simeq f_1$ $(-1,2) \simeq f_2$	$3\sigma_1 = 0$ $\sigma_2 = 2\sigma_1$	$\mathbb{Z}_3$

Table 2: *Toric sections corresponding to the reflexive polytopes and the toric subgroup of the Mordell-Weil group generated by them.*

taken from Braun, Grimm, Keitel '13