Bosonic String

bosonic.scrbl

Introduction to BV

Lightning intro to BV

Integration in LAG

String theory requires one more element:

- integration over families of Lagrangian submanifolds

BV BRST

Let me explain this in the context of BV-BRST. Let H be the gauge group with gauge Lie algebra $\mathbf{h}$. Remember that the space of fields is $\Pi\mathbf{h} \times X$, where $\Pi\mathbf{h}$ is parametrized by c and X by ϕ. The Main Action has the form:

$$S_{BV}(\phi, \phi^\star) = S_{cl}(\phi) + c^A v^i_A(\phi)\phi^\star_i + \frac{1}{2} c^A c^B f^C_{AB} c^\star_C$$

> In fact the bosonic string belongs to this class.

For quantization we need to restrict S_{BV} to a Lagrangian submanifold, and then take path integral over ϕ. How do we choose a Lagrangian submanifold? A very naive choice is $\phi^\star = 0$, then the restriction would be just S_{cl} — degenerate, can not quantize.

Constraint and conormal bundle

As the next attempt, let us try the following construction. Let X denote the space of fields ϕ. Consider some subspace $Y \subset X$ defined by some equations (“constraints”). We want Y to be transverse to the the gauge orbits. We choose:

$$L(Y) \;=\; \Pi(TY)^\perp \times [\text{c-ghosts}] \;=\; \Pi(TY)^\perp \times \Pi\mathbf{h} \qquad (1)$$

This is a Lagrangian submanifold.

Degeneracy on conormal bundle

Let us study the quadratic terms in the expansion of $S_{BV}|_{L(Y)}$. Suppose that the extremum of $S_{cl}(\phi)$ is at $\phi = 0$. We assume that the vector fields v^i_A do not vanish at $\phi = 0$:

$$v^i_A(0) = T^i_A$$

Then:

$$\left(S_{BV}|_{L(Y)}\right)_{\text{quadratic}} = \left(S_{cl}(\phi)\ |_{\phi\in Y}\right)_{\text{quadratic}} + T^i_A c^A \phi^\star_i\ |_{\phi^\star \in \Pi(TY)^\perp} \tag{2}$$

$\phi\phi$ terms

The quadratic terms for ϕ come from $S_{cl}(\phi)$. We assume that all the degeneracy of S_{cl} is due to gauge symmetry. It is usually not difficult to choose Y transverse to gauge orbits, so let us assume that. Then the quadratic term in $S_{cl}(\phi)\ |_{\phi\in Y}$ is a nondegenerate quadratic form.

$c\phi^\star$ terms

The quadratic term for c and $\phi^\star$ comes from the second term: $T^i_A c^A \phi^\star_i\ |_{\phi^\star\in\Pi(TY)^\perp}$. (It is perhaps useful to think of this term as a pairing $\langle \phi^\star, T\langle c\rangle\rangle$.) We assume that $T:\ \mathbf{h} \to T_0 X$ is an injection. Then $\langle \phi^\star, T\langle c\rangle\rangle$ is degenerate in $\phi^\star$, in the following sense: exist $\phi^\star_{(0)}$ such that:

$$\forall c:\ \left\langle \phi^\star_{(0)}, T\langle c\rangle \right\rangle = 0 \tag{3}$$

This degeneracy means that the path integral is not satisfactory. If $\mathbf{h}$ is bosonic, it would be zero. If gauge symmetries form a superalgebra, then it is ill-defined.

Interpretation of zero modes

Those $\phi^\star_{(0)}$ which satisfy Eq. (3) are precisely such elements of $\Pi T^* X$ which vanish both on TY **and** on the tangent space to the orbits of $\mathbf{h}$. This space has the following geometrical interpretation. Let us consider gauge transformations of $Y \subset X$:

$$\{hY | h \in H\}$$

Let us restrict to h belonging to a small neighborhood of the unit in H . Let us consider their union:

$$\bigcup_{h\in H} hY$$

Remember that we assume that the orbits of H are transverse to Y . This implies that $h_1 Y$ and $h_2 Y$ do not intersect for h_1 and h_2 close enough to the unit of H . The union $\bigcup_{h\in H} hY$ seems to be actually well defined. But let us ask the question: does $\bigcup_{h\in H} hY$

coincide with the whole X in the vicinity of $Y \subset X$? In fact, the existence of nonzero solutions to Eq. (3) is equivalent to $\bigcup_{h \in H} hY$ being strictly a subset of X. If the space of solutions is finite-dimensional, then the codimension $\bigcup_{h \in H} hY$ in X is equal to the number of linearly independent solutions of Eq. (3). In fact, the space of solutions to Eq. (3) is (as a linear space) **dual to** the fiber of the normal bundle at the point $0 \in X$:

$$N_0 \bigcup_{h \in H} hY$$

where $\bigcup_{h \in H} hY$ is considered a submanifold of X.

This means that our gauge slice $Y \subset X$, even after being translated by gauge transformations, **does not cover the whole** X. But we do want to integrate over the **whole** X ! At first sight, it seems that the problem can be easily solved. Simply admit the wrong choice of Y: our chosen $Y \subset X$ was too small! Let us extend it, pick a larger Y!

But it turns out that there are examples (bosonic string amonth them) when we cannot actually enlarge Y. The problem is, that we have to stick to particular class of constraints, which basically respects the requirement of **locality**. After we restrict to the Lagrangian submanifold, we should have a local QFT.

Need to integrate in LAG

The solution is to **integrate over families of gauge conditions** $Y \subset X$. Moreover, there is canonical integration measure (more precisely, a PDF) on any families of Lagrangan submanifolds, not only on families of conormal bundles. If gauge fermions $\Psi_1, \ldots, \Psi_n$ define n tangent vectors to LAG at the point $L \in LAG$ then the measure is:

$$\int_L \Psi_1 \cdots \Psi_n \ \exp(S_{BV})$$

How does this formula work in the case of family of conormal bundles? We combine n-forms into a PDF:

$$\Omega \ = \ \int_L \exp\left(S_{BV}|_L + \alpha|_L\right)$$
$$\text{where } \alpha = \phi^\star_i d\phi^i$$

In this formula, $\alpha = \phi^\star_i d\phi^i$ should be understood as follows. Given an infinitesimal variation of $Y \subset X$ we pick any $\dot{\phi}$ representing this variation, and compute $\langle \alpha, \dot{\phi} \rangle$. Remember that $\phi^\star$ is restricted to belong to the conormal fiber $\Pi(TY)^\perp$; therefore $\langle \alpha, \dot{\phi} \rangle$ does not depend on the choice of $\dot{\phi}$. (We can add to $\dot{\phi}$ any vector tangent to $Y \subset X$ and $\langle \alpha, \dot{\phi} \rangle$ will not change.)

Moreover, notice that the equations of motion of the action of Eq. (2) project $\phi^\star$ from $\Pi(TY)^\perp$ down to $\Pi(TY + T\mathcal{O})^\perp$. This means that the resulting PDF is $\mathbf{h}$-base.

Note on equivariant BV

The BV phase space is:

$$M = \Pi T^* \left(\frac{\Pi TH \times X}{H} \right)$$

Here we quotient by the action of H where H acts on ΠTH **from the right**. The Q_{BRST} comes from the canonical nilpotent vector field on ΠTH. We want Q_{BRST} to preserve the volume on $\frac{\Pi TH \times X}{H}$. This is some condition on the trace of the structure constant and the div's of generators.

But the **left** action of H on ΠTH remains. It is generated by an exact Hamiltonian:

$$H\langle\xi\rangle = \Delta(\xi^\alpha c^\star_\alpha)$$

This means that:

Gauge symmetries act on the BV phase space

An equivariant form is given by:

$$\int_L \exp\left(S_{BV} + \Psi + \mathbf{t}^\alpha c^\star_\alpha\right)$$

Notice that $c^\star$ vanishes on Lagrangian submanifolds which are obtained by the conormal bundle construction.

Bosonic string worldsheet theory

BV phase space and Main Equation

Bosonic string worldsheet theory is 2D gravity coupled to 2D matter.

The word “matter” means target space coordinates $X^0, \ldots, X^{D-1}$ (*i.e.* there are D matter fields).

The word “gravity” could mean two different things:

- Non-critical: dynamical metric with Liouville action
- Critical ($D = 26$): dynamical complex structure

In any case, there is gauge symmetry: $\mathrm{Diff}(\Sigma)$.

We therefore apply the Faddeev-Popov procedure. In the BV language, we add ghost fields $c^\alpha(z, \bar{z})$ corresponding to vector fields on the worldsheet with fermionic statistics.

Let us concentrate on **critical** string. The BV Main action is:

$$S_{BV} = \int_\Sigma \left(dx^a \wedge I.dx^a + (\mathcal{L}_c x)x^\star + (\mathcal{L}_c I)I^\star + \frac{1}{2}(\mathcal{L}_c c)c^\star \right)$$

It satisfies the classical Main Equation:

$$\{S_{BV}, S_{BV}\} = 0$$

In this case X is parameterized by (x, I, c) .

Lagrangian submanifold

We pick some fixed complex structure $I^{(0)}$ and introduce the following constraint:

$$I = I^{(0)}$$

This equation defines $Y \subset X$. In genus $g \geq 2$ this is a good gauge condition, in the sense that the orbits of $\mathbf{h}$ are transverse to Y . Our Lagrangian submanifold is $\Pi(TY)^\perp$. In this case Y is parameterized by:

$$\phi = x \text{ and } c$$

The conormal fiber is parametrized by:

$$\phi^\star = I^\star$$

Expansion of Main Action in the vicinity of Lagrangian submanifold

> This is joint work in progress with my students Eggon Viana and Vinícius Bernardes da Silva.

We will now work in a vicinity of a point in Σ . We may then introduce coordinates $(z,\bar{z})$ so that:

$$I^{(0)} \;:\; T\Sigma \to T\Sigma$$
$$I^{(0)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

At each point on Σ a complex structure is a matrix satisfying $I^2 = -\mathbf{1}$. It can be parametrized by $m \in \mathbf{CP}^1$:

$$I = \begin{pmatrix} I^z_z & I^z_{\bar{z}} \\ I^{\bar{z}}_z & I^{\bar{z}}_{\bar{z}} \end{pmatrix} = \begin{pmatrix} i\sqrt{1+m\bar{m}} & m \\ \bar{m} & -i\sqrt{1+m\bar{m}} \end{pmatrix}$$

Let $\mathcal{I}$ denote the manifold ($\simeq \mathbf{CP}^1$) of such matrices. Then $\Pi T^*\mathcal{I}$ is parametrized by I and $I^\star$. The odd symplectic form is:

$$\omega = \mathrm{tr}(dI^\star \wedge dI)$$

with the following gauge symmetry of $I^\star$:

$$\delta_\eta I^\star = \eta I + I\eta$$

Let us gauge fix $I^\star$ to:

$$I^\star = \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}$$

This is called " b -ghost". It is defined so that m and b are Darboux coordinates of the BV symplectic form:

$$\omega = dm \wedge d\bar{b} + d\bar{m} \wedge db$$

The expansion of the BV Main Action reads (renaming m as $b^\star$):

$$
\begin{aligned}
S_{BV} = & \int_\Sigma d^2z \left(\sqrt{1+b^\star\bar{b}^\star}\,\partial x\bar{\partial}x + \frac{i}{2}\,b^\star(\partial x)^2 - \frac{i}{2}\,\bar{b}^\star(\bar{\partial}x)^2\right) + \\
& + \int_\Sigma \left(\mathcal{L}_c x x^\star + \frac{1}{2}\,\mathcal{L}_c c c^\star\right) + \\
& + \int_\Sigma \left((c\partial + \bar{c}\bar{\partial} + \partial c - \bar{\partial}\bar{c})b^\star - 2i\bar{\partial}c\sqrt{1+b^\star\bar{b}^\star}\right) b \\
& + \int_\Sigma \left((c\partial + \bar{c}\bar{\partial} + \bar{\partial}\bar{c} - \partial c)\bar{b}^\star + 2i\partial\bar{c}\sqrt{1+b^\star\bar{b}^\star}\right) \bar{b}
\end{aligned}
$$

Zero order term (action)

The restriction to L is:

$$
S = \int \left(\partial x \wedge \bar{\partial}x + b\bar{\partial}c + \bar{b}\partial\bar{c}\right)
$$

Notice that b and $\bar{b}$ only enter linearly.

First order term (BRST operator)

The BRST operator is read from the term linear in antifields. In particular:

$$
\begin{aligned}
Qb &= T \\
Q\bar{b} &= \overline{T} \\
\text{where } T &= (\partial x)^2 + \dots \\
\overline{T} &= (\bar{\partial}x)^2 + \dots
\end{aligned}
$$

Second order term (a bracket)

$$
\pi = \int d^2z \left(\partial x\bar{\partial}x + 2i(b\bar{\partial}c - \bar{b}\partial\bar{c})\right)\frac{\delta}{\delta b}\wedge\frac{\delta}{\delta\bar{b}} \tag{4}
$$

This defines some bracket on the space of field configurations. This bracket being nonzero implies that the BRST operator is only nilpotent on-shell. It is highly degenerate (only contains derivatives in the b - and $\bar{b}$ -directions).

Nonlinearity of the space of deformations

The nonzero bracket leads to the following effect. Let us deform by gauge fermion $\int \mu b + \bar{\mu}\bar{b}$. This corresponds to the change of the worldsheet complex structure. For, example, the matter field kinetic term deforms as follows:

$$
\int d^2z \left(\partial x \wedge \bar{\partial}x\right) \mapsto \int d^2z \left(\partial x\bar{\partial}x + \mu(\partial x)^2 + \bar{\mu}(\bar{\partial}x)^2\right) \tag{5}
$$

But the nonzero bracket of Eq. (4) implies that Q **also deforms**. Therefore Eq. (5) is only valid in the first order in μ and $\overline{mu}$. The correct formula up to the second order is:

$$\int d^2z\,\left(\partial x\bar{\partial}x\right) \mapsto \int d^2z\,\left(\partial x\bar{\partial}x + \mu(\partial x)^2 + \overline{\mu}(\bar{\partial}x)^2 - \mu\overline{\mu}\partial x\bar{\partial}x\right) \tag{6}$$

This should be said as follows. The BRST-exact deformations sweep the space of kinetic terms of the form:

$$\int d^2z\sqrt{\det h_{\bullet\bullet}}h^{\alpha\beta}\partial_\alpha x\partial_\beta x \tag{7}$$

while the most general expression would be:

$$\int d^2z a^{\alpha\beta}\partial_\alpha x\partial_\beta x \tag{8}$$

This, in principle, has arbitrary symmetric tensor $a^{\alpha\beta}$. The ones of the form Eq. (7) satisfy the nonlinear equation:

$$\mathrm{Det}\ a = 1 \tag{9}$$

where Det should be understood in the following sense. Notice that a is geometrically a map $\Lambda^2T\Sigma \to S^2T\Sigma$. This uniquely defines a map $(\Lambda^2T\Sigma)^{\otimes 2} \to (\Lambda^2T\Sigma)^{\otimes 2}$ which we call $\mathrm{Det}\ a$. The space of BRST deformations is described by a nonlinear Eq. (9).

Partial on-shell condition

The bracket defined by Eq. (4) does not satisfy the Jacobi identity. Can we define a Poisson bracket satisfying the Jacobi identity, out of the expansion of S_{BV}?

One might try to say that satisfies Jacobi identity on-shell and up to Q of something, but this is very tricky. Although we can impose on-shell conditions, it is not reasonable to expect the Poisson bivector be tangent to on-shell conditions.

But at least in the particular case of bosonic string, we can impose partial on-shell condition, by asking that the action be stationary, but only under the variations of b and $\overline{b}$:

$$\partial\overline{c} = \bar{\partial}c = 0 \tag{10}$$

The key observation is that the action depends on b and $\overline{b}$ only linearly. Therefore the bivector defined by Eq. (4) is tangent to Eqs. (10). The on-partial-shell bracket satisfies the Jacobi identity, in a rather trivial way:

$$\pi_{os} = \int d^2z\,\left(\partial x(z,\overline{z})\bar{\partial}x(z,\overline{z})\right)\frac{\delta}{\delta b(z,\overline{z})}\wedge\frac{\delta}{\delta\overline{b}(z,\overline{z})} \tag{11}$$

(by being “essentially constant”). Moreover, they commute with Q, once we impose our partial on-shell conditions. (This is equivalent to the conformal invariance of Eq. (11).)

Higher order terms

Similarly there are higher brackets, which are all conformally invariant and trivially satisfy the Jacobi identities:

$$\int d^2z\ \partial x \bar{\partial} x\ \frac{\delta}{\delta b(z,\bar{z})} \wedge \cdots \wedge \frac{\delta}{\delta b(z,\bar{z})} \wedge \frac{\delta}{\delta \bar{b}(z,\bar{z})} \wedge \cdots \wedge \frac{\delta}{\delta \bar{b}(z,\bar{z})}$$

They are all conformally invariant, as required by commutating with Q.

Action of diffeomorphisms

A vector field $v \in \mathrm{Vect}\,\Sigma$ acts on the BV phase space as an infinitesimal diffeomorphism. The action is generated by the Hamiltonian:

$$H\langle v\rangle = \Delta \left(\int_\Sigma v^\alpha c^\star_\alpha \right) + \left\{ \int_\Sigma v^\alpha c^\star_\alpha \,,\, \int_\Sigma v^\alpha c^\star_\alpha \right\}$$

Notice that the second term is zero because $\{c^\star, c^\star\} = 0$.

We believe that this is a fundamental equation which should be present in any string worldseet theory:

$$\begin{aligned} H\langle v\rangle &= \Delta i(v) + \frac{1}{2}\{i(v), i(v)\} \\ \{H\langle v_1\rangle, i(v_2)\} &= \left.\frac{d}{dt}\right|_{t=0} i\left(e^{t[v_1,-]}v_2\right) \end{aligned}$$

where $i(v)$ does not have to be linear in v .