Graded geometry for mixed-symmetry tensor gauge theories and duality

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Graded Geometry Teaser

- $\mathbb{Z}_2$-graded geometry $\rightsquigarrow$ supersymmetry
- $\mathbb{Z}$-graded geometry $\rightsquigarrow$ BV-BRST formalism / AKSZ sigma models
- Q-manifolds $\rightsquigarrow$ Poisson geometry / Courant algebroids / L_∞ algebras

- In this talk $\rightsquigarrow$ some other uses/applications

Motivation & Goals

- Description of mixed-symmetry tensors, their kinetic, mass and topological terms.
  - Unified formalism for p-forms, gravitons, the Curtright field (2,1), &c. (linear)
- Interacting theories with higher derivatives but 2nd-order field equations.
  - Goal: Universal Lagrangians for mixed-symmetry tensors/classification of interactions.
- Dualities: standard, exotic, double, infinite chain.
  - Goal: Systematic dualization procedure for all types of dualities & equivalences.
- Mixed-symmetry fields appear in formulations of string/M theory; couple to branes.

Differential forms as functions

Basic idea: Tensor fields as functions on a graded supermanifold

$\mathbb{Z}_2$-graded geometry, even coordinates x^i and odd coordinates θ^i,

$$\theta^i\theta^j = -\theta^j\theta^i \ .$$

Functions on graded vector bundles $\rightsquigarrow$ p-forms or p-vector fields,

$$C^\infty(T[1]M) \simeq \Omega^\bullet(M) \quad \text{and} \quad C^\infty(T^*[1]M) \simeq \Gamma(\wedge^\bullet TM) \ .$$

A function on $T[1]M$ may be expanded as

$$\omega(x,\theta) = \sum_{k=0}^{D} \frac{1}{k!}\, \omega_{i_1\dots i_k}(x)\, \theta^{i_1} \dots \theta^{i_k} \ .$$

Integration is defined as usual for Grassmann variables, $\int \mathrm{d}^D\theta\, \theta^1\theta^2 \dots \theta^D = 1$.

Mixed symmetry tensor fields as functions

For bipartite tensors of degree $|\omega| = (p, q)$, consider functions on $T[1]M \oplus T[1]M$,

$$\omega_{p,q} = \frac{1}{p!q!}\,\omega_{i_1\dots i_p j_1\dots j_q}(x)\,\theta^{i_1}\dots\theta^{i_p}\chi^{j_1}\dots\chi^{j_q}\ .$$

Two separate sets of odd coordinates θ^i and χ^i which mutually commute by convention,

$$\theta^i\theta^j = -\theta^j\theta^i\ , \quad \chi^i\chi^j = -\chi^j\chi^i\ , \quad \theta^i\chi^j = \chi^j\theta^i\ .$$

The components of the tensor field have manifest mixed index symmetry

$$\omega_{i_1\dots i_p j_1\dots j_q} = \omega_{[i_1\dots i_p][j_1\dots j_q]}\ .$$

N.B. Useful to think of differential forms as bipartite tensors with 1 empty slot (p or q).

N-partite tensors for $\mathcal{M} = \bigoplus^N T[1]M$ with $k := 2\left\lfloor\frac{N+1}{2}\right\rfloor$ sets of degree-1 θ^i_A.

Graded (Bipartite) Calculus

- Exterior derivatives $\mathrm{d} : \omega_{p,q} \mapsto \omega_{p+1,q}$ and $\widetilde{\mathrm{d}} : \omega_{p,q} \mapsto \omega_{p,q+1}$

$$\mathrm{d} = \theta^i \partial_i \quad \text{and} \quad \widetilde{\mathrm{d}} = \chi^i \partial_i \quad \text{with} \quad \mathrm{d}^2 = \widetilde{\mathrm{d}}^2 = 0 \quad \text{and} \quad \mathrm{d}\,\widetilde{\mathrm{d}} = \widetilde{\mathrm{d}}\,\mathrm{d}\,.$$

- Transposition $\theta \leftrightarrow \chi$ (n.b.: applies to diff. forms too)

$$\omega_{p,q} \;\mapsto\; \omega^{\top\theta\chi} \equiv \widetilde{\omega}_{q,p} = \frac{1}{p!q!}\omega_{i_1\ldots i_p j_1\ldots j_q}\theta^{j_1}\ldots\theta^{j_q}\chi^{i_1}\ldots\chi^{i_p}\,.$$

- Partial Hodge stars $* : \omega_{p,q} \mapsto \omega_{D-p,q}$ and $\widetilde{*} : \omega_{p,q} \mapsto \omega_{p,D-q}$ (ψ^i: auxiliary odd set.)

$$*\,\omega \quad = \quad \frac{1}{(D-p)!}\int_\psi \omega^{\top\theta\psi}\,(\eta^{\top\chi\psi})^{D-p}\,, \qquad (\int_\psi = \int \mathrm{d}^D\psi)\,.$$

Here, $\eta = \eta_{ij}\theta^i\chi^j$ is the Minkowski metric, whereas $\eta^{\top\chi\psi} = \eta_{ij}\theta^i\psi^j$.

Essentially, $|\omega| = (0, p, q)$ and $|\eta| = (0, 1, 1)$.

cf. Hull '01; de Medeiros, Hull '02

Dual operations

- Bipartite tensors have traces, unlike differential forms.

$$\mathrm{tr} = \eta^{ij}\bar{\theta}_i\bar{\chi}_j\,, \quad \text{where} \quad \bar{\theta}_i = \frac{\partial}{\partial\theta^i} \quad \text{and} \quad \bar{\chi}_i = \frac{\partial}{\partial\chi^i}\,.$$

- Codifferentials $\mathrm{d}^\dagger : \omega_{p,q} \mapsto \omega_{p-1,q}$ and $\widetilde{\mathrm{d}}^\dagger : \omega_{p,q} \mapsto \omega_{p,q-1}$:

$$\mathrm{d}^\dagger := (-1)^{1+D(p+1)} * \mathrm{d} * = \eta^{ij}\bar{\theta}_i\partial_j\,.$$

- Cotraces $\sigma : \omega_{p,q} \mapsto \omega_{p+1,q-1}$ and $\widetilde{\sigma} : \omega_{p,q} \mapsto \omega_{p-1,q+1}$:

$$\sigma \;:= (-1)^{1+D(p+1)} * \mathrm{tr} * = -\theta^i\bar{\chi}_i\,.$$

Criterion for $GL(D)$-irreducibility: for $p \geq q$: $\sigma\,\omega = 0$ and also $\widetilde{\omega} = \omega$ when $p = q$.

Irreducible field: $\omega_{[p,q]} = \mathcal{P}_{[p,q]}\omega_{p,q}$, with Young projector

$$\mathcal{P}_{[p,q]} = \begin{cases} \mathbb{I} + \sum_{n=1}^{q} c_n(p,q)\widetilde{\sigma}^n\sigma^n\,, & p \geq q \\ \mathbb{I} + \sum_{n=1}^{p} c_n(q,p)\sigma^n\widetilde{\sigma}^n\,, & p \leq q \end{cases}\,, \quad c_n(p,q) = \frac{(-1)^n}{\prod_{r=1}^{n} r(p-q+r+1)}\,.$$

cf. de Medeiros, Hull '02

Generalized Hodge duality

To construct Lagrangians, we need a suitable inner product. (For p-forms controlled by $*$).

Generalized Hodge star operator for bipartite tensor fields,

$$(\star\,\omega)_{D-p,D-q} = \frac{1}{(D-p-q)!}\,\eta^{D-p-q}\,\widetilde{\omega}_{q,p}\,.$$

Note that the combination $*\widetilde{*}$ is different than $\star$:

$$\star\,\omega = *\,\widetilde{*}\,(-1)^{\epsilon}\sum_{n=0}^{\min(p,q)}\frac{(-1)^n}{(n!)^2}\,\eta^n\,\mathrm{tr}^n\,\omega\;, \qquad \big(\epsilon = (D-1)(p+q)+pq+1\big)\,.$$

Very welcome that $\star$ also encodes all traces of the mixed-symmetry tensor.

A symmetric inner product of some ω and ω' is then defined by $\int_{\theta,\chi}\omega\star\omega'$.

Kinetic terms

- For differential forms ω_p, we know that $S_{\text{kin}} = \int \mathrm{d}\omega \wedge * \mathrm{d}\omega$.
- For $p = q = 1$, the linearized Einstein-Hilbert Lagrangian is

$$\mathcal{L}_{\text{LEH}}(h_{[1,1]}) = -\frac{1}{4} h^i{}_i \Box h^j{}_j + \frac{1}{2} h^k{}_k \partial_i \partial_j h^{ij} - \frac{1}{2} h_{ij} \partial^j \partial_k h^{ik} + \frac{1}{4} h_{ij} \Box h^{ij}.$$

- For $p = 2$, $q = 1$, there exists a gauge theory for the hook Young tableaux Curtright '80

$$\begin{aligned} \mathcal{L}_{\text{Curtright}}(\omega_{[2,1]}) \quad = \quad & \frac{1}{2} \Big(\partial_i \omega_{jk|l} \partial^i \omega^{jk|l} - 2 \partial_i \omega^{ij|k} \partial^l \omega_{lj|k} - \partial_i \omega^{jk|i} \partial^l \omega_{jk|l} - \\ & - 4 \omega_i{}^{j|i} \partial^k \partial^l \omega_{kj|l} - 2 \partial_i \omega_j{}^{k|j} \partial^i \omega^l{}_{k|l} + 2 \partial_i \omega_j{}^{i|j} \partial^k \omega^l{}_{k|l} \Big) \ . \end{aligned}$$

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For any $\omega_{p,q}$ in Minkowski spacetime $\mathbb{R}^{1,D-1}$, a universal kinetic term:

$$\boxed{\mathcal{L}_{\text{kin}}(\omega_{p,q}) = \int_{\theta,\chi} \mathrm{d}\omega \star \mathrm{d}\omega \,.}$$

- Gauge invariance $\delta\omega = \mathrm{d}\lambda_{p-1,q} + \widetilde{\mathrm{d}}\lambda_{p,q-1}$ is basically obvious.
- Mass term is $m^2 \int_{\theta,\chi} \omega \star \omega$. E.g., Fierz-Pauli term, $m^2 \left(h^{ij} h_{ij} - (h^i{}_i)^2 \right)$.

“Galileon” higher derivative interaction terms

What is the most general theory in flat spacetime with field equations being polynomial in (strictly) 2$^{\text{nd}}$ order derivatives of ω?

For scalar fields, the answer is given by Galileons. Nicolis, Rattazzi, Trincherini ’08

They are invariant under the characteristic symmetry $\phi \to \phi + c$ and $\partial_i\phi \to \partial_i\phi + b_i$.

Originally they were found as (in 4D):

3 fields : $\partial^i\phi\,\partial^j\phi\,\partial_i\partial_j\phi - \partial^i\phi\,\partial_i\phi\,\Box\phi$

4 fields : $$-(\Box\phi)^2\,\partial_i\phi\,\partial^i\phi + 2\,\Box\phi\,\partial_i\phi\,\partial_j\phi\,\partial^i\partial^j\phi + \\ +\partial_i\partial_j\phi\,\partial^i\partial^j\phi\,\partial_k\phi\,\partial^k\phi - 2\,\partial_i\phi\,\partial^i\partial^j\phi\,\partial_j\partial_k\phi\,\partial^k\phi$$

5 fields : $$-(\Box\phi)^3\,\partial_i\phi\,\partial^i\phi + 3\,(\Box\phi)^2\,\partial_i\phi\,\partial_j\phi\,\partial^i\partial^j\phi + \\ +3\,\Box\phi\,\partial_i\partial_j\phi\,\partial^i\partial^j\phi\,\partial_k\phi\,\partial^k\phi - 6\,\Box\phi\,\partial_i\phi\,\partial^i\partial^j\phi\,\partial_j\partial_k\phi\,\partial^k\phi + \\ -2\,\partial_i\partial^j\phi\,\partial_j\partial^k\phi\,\partial_k\partial^i\phi\,\partial_l\phi\,\partial^l\phi - 3\,\partial_i\partial_j\phi\,\partial^i\partial^j\phi\,\partial_k\phi\,\partial_l\phi\,\partial^k\partial^l\phi + \\ +6\,\partial_i\phi\,\partial^i\partial^j\phi\,\partial_j\partial_k\phi\,\partial^k\partial^l\phi\,\partial_l\phi$$

Later cast in more controllable form & generalised to p-forms Deffayet, Deser, Esposito-Farese ’10

Bipartite tensor Galileons as "generalised kinetic terms"

Including $\mathcal{L}_{\text{kin}}$, such interactions for any bipartite tensor in any D are included in

$$\mathcal{L}_{\text{Gal}}(\omega_{p,q}) = \sum_{n=0}^{n_{max}} \int_{\theta,\chi} \mathrm{d}\omega_{(n+1)} \star \mathrm{d}\omega_{(n+1)} \,,$$

where we defined $\omega_{(n+1)} \equiv \omega \left(\mathrm{d}\widetilde{\mathrm{d}}\,\omega\right)^n$. Note: only even field appearances here.

In the special case of $p = q$ (scalars, gravitons &c.), an enhancement to odd fields

$$\widetilde{\mathcal{L}}_{\text{Gal}}(\omega_{p,p}) = \mathcal{L}_{\text{Gal}}(\omega) + \sum_n \int_{\theta,\chi} \eta^{p+1} \mathrm{d}\omega_{(n)} \star \mathrm{d}\omega_{(n+1)} = \sum_{n=1}^{n_{\text{max}}} \int_{\theta,\chi} \eta^{D-(p+1)n-p}\, \omega_{(n+1)} \,.$$

- Bound on field appearances: $n_{\text{max}}^{(p,q)} = \left\lfloor \frac{D+1}{p+q+2} \right\rfloor$ and $n_{\text{max}}^{[p,p]} = \left\lfloor \frac{D-p}{p+1} \right\rfloor$.
- "Evenophilic": $(\mathrm{d}\widetilde{\mathrm{d}}\,\omega_{p,q})^2|_{p+q=\text{odd}} = 0 = (\mathrm{d}\widetilde{\mathrm{d}}\,\widetilde{\omega}_{q,p})^2|_{p+q=\text{odd}}$. For odd, higher-∂ topol. terms.
- For graviton $\rightsquigarrow$ correspondence to Lovelock invariants. Exist for 2-form too.

Generalizations

The symmetry is (with b fully antisymmetric (and constant)):

$$\delta\omega_{p,q} = \begin{cases} \mathrm{d}\lambda_{p-1,q} + \tilde{\mathrm{d}}\lambda'_{p,q-1} + b_{i_0 i_1 \dots i_{p+q}} x^{i_0} \theta^{i_1} \cdots \theta^{i_p} \chi^{i_{p+1}} \cdots \chi^{i_{p+q}} & (p, q > 0) \\ \mathrm{d}\lambda_{p-1,0} + b_{i_0 i_1 \dots i_p} x^{i_0} \theta^{i_1} \cdots \theta^{i_p} & (p > 0, q = 0) \\ \tilde{\mathrm{d}}\lambda'_{0,q-1} + b_{i_0 i_1 \dots i_q} x^{i_0} \chi^{i_1} \cdots \chi^{i_q} & (p = 0, q > 0) \\ c + b_i x^i & (p = q = 0) \end{cases}$$

A number of generalizations exist, elegantly captured in the graded formalism:

cf. Deffayet, Deser, Esposito-Farese ’09, Deffayet, Esposito-Farese, Vikman ’09

- Multiple interacting species of any type; allows Galileons with odd total degree too.
- Field equations up to second order.
- Curved spacetime; e.g. Horndeski for 4D scalar (more tricky for bipartite tensors).

Standard duality and parent actions

Typically, duality is realised at Lagrangian level via a parent $\mathcal{L}$ for 2 independent fields.

Integrating out each of them leads to 2 dual theories and implements a duality relation.

- Dualization of a $(p-1)$-form to a $(D-p-1)$-form.

$$\mathcal{L}_{\mathrm{P}}(F_p, \lambda_{p+1}) = -\frac{1}{2(p+1)!} F_{i_1 \dots i_p} F^{i_1 \dots i_p} - \frac{1}{(p+1)!} \lambda^{i_1 \dots i_{p+1}} \partial_{i_1} F_{i_2 \dots i_{p+1}} \,.$$

λ-EOM $\rightsquigarrow \mathcal{L}(\omega_{p-1})$. F-EOM $\rightsquigarrow$ Duality relation $\rightsquigarrow \mathcal{L}(\widehat{\omega}_{D-p-1} = *\lambda_{p+1})$.

  - E.g. $D = 2$, $p = 1 \rightsquigarrow R \leftrightarrow 1/R$ (T-)duality; $D = 4$, $p = 2 \rightsquigarrow e \leftrightarrow 1/e$ (S-)duality
  - Duality relation:

$$\mathrm{d}\omega_{p-1} = F \propto *\widehat{F} = \mathrm{d}\widehat{\omega}_{D-p-1} \,.$$

  - BI/EOM $\mathrm{d}F = 0 = \mathrm{d}^\dagger F$ are mapped to EOM/BI for the dual field $\mathrm{d}^\dagger \widehat{F} = 0 = \mathrm{d}\widehat{F}$.

Duality for the graviton

- Dualization of the graviton $h_{[1,1]}$ West '01; Boulanger, Cnockaert, Henneaux '03

$$\mathcal{L}_{\mathrm{P}}(f_{2,1},\lambda_{3,1}) = f_{ij}{}^{j} f^{ik}{}_{k} - \frac{1}{2} f_{ijk} f^{ikj} - \frac{1}{4} f_{ijk} f^{ijk} + \frac{1}{2} \lambda_{ijkl} \partial^{i} f^{jkl} .$$

λ-EOM $\rightsquigarrow$ Linearised Einstein-Hilbert (antisymmetric part cancels out).

f-EOM $\rightsquigarrow$ Duality relation $\rightsquigarrow$ $\mathcal{L}(\hat{\omega}_{[D-3,1]} = *\widehat{\lambda}_{3,1})$ s.t. tr $\widehat{\lambda} = 0$.

  - E.g. D=4, graviton $\leftrightarrow$ graviton; D=5, graviton $\leftrightarrow$ Curtright; D=10, [7,1] (couple KKM).
  - Duality relation: Hull '01

$$\mathrm{d}\widetilde{\mathrm{d}}\omega_{[1,1]} = R \propto *\widehat{R} = \mathrm{d}\widetilde{\mathrm{d}}\widehat{\omega}_{[D-3,1]} .$$

  - BI $\widetilde{\mathrm{d}}R = 0 \mapsto$ BI $\widetilde{\mathrm{d}}\widehat{R} = 0$.
    Irreducibility $\sigma R = 0 \mapsto$ EOM tr$\widehat{R} = 0$.
    BI $\mathrm{d}R = 0$ & EOM tr$R = 0 \mapsto$ BI $\mathrm{d}\widehat{R} = 0$.

“Exotic” duality

- “Exotic” dualization of 2-form, Boulanger, Cook, Ponomarev ’12, Bergshoeff, Hohm, Penas, Riccioni ’16

  Essentially seen as bipartite with a trivial slot...

$$\mathcal{L}_{\mathrm{P}}(Q_{1,2},\lambda_{2,2}) = -\frac{1}{6}\,Q_{i|jk}Q^{i|jk} + \frac{1}{3}\,Q_{i|}{}^{ij}Q^{k|}{}_{kj} + \frac{1}{2}\,\lambda_{ij|kl}\partial^i Q^{j|kl}\,.$$

  λ-EOM $\rightsquigarrow \mathcal{L}(\omega_2)$. Q-EOM $\rightsquigarrow$ a theory for a $\widehat{\omega}_{[D-2,2]} = *\widehat{\lambda}_{2,2}$ s.t. $\mathrm{tr}\,\widehat{\lambda} = 0$.

  - Duality relation:

$$\mathrm{d}\widetilde{\mathrm{d}}\omega_{[0,2]} = R \propto *\widehat{R} = \mathrm{d}\widetilde{\mathrm{d}}\widehat{\omega}_{[D-2,2]}\,.$$

  - But now, irreducibility $\widetilde{\sigma}R = 0 \mapsto \mathrm{tr}\widehat{R} \neq 0$.
  - Instead $\mathrm{tr}^3\widehat{R} = 0$, but cannot be EL of any $\mathcal{L} \rightsquigarrow$ additional fields off-shell.

- Also double dual graviton, duals for Curtright and higher $(p,1)$ tensors &c.

A unified treatment of all these dualizations?

A universal first order action

A single two-parameter parent Lagrangian simultaneously accounting for

- the standard and exotic duals for any differential p-form, and
- the standard and double duals for any "generalized graviton" $(p, 1)$.

$$\boxed{\mathcal{L}_{\mathrm{P}}^{(p,q)}(F,\lambda) = \int_{\theta,\chi} F_{p,q} \star \mathcal{O}\, F_{p,q} + \int_{\theta,\chi} \mathrm{d}F_{p,q} * \widetilde{*}\, \lambda_{p+1,q}} \quad \text{for} \quad D \geq p+q+1\,.$$

- F and λ are independent $GL(D)$-reducible bipartite tensors.
- $\mathcal{O} = \mathcal{O}^{(p,q)}$ is a (known in closed form) operator acting on (p, q) tensors s.t.

$$\mathcal{O}\,\mathrm{d}\omega = \mathrm{d}\,[\omega] + \widetilde{\mathrm{d}}(\dots)\,.$$

Role: Yield the kinetic term for *irreducible* potential $[\omega]$ upon taking the λ-EOM.

- E.g. $\mathcal{O}^{(2,1)} = \mathbb{I} - \frac{1}{2}\,\widetilde{\sigma}\,\sigma$ (graviton), $\mathcal{O}^{(3,1)} = \mathbb{I} - \frac{1}{3}\,\widetilde{\sigma}\,\sigma$, $\mathcal{O}^{(2,2)} = \frac{4}{3}\,\mathbb{I} - \frac{1}{3}\,\sigma\,\widetilde{\sigma}$ (Curtright)

Admissible domains

For four domains of values, this Lagrangian yields all possible dual theories.

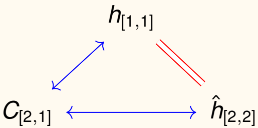
p	q	Original field	Dual field
$\in [1, D-1]$	0	$[p-1, 0]$	$[D-p-1, 0]$
$\in [2, D-2]$	1	$[p-1, 1]$	$[D-p-1, 1]$
1	$\in [1, D-2]$	$[0, q]$	$[D-2, q]$
2	$\in [2, D-3]$	$[1, q]$	$[D-3, q]$

- For the first 2, dual dynamics follow from $\mathcal{L}$. For the latter 2, extra off-shell fields.
  see also: Bergshoeff, Hohm, Penas, Riccioni '16
- All necessary cancellations follow from general identities.
- Extremal case $p = 0$ also admissible (domain walls).
- In suitable dimensions, topological θ terms also fit in this setting.
  e.g. in 4D, the $\tau \mapsto -\frac{1}{\tau}$ ("S" of $SL(2;\mathbb{Z})$) for the coupling $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$.
  Multiple field generalization $\rightsquigarrow$ higher "Buscher rules" in coupling space.
  In progress with Karagiannis & Ranjbar

The Fate of the Double Dual Graviton

More recently $\rightsquigarrow$ the double dual graviton does not provide a truly new description.
Henneaux, Lekeu, Leonard '19

In 5D, out of the three candidate duals, $h_{[1,1]}, C_{[2,1]}, \hat{h}_{[2,2]}$ two are algebraically related.

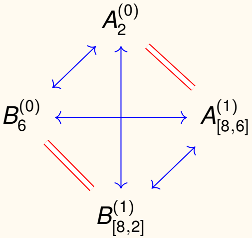


No exchange of EOMs and Bianchi identities between h and $\hat{h}$.

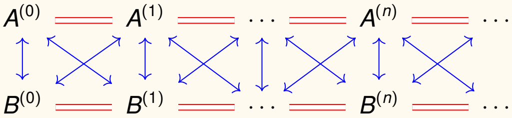
No new "doubly magnetic" solutions, only two (electric and magnetic) sources.
see also Hull '01

The Fate of the Exotic Dual of a differential form?

Take for example a 2-form in 10D. How many out of the 4 duals are independent?



Only 2! This is also true for all “infinite chain of duals” of Boulanger, Sundell, West ’15.



Nonstandard approach to standard duality of $A^{(0)}$ and $B^{(0)}$

Think of a p-form as a bipartite tensor of type $(p, 0)$ with a "Riemann" tensor

$$R^{A^{(0)}}_{[p+1,1]} := \mathrm{d}\widetilde{\mathrm{d}}A^{(0)}_p \,.$$

Its field equations and Bianchi identities are

$$\mathrm{d} * \mathrm{d}A^{(0)}_p = 0 \Leftrightarrow \mathrm{tr}\, R^{A^{(0)}} = 0 \quad \text{and} \quad \mathrm{d}R^{A^{(0)}} = 0 = \widetilde{\mathrm{d}}R^{A^{(0)}} \,.$$

(The identity $\mathrm{d}\mathrm{tr} + \mathrm{tr}\mathrm{d} = \widetilde{\mathrm{d}}^{\dagger}$ turns "Maxwell" into "Einstein".)

The dual tensor is defined in the standard way and satisfies BIs, therefore:

$$R^{B^{(0)}}_{[D-p-1,1]} := *R^{A^{(0)}}_{[p+1,1]} \,, \quad \mathrm{d}R^{B^{(0)}} = 0 = \widetilde{\mathrm{d}}R^{B^{(0)}} \rightsquigarrow R^{B^{(0)}}_{[D-p-1,1]} = \mathrm{d}\widetilde{\mathrm{d}}B^{(0)}_{D-p-2} \,, \quad \mathrm{tr}\, R^{B^{(0)}} = 0 \,.$$

The potentials are related by the expected Hodge duality (not related algebraically):

$$\mathrm{d}B^{(0)} = *\mathrm{d}A^{(0)} \,.$$

Relation of $B^{(0)}$ and $B^{(1)}$

Define a new irreducible tensor

$$R^{B^{(1)}}_{[D-1,p+1]} := (\widetilde{*} R^{A^{(0)}}_{[p+1,1]})^{\top} = *\widetilde{*}(R^{B^{(0)}}_{[D-p-1,1]})^{\top} \, .$$

It is easily shown that it satisfies the BIs $\mathrm{d}R^{B^{(1)}} = 0 = \widetilde{\mathrm{d}} R^{B^{(1)}}$. Therefore, locally it is

$$R^{B^{(1)}}_{[D-1,p+1]} = \mathrm{d}\widetilde{\mathrm{d}} B^{(1)}_{[D-2,p]} \, .$$

Using the definition of $R^{B^{(1)}}$ and the $B^{(0)}$-EOMs, it is shown that $R^{B^{(1)}} \propto \eta^p R^{B^{(0)}}$, thus

$$\mathrm{d}\widetilde{\mathrm{d}}(B^{(1)} - \eta^p B^{(0)}) = 0 \quad \Rightarrow \quad B^{(1)} \approx \eta^p B^{(0)} \, .$$

Also true for the associated currents/sources. Easily extended to “higher duals”.

Epilogue & Outlook

Concluding remarks

- Unified approach to mixed-sym. tensor gauge theories via graded coordinates.
- Offers a tractable path to search for generalizations of interacting HD theories.
- General treatment of (many, all in certain domain) standard and exotic dualities.

Outlook

- Universal parent $\mathcal{L}$ for multiple fields. Higher (non-stringy) “Buscher rules”?
- Topological terms? Applications? e.g. in gravitoelectromagnetism, Th Ch, Karagiannis, Schupp ’20
- Higher gauge theory approach to mixed-sym. tensors? perhaps á la Grützmann, Strobl ’14

THANKS

Finding $\mathcal{O}$

The operator $\mathcal{O}$ has the role of selecting the irreducible field. The requirement is

$$\mathcal{O}\,\mathrm{d}\omega_{p-1,q} \stackrel{!}{=} \mathrm{d}\omega_{[p-1,q]} + \widetilde{\mathrm{d}}(\dots)\ .$$

We find $(c_n(p,q) = \frac{(-1)^n}{\prod_{r=1}^{n} r(p-q+r+1)})$

$$\mathcal{O} = \begin{cases} \mathbb{I} + \sum_{n=1}^{q} c_n(p-1,q)\,\widetilde{\sigma}^n\,\sigma^n\,, & p \geq q+1 \\ \mathbb{I} + \sum_{n=1}^{p-1} c_n(q,p-1)\left(\sigma^n\,\widetilde{\sigma}^n + \sum_{k=1}^{n}(-1)^k \prod_{m=0}^{k-1}(n-m)^2 \sigma^{n-k}\,\widetilde{\sigma}^{n-k}\right)\,, & p < q+1 \end{cases}\ .$$

N.B.: For the domains of interest, only one term in the sum is relevant.

In fact, the domains are such that solving for λ with this $\mathcal{O}$ leads to the 2nd order theory

$$\mathcal{L}^{(p,q)}_{\lambda\text{-on-shell}} = \int_{\theta,\chi} \mathrm{d}\omega_{[p-1,q]} \star \mathrm{d}\omega_{[p-1,q]}\ .$$

This guarantees that the first side of the duality is correctly obtained.

Comments on the dualization

Establishing the duality requires varying with respect to $F_{p,q}$. We first show that

$$\int_{\theta,\chi} \delta\,(F \star \mathcal{O} F) = 2 \int_{\theta,\chi} \delta F \star \mathcal{O} F\,.$$

The F-variation then yields a duality relation, and $\mathcal{O}^{-1}$ is needed to solve it. We find

$$\begin{aligned}(\mathcal{O}^{(p,1)})^{-1} &= \mathbb{I} - \widetilde{\sigma}\,\sigma\,,\\ (\mathcal{O}^{(2,q)})^{-1} &= b_1\,\mathbb{I} + b_2\,\sigma\,\widetilde{\sigma} + b_3\,\sigma^2\,\widetilde{\sigma}^2\,,\end{aligned}$$

or trivial for the rest of the cases; b coefficients are given by

$$b_1 = \frac{q+1}{q+2}\,, \qquad b_2 = \frac{q+1}{2(q+2)}\,, \qquad b_3 = -\frac{q+1}{2q(q+2)}\,.$$

Further comments on the dualization

- Domain I: straightforward (dual field is a differential form).
- Domain II: decompose the Lagrange multiplier

$$\lambda_{p+1,1} = \widehat{\lambda}_{p+1,1} + \eta\, \mathring{\lambda}_{p,0}\ , \qquad \mathrm{tr}\, \widehat{\lambda} = 0\,.$$

  Define $\widehat{\omega} = *\widehat{\lambda}$ (irreducible dual field). The dual $\mathcal{L}$ depends only on $\widehat{\omega}$.

- Domain III: decompose the Lagrange multiplier

$$\lambda_{2,q} = \widehat{\lambda}_{2,q} + \eta \mathring{\lambda}_{1,q-1}\ , \qquad \mathrm{tr}\, \widehat{\lambda} = 0\,.$$

  Define $\widehat{\omega} = *\widehat{\lambda}$. The dual $\mathcal{L}$ depends not only on $\widehat{\omega}$, but also on $\mathring{\lambda}$. The correct dual EOM is obtained by taking a suitable trace:

$$\mathrm{tr}^{q+1}\, \mathrm{d}\widetilde{\mathrm{d}}\, \widehat{\omega}_{[D-2,q]} = 0\,.$$

- Domain IV: decompose the Lagrange multiplier

$$\lambda_{3,q} = \widehat{\lambda}_{3,q} + \eta \mathring{\lambda}_{2,q-1}\,, \qquad \mathrm{tr}\, \widehat{\lambda} = 0\,.$$

  Define $\widehat{\omega} = *\widehat{\lambda}$. The dual $\mathcal{L}$ depends not only on $\widehat{\omega}$, but also on $\mathring{\lambda}$. The correct dual EOM is obtained by taking a suitable trace:

$$\mathrm{tr}^{q}\mathrm{d}\widetilde{\mathrm{d}}\widehat{\omega}_{[D-3,q]} = 0\,.$$