Geometry of Q-manifolds and Gauge Theories I

Alexei Kotov

University of Hradec Králové
Faculty of Science

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Part I. Q-manifolds: a general geometric approach to the classical BV-BRST

The category of Q-manifolds

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The category of Q-manifolds QMan consists of:

- Q-manifolds;
- $Q-$morphisms between two Q-manifolds (M_1, Q_1) and (M_2, Q_2) - degree preserving maps $\phi\colon M_1 \to M_2$ with the vanishing field strength

$$F := Q_1\phi^* - \phi^* Q_2 = 0$$

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The internal homomorphisms $\underline{Hom}(-,-)$ (or the super space of maps $M_1 \to M_2$) - in good cases is a new, possibly infinite-dimensional Q-manifold (G.Bonavolonta, A. K, 2013, the smooth structure on $\underline{Hom}(-,-)$)

Stable deformation $Q-$retract

Let (M, Q) be a $Q-$manifold. A stable deformation $Q-$retract of M is a $Q-$submanifold $N \subset M$ together with a projection $\mathrm{pr}_N \colon M \to N$ in the category of $Q-$manifolds and a $Q-$morphism $\widetilde{\phi} \colon M \times T[1]I \to M$, where I is the interval $[0, 1]$ parameterized by t, which satisfies the following properties:

1. $\phi_0 = \mathrm{pr}_N$
2. $\phi_1 = \mathrm{Id}_M$
3. $\phi_t|_N = \mathrm{Id}_N$

Here $\phi_t = \widetilde{\phi}|_{\mathrm{d}t=0}$; the third condition holds for all $t \in I$; the $Q-$structure on $M \times T[1]I$ is the product of the $Q-$structure on M and the canonical de Rham $Q-$structure on $T[1]I$.

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- $T[1]M$ for a graded supermanifold; the Q-field is the de Rham operator
- Lie-infinity algebroids (non-negatively or $\mathbb{N}-$graded Q-manifolds)

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- The differential graded resolution of a (possibly) singular variety, an example of a non-positively graded Q-manifold

Example of a differential graded (Koszul) resolution

Let $\chi : X \to \Sigma$ be a vector bundle. Consider the pull-back bundle $V :\ = \chi^*(X) = X \times_\Sigma X$ over the total space X; V admits the canonical section F, induced by the diagonal embedding $X \hookrightarrow X \times_\Sigma X$, the zero locus of which, $\Sigma = F^{-1}(0)$, coincides with the zero section of χ. One can easily verify that $(\Lambda^\bullet V^*, \delta = \iota_F)$ is the Koszul resolution of Σ, i.e.

$$H^i_\delta(\Gamma(\Lambda^\bullet V^*)) = \begin{cases} C^\infty(\Sigma)\,, & i = 0 \\ 0\,, i \neq 0 \end{cases}$$

EXAMPLE OF A DIFFERENTIAL GRADED (KOSZUL) RESOLUTION

If we impose that sections of $\Lambda^i V^*$ have the degree $-i$, so that the whole space of sections $\Gamma(\Lambda^\bullet V^*)$ becomes isomorphic to the algebra of functions on $M = V[-1]$, then (M, δ) is a non-positevely graded $Q-$manifold.

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Let U be an open subset of Σ, z^a be local coordinates and q^μ be some linear fiber coordinates on $X|_U$. The associated local coordinates on V are $(z^a, q^\mu, (q')^\nu)$, such that the canonical section F is given by $(z^a, q^\mu) \mapsto (z^a, q^\mu, q^\mu)$. Let p^μ be local fiber coordinates on $V[-1]$ corresponding to $(q')^\mu$, so that the degree of p^μ is equal to -1. Then δ will take the form

$$\delta = \sum_\mu q^\mu \partial_{p^\mu}$$

CONTRACTIBLE $Q-$MANIFOLDS

Let V be a $\mathbb{Z}-$graded vector space (for simplicity, we assume that the grading is bounded either from above or from below). We shall call $(T[1]V, \mathrm{d}_V)$ a contractible $Q-$manifold.

By the definition, a contractible $Q-$manifold possesses a homogeneous coordinate system (w^α, v^α), such that $Qw^\alpha = v^\alpha$.

Q-bundles

A $Q-$bundle (A.Kotov, T.Strobl, 2007) is a fibered bundle in the category QMan, that is, a locally trivial $\mathbb{Z}-$graded bundle $\pi\colon E \to M$ over a $Q-$manifold M, supplied with a total $Q-$structure, such that the projection map is a $Q-$morphism.

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Example of a Q-bundle

Let $\pi_X\colon E \to X$ be a fibered bundle over a smooth manifold, then $\pi = \mathrm{d}\pi_X\colon (T[1]E, \mathrm{d}_E) \to (T[1]X, \mathrm{d}_X)$ is a $Q-$bundle. The tangent map to any section of π_X gives us a Q-section of π.

Equivalent $Q-$manifolds

A $Q-$bundle $\lambda\colon M' \to M$ is called an equivalent reduction of $Q-$manifolds (or an equivalence $Q-$reduction) if λ admits a global $Q-$section $\sigma\colon M \to M'$ and a local trivialization over some open cover $\{U_i\}$ of M with a trivial $Q-$fiber $T[1]V$ for some V.

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The minimal equivalence relation generated by the equivalence $Q-$reduction is called an equivalence of $Q-$manifolds.

Example of equivalent $Q-$manifolds

Let $\mathfrak{g}$ be a Lie algebra, $X \to \Sigma$ be a $\mathfrak{g}-$equivariant bundle. Then F from the Example of Koszul resolution is an equivariant section of a $\mathfrak{g}-$equivariant vector bundle $V \to X$.

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Let $(\Gamma\left(\Lambda^\bullet V^* \otimes \Lambda^\bullet \mathfrak{g}^*\right), \gamma)$ be the Chevalley-Eilenberg complex, corresponding to the $\mathfrak{g}-$action on sections of $\Lambda^\bullet V^*$ and $s = \delta + \gamma$, where the Koszul operator δ is extended to the whole space by linearity.

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Then $M = V[-1]_X \times (\mathfrak{g}[1] \times X)$ is a $Q-$manifold, which is equivalent to $(\mathfrak{g}[1] \times \Sigma, \gamma_0)$. Here γ_0 is the Chevalley-Eilenberg differential, corresponding to the $\mathfrak{g}-$action on Σ.

Properties of equivalent Q–manifolds

- Given an equivalent reduction $\lambda\colon M' \to M$, we can always find a local trivialization (U_i) of M which is compatible with the section σ in the sense that $\sigma|_{U_i}$ coincides with the canonical inclusion $U_i \hookrightarrow U_i \times T[1]V$

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- Two Q-manifolds (M_1, Q_1) and (M_2, Q_2) are equivalent if and only if there is a third one (M, Q) together with equivalence reductions $\lambda_1\colon M \to M_1$ and $\lambda_2\colon M \to M_2$.

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- Equivalent $Q-$manifolds have the same $Q-$cohomology in all natural $Q-$complexes
- (under some topological properties) the equivalence relation generated by the equivalence reduction coincides with the equivalence relation generated by stable deformation $Q-$retracts

Infinitesimal gauge symmetries

Let (M, Q) be a $Q-$manifold, ξ be a degree -1 vector field on M. An infinitesimal gauge symmetry generated by ξ is the degree zero vector field $\delta_\xi = [\xi, Q] = \xi Q + Q\xi$.

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Let $\iota\colon (N, Q_N) \hookrightarrow (M, Q)$ be a $Q-$submanifold. Consider $T_N = (TM)|_N$ as a graded vector bundle over N. A section of degree k of T_N can be viewed as a $\iota-$derivation of functions on M, $\mathcal{F}(M)$, with values in functions on N, $\mathcal{F}(N)$, that is, a degree k linear operator

$$v\colon \mathcal{F}(M) \to \mathcal{F}(N)$$

which satisfies the $\iota-$Leibniz rule

$$v(fh) = v(f)\iota^*(h) + (-1)^{k \deg f}\iota^*(f)v(h)$$

for any two functions f, h on M, where the first function is of pure degree.

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Let ϵ be a degree -1 section of T_N. An infinitesimal gauge symmetry at N generated by ϵ is the degree zero $\iota-$derivation $\delta_\epsilon = \epsilon \circ Q + Q_N \circ \epsilon$.

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- Given a $Q-$submanifold N of M, the restriction of any gauge symmetry δ_ξ onto N is an infinitesimal gauge symmetry at N, generated by $\epsilon = \delta_\xi|_N$.

The generalized Cartan map

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The map $\tilde{\phi}\colon M_1 \to T[1]M_2$ is a Q-morphism (A.K., T.Strobl, 2007)

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Then ϕ can be canonically extended (in the same way as above) to a $G-$equivariant Q-morphism, $\tilde{\phi} : M_1 \to T[1]M_2$, where $T[1]M_2$ is supplied with the total differential $\mathrm{d} + L_{Q_2}$ (A.K., T. Strobl, 2007).

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We shall use that $\Omega(M)$ is canonically isomorphic to $\mathcal{F}(T[1]M)$.

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Then σ induces a chain map

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We identify σ with a $G-$equivariant map $P \to N$, which induces (by the previous Proposition) a chain map

$$(\Omega^*(N)^G, \mathrm{d} + L_Q) \to (\mathcal{F}(P)^G, Q_P) \simeq (\mathcal{F}(M), Q_M)$$

Chern-Weil characteristic classes

Let G be a Lie group and $Y \to X$ be a principal $G-$bundle. Then $T[1]Y \to T[1]X$ is a principal $T[1]G-$bundle. $T[1]G/G \simeq \mathfrak{g}[1]$ is contractible, therefore $T[1]Y/G \simeq T[1]Y \times_{T[1]G} \mathfrak{g}[1]$ always admits a section and all sections belong the same homotopy class.

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While $T[1]Y/T[1]G \simeq T[1](Y/G) \simeq T[1]X$, the quotient $T[1]Y/G$ gives us the structure of a Q-bundle over $T[1]X$ with the fiber isomorphic to $\mathfrak{g}[1]$. This $Q-$bundle corresponds to the Lie algebroid of infinitesimal symmetries of $Y \to X$, known as the **Atiyah algebroid**. Sections of this bundle are in one-to-one correspondence with $G-$connections on $Y \to X$.

$$\Omega^*(\mathfrak{g}[1]) \simeq W\mathfrak{g}\,, \quad (\Omega^*(\mathfrak{g}[1]))^{T[1]G} \simeq (S^*\mathfrak{g}^*)^G$$

The obtained map is the Chen-Weil map, which gives complex (or real) Chern classes.

The AKSZ sigma model

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The BV formalism for the AKSZ sigma model is immediately produced by the Q-structure on $\underline{Hom}(N, M)$, which is now endowed with a compatible symplectic structure of degree -1

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- Examples of the AKSZ sigma model: Chern-Simons theory; Poisson sigma model (N. Ikeda and P.Schaller, T.Strobl)
- There exists a presymplectic version of the AKSZ action (K. Alkalaev, M. Grigoriev; M. Grigoriev, A.K. for $Q-$bundles, in preparation)

The transgression formula for AKSZ

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The generalization of this fact in terms of the AKSZ action is given by the following theorem (A.K., T.Strobl, 2007)

Let (M, ω) be a symplectic Q-manifold of degree p, N_{p+2} a (p+2)-dimensional manifold with boundary $\partial N_{p+2} = N_{p+1}$ and ϕ a (degree-preserving) map from $T[1]N_{p+2}$ to $\mathcal{M}$. Then

$$\int_{T[1]N_{p+2}} \tilde{\phi}^*(\omega)$$

gives us the (classical part of the) AKSZ sigma model for $\mathcal{N} = T[1]N_{p+1}$ and M.

The transgression formula for AKSZ

One can reformulate it as follows: for a symplectic $Q-$manifold (M, Q, ω) with the symplectic form ω of degree $p > 0$ one has

- $\mathrm{d}\omega = 0$
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where ϵ is the Euler vector field on M, which determines the grading.

This implies that

$$\omega = (\mathrm{d} + L_Q)(\chi + I)$$

where $\chi = \frac{1}{p}\iota_\epsilon\omega$ and $I = \frac{1}{p+1}\iota_Q\chi$.

The transgression formula for AKSZ

In particular, one has

- $\omega = \mathrm{d}\chi$ and
- $\iota_Q\omega = \mathrm{d}h$, where $h = \frac{p}{p+1}\iota_Q\chi$.

The transgression formula for AKSZ

In particular, one has

- $\omega = \mathrm{d}\chi$ and
- $\iota_Q \omega = \mathrm{d}h$, where $h = \frac{p}{p+1} \iota_Q \chi$.

Let (M, ω) be a symplectic Q-manifold of degree p, N_{p+1} a (p+1)-dimensional manifold and ϕ a (degree-preserving) map from $T[1]N_{p+1}$ to M. Then the (classical part of the) AKSZ sigma model action for the source space $T[1]N_{p+1}$ and the target M is

$$S_{AKSZ}[\phi] = \int_{T[1]N_{p+1}} \tilde{\phi}^*(\chi + I)$$