Dimensional reduction for Manin triples

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AKSZ Topological σ-models

Alexandrov, Kontsevich, Schwarz, Zaboronsky

$$\text{Geometric data} = \begin{cases} \Sigma \quad d\text{-dimensional manifold}, \\ (\mathcal{M}, \omega_{\mathcal{M}} = d\alpha_{\mathcal{M}}, \theta) \; d-1 \text{ symplectic } Q\text{-manifold.} \end{cases}$$

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$d = 2$, Poisson σ-model

Ikeda, Schaller, Strobl

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- Some boundary conditions: are given by Coisotropic submanifolds of (M, π).

When $\Sigma =$Disk Cattaneo, Felder obtain the Kontsevich $\star$-product.

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Roytenberg, Severa

The correspondence asserts that:

- Degree 2 symplectic Q-manifodls $\leftrightarrow$ Courant algebroids.
- Lagrangian Q-submanifolds $\leftrightarrow$ Dirac structures.

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- Some boundary conditions: are Dirac strucures.

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Yes, related to Lie theory!!

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Using the pairing we identify $A' \cong A^*$, and Liu, Weinstein, Xu proved that

$$(A \to M, [\cdot, \cdot], \rho, \pi_A, \eta_A) \quad \pi_A \in \mathfrak{X}^2(A),\ \eta_A \in \mathfrak{X}^3(A)$$

becomes a Lie quasi-bialgebroid and E is the "Drinfeld double".

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If A' is also Dirac $\rightsquigarrow (E, A, A')$ forms a Manin triple and $(A \to M, [\cdot, \cdot], \rho, \pi_A)$ is a Lie bialgebroid.

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Supergeometric perspective: $A[1] \subset \mathcal{M}$ lagrangian Q-submanifold.
Choosing A' is equivalent to find a Weinstein tubular neighbourhood $\mathcal{M} \cong T^*[2]A[1]$, then $\theta \cong Q + \pi_A + \eta_A$.
A' Dirac $\leftrightarrow \theta \cong Q + \pi_A$.

Lie bialgebroids

Mackenzie, Xu

A Lie bialgebroid $(A \to M, [\cdot,\cdot]_A, \rho_A, \pi_A)$ is a Lie algebroid with $\pi_A \in \mathfrak{X}^2_{lin}(A)$ such that is Poisson and "infinitesimally multiplicative" (cocycle condition).

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Lie algebroids are infinitesimal versions of Lie groupoids.
We denote by $G(A) \rightrightarrows M$ the unique, if it exists, source simply connected Lie groupoid with Lie algebroid $A \to M$.

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Theorem (Mackenzie, Xu)

Let $(A \to M, [\cdot,\cdot]_A, \rho_A, \pi_A)$ be a Lie bialgebroid and suppose that $(A \to M, [\cdot,\cdot]_A, \rho_A)$ is integrable. Then the ssc integration $(G(A) \rightrightarrows M, \pi_G)$ is a Poisson groupoid.

Examples: Poisson-Lie groups, Symplectic groupoids.

The result

Result [Cabrera, -]

Let (E, A, A') be a Manin triple, Σ closed surface and $I = [0, 1]$. The Courant σ-model defined by

$$\begin{cases} \text{Source: } \Sigma \times I \\ \text{Target: } (E \to M, \langle\cdot,\cdot\rangle, \rho, [\![\cdot,\cdot]\!]) \\ \text{Boundary conditions: fields take values in a supermanifold} \\ \text{defined by the splitting } A' \subset E \text{ at } t = 0, 1. \end{cases}$$

admits a gauge fixing depending on A such that it leads to a "dimensional reduction" onto the Poisson sigma model defined by

$$\begin{cases} \text{Source: } \Sigma \\ \text{Target: } (G(A) \rightrightarrows M, \pi_G) \end{cases}$$

Our result is of Field-theoretic nature.

Related works

- In 2D and 3D topological field theories for generalized complex geometry Cattaneo, Qiu, Zabzine prove the lienar case, i.e. when $(A' \to M, [\cdot,\cdot], \rho, \pi_A = 0)$. Here $G(A) = A$.

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- In Towards holography in the BV-BFV setting Mnev, Schiavina, Wernli similar result for Chern-Simons, i.e. $M = *$.

Examples of Manin triples

Geometric objects that are codified by a Manin triple:

- r-matrix: $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, graph(r))$.

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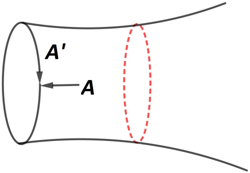
- Complex ss Lie group G: $(TG \oplus T^*G, E_G, \widehat{F}_G)$ Cartan-Dirac and Gauss-Dirac structures Alekseev, Bursztyn, Meinrenken.

Consequences

1. Collar neighbourhood: for a 3-dim manifold with 1 boundary



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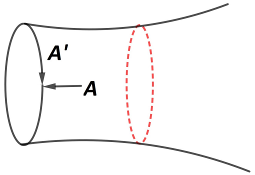

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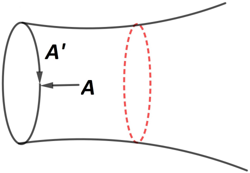

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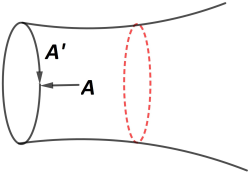

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4. Switching $A \leftrightarrow A'$: provides a "duality" between Poisson σ-models on dual Poisson-Lie groupoids.

Main ingredients I: Mapping space

Recall that $\mathcal{GM}$, the category of $\mathbb{Z}$-graded manifolds, has

- Objects: graded manifolds $\mathcal{M} = (M, \mathcal{O}_\mathcal{M}), \mathcal{N} = (N, \mathcal{O}_\mathcal{N})$...
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Associated to any pair of graded manifolds $(\mathcal{M}, \mathcal{N})$, formally, one can construct a new graded manifold (inner Hom) $Maps(\mathcal{M}, \mathcal{N})$ satisfying

$$Mor(\mathcal{Z} \times \mathcal{M}, \mathcal{N}) \cong Mor(\mathcal{Z}, Maps(\mathcal{M}, \mathcal{N})) \quad \forall \mathcal{Z}.$$

For graded vector spaces, deg k elements of $Maps(V, W)$ are $Mor(V, W[k])$.

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Associated to any pair of graded manifolds $(\mathcal{M}, \mathcal{N})$, formally, one can construct a new graded manifold (inner Hom) $Maps(\mathcal{M}, \mathcal{N})$ satisfying

$$Mor(\mathcal{Z} \times \mathcal{M}, \mathcal{N}) \cong Mor(\mathcal{Z}, Maps(\mathcal{M}, \mathcal{N})) \quad \forall \mathcal{Z}.$$

For graded vector spaces, deg k elements of $Maps(V, W)$ are $Mor(V, W[k])$.
A remarkable finite dimensional example is

$$Maps(\mathbb{R}[1], \mathcal{M}) \cong T[-1]\mathcal{M}.$$

Main ingredients II: Integration of Lie algebroids

Weinstein groupoid

Let $(A \xrightarrow{p} M, [\cdot,\cdot], \rho)$ be a Lie algebroid. Then we can construct the ssc groupoid $G(A) \rightrightarrows M$ by the following procedure:

- $\gamma \in Maps(I, A)$ is an "A-path" if

$$\rho(\gamma(t)) = \frac{d}{dt} p(\gamma(t)).$$

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$$\begin{cases} p \circ h_\epsilon(0) = p \circ \gamma_0(0), \quad p \circ h_\epsilon(1) = p \circ \gamma_0(1) \\ hdt + \widehat{h} d\epsilon : TI \times TI \to A \text{ algebroid morphism} \end{cases}$$

  for $\widehat{h}$ solution of $\partial_t \widehat{h} = \partial_\epsilon h + T_\nabla(h, \widehat{h})$; $\widehat{h}_\epsilon(0) = 0$.

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Theorem

If A is integrable then $G(A) = \frac{A-path}{A-homotopies}$

Steps of the proof

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1. We identify a symplectic fibration and integrate over the fibres.

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   Suppose $\mathcal{F}_{BV} = \mathcal{F}_1 \times \mathcal{F}_2$ with $\Phi = (\psi, \lambda)$.
   Choose a gauge fixing

   $$\mathcal{L}_{BV} = \mathcal{L}_1 \times \mathcal{L}_2$$

   and performing partial integration over $\mathcal{F}_2$

   $$\int_{\mathcal{L}_{BV} \subset \mathcal{F}_{BV}} \mathcal{O}\; e^{\frac{i}{\hbar} S_{BV}(\Phi)} \text{``}\mathcal{D}\Phi\text{''} = \int_{\mathcal{L}_1 \subset \mathcal{F}_1} \delta(P(\psi))\; \widehat{\mathcal{O}}\; e^{\frac{i}{\hbar} \widehat{S}(\psi)} \text{``}\mathcal{D}\psi\text{''}$$

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2. The effective theory of **1.** it has a remaining gauge symmetry that we can quotient out to obtain the Poisson groupoid.
   Our case: Quotient out A-homotopies.

Step 1.a: Structure space of fields

If (E, A, A') is a Manin triple

- $(\mathcal{M}, \omega_{\mathcal{M}}, \theta) \cong (T^*[2]A^*[1], \omega_{can} = d\alpha_{can}, Q + \pi_A)$,

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- $\mathcal{F}_{BV} = \{\phi \in Maps(T[1](\Sigma \times I), \mathcal{M}) | \phi(T[1]\partial(\Sigma \times I)) \subset A^*[1]\}$,
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$$\begin{aligned} \mathcal{F}_{BV} \subset Maps(T[1](\Sigma \times I), \mathcal{M}) = & \; Maps(T[1]\Sigma \times I \times \mathbb{R}[1], \mathcal{M}) \\ = & \; Maps(T[1]\Sigma \times I, T[-1]\mathcal{M}) \end{aligned}$$

$$\begin{array}{ccc} T[-1]\mathcal{M} & \xrightarrow{p} & T[-1]A^*[1] \\ \downarrow q & & \downarrow \\ T^*[1]A & \longrightarrow & A[1] \end{array}$$

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Proposition

$q : T[-1]\mathcal{M} \to T^*[1]A$ is a symplectic fibration. Moreover $\mathcal{L}_f = ker(p)$ induces a lagrangian submanifold on each fibre.

Step 1.b: The appearance of the δ

$T^*[1]A \to A^*[1]$ is a graded VB-algebroid over $A \to M$,

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Main field-theoretic claim

Let $\mathcal{K} \subset \mathcal{F}_A$ be a Lagrangian submanifold and $\mathcal{O} \in C^\infty(\mathcal{F}_A)$ then

$$\int_{\mathcal{L}_\mathcal{K}} (r^*\mathcal{O}) e^S = \int_\mathcal{K} \mathcal{O}\ \delta_\mathcal{C}\ e^{S_A}$$

where

$$S_A = \int_{T[1]\Sigma \times I} i_{\widehat{d}} ev^* \lambda_{can} + ev^* \pi_A$$

Step 1.b': Coordinate approach

Space	Coordinates
$A^*[1]$	$x^i(0), b_\alpha(1)$
$\mathcal{M} = T^*[2]A^*[1]$	$x^i, b_\alpha, a^\alpha(1), p_i(2)$
$T[-1]\mathcal{M}$	$x^i, b_\alpha, a^\alpha, p_i, \dot{x}^i(-1), \dot{b}_\alpha(0), \dot{a}_\alpha(0), \dot{p}_i(1)$
$T^*[1]A$	$x^i, b_\alpha, \dot{a}_\alpha, \dot{p}_i$

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Coordinates on $\mathcal{F}_{BV} = Maps(T[1](\Sigma \times I), \mathcal{M})$ capital letters.

$$S = \int P_i DX^i + A^\alpha DB_\alpha - \rho^i_\alpha A^\alpha P_i - \widehat{\rho}^{i\alpha} B_\alpha P_i + c^\gamma_{\alpha\beta} A^\alpha A^\beta B_\gamma + \widehat{c}^{\alpha\beta}_\gamma B_\alpha B_\beta A^\gamma$$

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Locally $\mathcal{F}_{BV} = \mathcal{F}_1 \times \mathcal{F}_2$ with

$$\mathcal{F}_1 = \{X^i, B_\alpha, \dot{A}_\alpha, \dot{P}_i\}, \; \mathcal{F}_2 = \{A^\alpha, P_i, \dot{X}^i, \dot{B}_\alpha\}$$

choose the gauge fixing $\mathcal{L}_2 = \{\dot{X}^i = 0, \dot{B}_\alpha = 0\}$ then

$$S_{|\mathcal{L}_2} = S_A + \int_{T[1]\Sigma} \int_I P_i \left(\partial_t X^i - \rho^i_\alpha \dot{A}^\alpha\right) + A^\alpha \left(-\partial_t B_\alpha + \rho^i_\alpha \dot{P}_i - c^\gamma_{\beta\alpha} \dot{A}^\beta B_\gamma\right)$$

Step 2.a: Lie-theoretic identities

Recall that we have the diagram

$$\begin{array}{ccc} C & \xrightarrow{\psi} & Maps(I, A) \\ \downarrow{\scriptstyle \tau} & & \\ G(A) & & \end{array}$$

Thm [Iglesias-Ponte, Laurent-Gengoux, Xu]

Let $(A \to M, [\cdot, \cdot], \rho)$ be a Lie algebroid with ssc groupoid $G(A)$. If $\omega \in \Omega(A)$ is an IM form then

$$\psi^* \int_I ev^* \omega = \tau^* \omega_{mul}$$

for some $\omega_{mul} \in \Omega(G(A))$ multiplicative form

Step 2.b: Reducing by symmetries

It is well known that $G(T^*A) = T^*G(A)$. In our graded context we obtain that $G(T^*[1]A) = T^*[1]G(A)$.

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Using the Theorem and the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\widehat{\psi}\quad} & Maps(T[1]\Sigma, Maps(I, T^*[1]A)) \\ \Big\downarrow{\scriptstyle \widehat{\tau}} & & \\ \mathcal{F}_G = Maps(T[1]\Sigma, T^*[1]G(A)) & & \end{array}$$

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Main Lie-Theoretic result

$$\widehat{\psi}^* S_A = \widehat{\tau}^* S_G \text{ where } S_G = \int_{T[1]\Sigma} i_{\widehat{d}} ev^* \Lambda_{can} + ev^* \pi_G$$

Final step: Putting everything together

It is easy to recognize that

$$\begin{cases} \mathcal{F}_G = Maps(T[1]\Sigma, T^*[1]G(A)) \\ S_G = \int_{T[1]\Sigma} i_{\widehat{d}} ev^*\Lambda + ev^*\pi_G \end{cases}$$

Give the Poisson σ-model with source Σ and target $(G(A), \pi_G)$.

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Give the Poisson σ-model with source Σ and target $(G(A), \pi_G)$.

If the observable is also homotopy invariant we have the following relation between the path integrals:

$$\int_{\mathcal{L}_{\mathcal{K}}} (r^*\mathcal{O}) e^S = \int_{\mathcal{K}} \mathcal{O}\ \delta_{\mathcal{C}}\ e^{S_A} = \int_{\mathcal{K}_{red}} \mathcal{O}_1\ e^{S_G}$$

Final remark: Formulation with sprays

One can gauge A-homotopies by using a Lie algebroid spray:
$V \in \mathfrak{X}(A)$ that is linear and $dp(V_a) = \rho(a),\ a \in A$.

The Lie algebroid spray identify some A-paths with a neighbourhood of the units in $G(A)$.

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The Lie algebroid spray identify some A-paths with a neighbourhood of the units in $G(A)$.

In particular, if $E = TM \oplus T^*M$ and $A = graph(\pi)$ for $\pi \in \mathfrak{X}^2(M)$ Poisson, the multiplicative symplectic form around the units of $G(T^*M)$ yields

$$\omega_\pi = \int_0^1 \varphi_t^* \omega_{can} dt$$

where $\varphi : I \times T^*M \to T^*M$ is the flow of the chosen Poisson spray. So we recover the Crainic, Marcut formula.

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- Is some version of this result related to Morita equivalence?

Thanks !!