Dimensional reduction for Manin triples

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AKSZ Topological $\sigma$-models

Alexandrov, Kontsevich, Schwarz, Zaboronsky

Geometric data = \[
\left\{ \begin{array}{l}
\Sigma \text{ } d\text{-dimensional manifold,} \\
(\mathcal{M}, \omega_\mathcal{M} = d\alpha_\mathcal{M}, \theta) \text{ } d - 1 \text{ symplectic } Q\text{-manifold.}
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- **Action:** $S: \mathcal{F}_{BV} \to \mathbb{R}$ given by $S = \int_{T[1]\Sigma} i_{d\Sigma} ev^* \alpha_M + ev^* \theta$. 
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- **Path integral:** \( \langle \mathcal{O} \rangle = \int_{\Phi \in \mathcal{L}_{BV} \subset \mathcal{F}_{BV}} \mathcal{O} \ e^{\frac{i}{\hbar} S(\Phi)} \) “\( \mathcal{D}\Phi \)”. 
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- **Gauge fixing:** $\mathcal{L}_{BV}$ Lagrangian submanifold of $(\mathcal{F}_{BV}, \omega_{BV})$. 
Geometric data = \{ \Sigma \text{ Surface}, (T^*[1]M, \omega_{can} = d\alpha_{can}, \theta = \pi) \text{ Poisson manifold.} \}
The classical space of fields

\[ \mathcal{F}^{cl} = \text{Hom}(T\Sigma, T^* M) = \{ Y : \Sigma \to M, \eta \in \Omega^1(\Sigma; Y^* T^* M) \} . \]
\(d = 2, \textbf{Poisson } \sigma\)-model

\textit{Ikeda, Schaller, Strobl}

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Some boundary conditions: are given by Coisotropic submanifolds of \( (M, \pi) \).

When \( \Sigma = \text{Disk} \) Cattaneo, Felder obtain the Kontsevich \( \star \)-product.
$d = 3$, **Courant $\sigma$-model**

Roytenberg, Severa

The correspondence asserts that:

- Degree 2 symplectic $Q$-manifolds $\leftrightarrow$ Courant algebroids.
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where $\langle e^a, e^b \rangle = g_{ab}$ and $\langle [e^a, e^b], e^c \rangle = T_{abc}$. 
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\item Some boundary conditions: are Dirac structures.
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Yes, related to Lie theory!!
(E → M, ⟨·, ·⟩, [·, ·], ρ) Courant algebroid.

A → M ⊂ E → M a Dirac structure.
Secret Poisson geometry of Dirac structures

- \((E \to M, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)\) Courant algebroid.
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choose \(A' \to M \subset E \to M\) a complement, i.e. \(E = A \oplus A'\).
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Using the pairing we identify \(A' \cong A^*\), and Liu, Weinstein, Xu proved that

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(A \to M, [\cdot, \cdot], \rho, \pi_A, \eta_A) \quad \pi_A \in \mathfrak{x}^2(A), \quad \eta_A \in \mathfrak{x}^3(A)
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becomes a Lie quasi-bialgebroid and \(E\) is the “Drinfeld double”.
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If $A'$ is also Dirac $\rightsquigarrow (E, A, A')$ forms a Manin triple and
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Supergeometric perspective: $A[1] \subset \mathcal{M}$ lagrangian $Q$-submanifold. Choosing $A'$ is equivalent to find a Weinstein tubular neighbourhood $\mathcal{M} \cong T^*[2]A[1]$, then $\theta \cong Q + \pi_A + \eta_A$.

$A'$ Dirac $\leftrightarrow \theta \cong Q + \pi_A$. 
A Lie bialgebroid \((A \to M, [\cdot, \cdot]_A, \rho_A, \pi_A)\) is a Lie algebroid with \(\pi_A \in \mathfrak{X}_\text{lin}^2(A)\) such that is Poisson and “infinitesimally multiplicative” (cocycle condition).
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Lie algebroids are infinitesimal versions of Lie groupoids. We denote by \(G(A) \rightrightarrows M\) the unique, if it exists, source simply connected Lie groupoid with Lie algebroid \(A \rightarrow M\).
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**Theorem (Mackenzie, Xu)**

Let \((A \to M, [\cdot, \cdot]_A, \rho_A, \pi_A)\) be a Lie bialgebroid and suppose that \((A \to M, [\cdot, \cdot]_A, \rho_A)\) is integrable. Then the ssc integration \((G(A) \rightrightarrows M, \pi_G)\) is a Poisson groupoid.

Examples: Poisson-Lie groups, Symplectic groupoids.
The result

Result [Cabrera, -]

Let \((E, A, A')\) be a Manin triple, \(\Sigma\) closed surface and \(I = [0, 1]\). The Courant \(\sigma\)-model defined by

\[
\begin{cases}
\text{Source: } \Sigma \times I \\
\text{Target: } (E \to M, \langle \cdot, \cdot \rangle, \rho, [\cdot, \cdot]) \\
\text{Boundary conditions: fields take values in a supermanifold defined by the splitting } A' \subset E \text{ at } t = 0, 1.
\end{cases}
\]

admits a gauge fixing depending on \(A\) such that it leads to a “dimensional reduction” onto the Poisson sigma model defined by

\[
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\text{Source: } \Sigma \\
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Our result is of Field-theoretic nature.
In 2D and 3D topological field theories for generalized complex geometry Cattaneo, Qiu, Zabzine prove the linear case, i.e. when \((A' \to M, [\cdot, \cdot], \rho, \pi_A = 0)\). Here \(G(A) = A\).
Related works

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- In Towards holography in the BV-BFV setting Mnev, Schiavina, Wernli similar result for Chern-Simons, i.e. \(M = \ast\).
Examples of Manin triples

Geometric objects that are codified by a Manin triple:

» **r-matrix**: \((\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \text{graph}(r))\).
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- **Mixing both:** \((TM \oplus T^*M, graph(\omega), graph(\pi))\).

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- **Twisted Poisson:** \(((TM \oplus T^*M) \overset{H}{\rightarrow} TM, TM, graph(\pi))\).

- **Complex ss Lie group \(G\):** \((TG \oplus T^*G, E_G, \hat{F}_G)\) Cartan-Dirac and Gauss-Dirac structures Alekseev, Bursztyn, Meinrenken.
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Consequences

1. **Collar neighbourhood:** for a 3-dim manifold with 1 boundary

   Transversal direction good ⇒ Dimensional reduction.
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   ![Diagram](image)

   Transversal direction good $\Rightarrow$ Dimensional reduction.

2. **Gauge fixing (in)dependence**: $\pi_A \in \mathcal{X}^2(A)$ is independent of $A$ but the topology of $G(A)$ strongly depends on $A$. 
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<table>
<thead>
<tr>
<th>Source</th>
<th>Poisson $\sigma$-model</th>
<th>Chern-Simons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target</td>
<td>$\Sigma$</td>
<td>$\Sigma \times I$</td>
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<tr>
<td>$(G, \pi)$</td>
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   \hline
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4. **Switching $A \leftrightarrow A'$:** provides a "duality" between Poisson $\sigma$-models on dual Poisson-Lie groupoids.
Main ingredients I: Mapping space

Recall that $\mathcal{G}M$, the category of $\mathbb{Z}$-graded manifolds, has

- **Objects**: graded manifolds $\mathcal{M} = (M, O_M), \mathcal{N} = (N, O_N), ...$
- **Morphisms**: degree preserving morphisms $\text{Mor}(\mathcal{M}, \mathcal{N})$

The cartesian product $M \times N$ makes $(\mathcal{G}M, \times)$ into a monoidal category. Associated to any pair of graded manifolds $(M, N)$, formally, one can construct a new graded manifold (inner Hom) $\text{Maps}(M, N)$ satisfying

$$\text{Mor}(\mathbb{Z} \times M, N) \cong \text{Mor}(\mathbb{Z}, \text{Maps}(M, N)) \quad \forall \mathbb{Z}.$$
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$$\text{Mor}(\mathbb{Z} \times \mathcal{M}, \mathcal{N}) \cong \text{Mor}(\mathbb{Z}, \text{Maps}(\mathcal{M}, \mathcal{N})) \quad \forall \mathbb{Z}.$$ 

For graded vector spaces, deg $k$ elements of $\text{Maps}(V, W)$ are $\text{Mor}(V, W[k])$. 
Main ingredients I: Mapping space

Recall that $\mathcal{GM}$, the category of $\mathbb{Z}$-graded manifolds, has

- **Objects**: graded manifolds $\mathcal{M} = (M, O_M), \mathcal{N} = (N, O_N)$...
- **Morphisms**: degree preserving morphisms $\text{Mor}(\mathcal{M}, \mathcal{N})$

The cartesian product $\mathcal{M} \times \mathcal{N}$ makes $(\mathcal{GM}, \times)$ into a monoidal category.

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For graded vector spaces, deg $k$ elements of $\text{Maps}(V, W)$ are $\text{Mor}(V, W[k])$.

A remarkable finite dimensional example is

$$\text{Maps}(\mathbb{R}[1], \mathcal{M}) \cong T[-1]\mathcal{M}.$$
Main ingredients II: Integration of Lie algebroids

Weinstein groupoid

Let \((A \xrightarrow{p} M, [\cdot, \cdot], \rho)\) be a Lie algebroid. Then we can construct the ssc groupoid \(G(A) \rightrightarrows M\) by the following procedure:

- \(\gamma \in Maps(I, A)\) is an “\(A\)-path” if
  \[
  \rho(\gamma(t)) = \frac{d}{dt}p(\gamma(t)).
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- \(\gamma_0, \gamma_1 \in Maps(I, A)\) are “\(A\)-homotopic” if it exists a variation \(h_\epsilon\) satisfying
  \[
  \begin{cases}
  p \circ h_\epsilon(0) = p \circ \gamma_0(0), & p \circ h_\epsilon(1) = p \circ \gamma_0(1) \\
  hdt + \hat{h}d\epsilon : TI \times TI \to A \text{ algebroid morphism}
  \end{cases}
  \]
  for \(\hat{h}\) solution of \(\partial_t \hat{h} = \partial_\epsilon h + T_\nabla (h, \hat{h}); \hat{h}_\epsilon(0) = 0.\)
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Theorem

If \(A\) is integrable then \(G(A) = \frac{\text{\(A\)-path}}{\text{\(A\)-homotopies}}\)
Steps of the proof

It has two parts:

1. We identify a symplectic fibration and integrate over the fibres.

2. The effective theory of 1. it has a remaining gauge symmetry that we can quotient out to obtain the Poisson groupoid.

Our case: Localization over A-path.

Our case: Quotient out A-homotopies.
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1. We identify a symplectic fibration and integrate over the fibres. Fibre version of the Losev’s trick:

Suppose $\mathcal{F}_{BV} = \mathcal{F}_1 \times \mathcal{F}_2$ with $\Phi = (\psi, \lambda)$. Choose a gauge fixing

$$\mathcal{L}_{BV} = \mathcal{L}_1 \times \mathcal{L}_2$$

and performing partial integration over $\mathcal{F}_2$

$$\int_{\mathcal{L}_{BV} \subset \mathcal{F}_{BV}} \mathcal{O} \ e^{i \hbar S_{BV}(\Phi)} \ "D\Phi" = \int_{\mathcal{L}_1 \subset \mathcal{F}_1} \delta(P(\psi)) \ \hat{\mathcal{O}} \ e^{i \hbar \hat{S}(\psi)} \ "D\psi"$$

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Step 1.a: Structure space of fields

If \((E, A, A')\) is a Manin triple

\[
\begin{align*}
\mathcal{M}, \omega_{\mathcal{M}}, \theta & \cong (T^*[2]A^*[1], \omega_{\text{can}} = d\alpha_{\text{can}}, Q + \pi_A),
\end{align*}
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\begin{align*}
\mathcal{M}, \omega_\mathcal{M}, \theta &\cong (T^*[2]A^*[1], \omega_{\text{can}} = d\alpha_{\text{can}}, Q + \pi_A), \\
\mathcal{F}_{BV} &= \{\phi \in \text{Maps}(T[1](\Sigma \times I), \mathcal{M}) | \phi(T[1]\partial(\Sigma \times I)) \subset A^*[1]\}, \\
S &= \int_{T[1](\Sigma \times I)} i_D^*ev^*\alpha_{\text{can}} + ev^*Q + ev^*\pi_A
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\[\mathcal{F}_{BV} \subset \text{Maps}(T[1](\Sigma \times I), \mathcal{M}) = \text{Maps}(T[1]\Sigma \times I \times \mathbb{R}[1], \mathcal{M}) = \text{Maps}(T[1]\Sigma \times I, T[-1]\mathcal{M})\]

\[\begin{array}{ccc}
T[-1]\mathcal{M} & \xrightarrow{p} & T[-1]A^*[1] \\
q \downarrow & & \downarrow \\
\end{array}\]

is a double vector bundle.
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$= Maps( T[1]\Sigma \times I, T[-1]\mathcal{M})$

$T[-1]\mathcal{M} \xrightarrow{p} T[-1]A^*[1]$

$\downarrow q \hspace{3cm} \downarrow$

$T^*[1]A \xrightarrow{q} A[1]$

is a double vector bundle.

**Proposition**

$q : T[-1]\mathcal{M} \to T^*[1]A$ is a symplectic fibration. Moreover $\mathcal{L}_f = \ker(p)$ induces a lagrangian submanifold on each fibre.
Step 1.b: The appearance of the $\delta$

$T^*[1]A \to A^*[1]$ is a graded VB-algebroid over $A \to M$,
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Denote by $\mathcal{F}_A = Maps(T[1]\Sigma \times I, T^*[1]A)$ and by

$$C = \{\phi \in \mathcal{F}_A | \phi(T[1]\Sigma) \subset C\}.$$
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Main field-theoretic claim

Let $\mathcal{K} \subset \mathcal{F}_A$ be a Lagrangian submanifold and $\mathcal{O} \in C^\infty(\mathcal{F}_A)$ then

$$\int \mathcal{L}_\mathcal{K} (r^*\mathcal{O}) e^S = \int_{\mathcal{K}} \mathcal{O} \delta_C e^{S_A}$$

where

$$S_A = \int_{T[1]\Sigma \times I} i_d e^* \lambda_{can} + e^* \pi_A$$
Step 1.b': Coordinate approach

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Coordinates on $\mathcal{F}_{BV} = Maps(T[1](\Sigma \times I), \mathcal{M})$ capital letters.

$$S = \int P_i DX^i + A^\alpha DB_\alpha - \rho^i_\alpha A^\alpha P_i - \tilde{\rho}^i_\alpha B_\alpha P_i + c^\gamma_{\alpha\beta} A^\alpha A^\beta B_\gamma + \tilde{c}^\gamma_{\alpha\beta} B_\alpha B_\beta A^\gamma$$
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Locally $\mathcal{F}_{BV} = \mathcal{F}_1 \times \mathcal{F}_2$ with

$$\mathcal{F}_1 = \{X^i, B_\alpha, \dot{A}_\alpha, \dot{P}_i\}, \mathcal{F}_2 = \{A^\alpha, P_i, \dot{X}^i, \dot{B}_\alpha\}$$

choose the gauge fixing $\mathcal{L}_2 = \{\dot{X}^i = 0, \dot{B}_\alpha = 0\}$ then

$$S_{|\mathcal{L}_2} = S_A + \int_{T[1]\Sigma} \int_I P_i \left( \partial_t X^i - \rho^i_\alpha \dot{A}_\alpha \right) + A^\alpha \left( -\partial_t B_\alpha + \rho^i_\alpha \dot{P}_i - c^{\gamma}_{\beta\alpha} \dot{A}^\beta B_\gamma \right)$$
Step 2.a: Lie-theoretic identities

Recall that we have the diagram

\[ C \xrightarrow{\psi} \text{Maps}(I, A) \]
\[ \downarrow \tau \]
\[ G(A) \]

**Thm [Iglesias-Ponte, Laurent-Gengoux, Xu]**

Let \((A \to M, [\cdot, \cdot], \rho)\) be a Lie algebroid with ssc groupoid \(G(A)\). If \(\omega \in \Omega(A)\) is an IM form then

\[ \psi^* \int_I \text{ev}^* \omega = \tau^* \omega_{mul} \]

for some \(\omega_{mul} \in \Omega(G(A))\) multiplicative form.
Step 2.b: Reducing by symmetries

It is well known that $G(T^*A) = T^*G(A)$. In our graded context we obtain that $G(T^*[1]A) = T^*[1]G(A)$. 
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By construction $\lambda_{can} \in \Omega^1(T^*[1]A)$ and $\pi_A \in \Omega^0(T^*[1]A)$ are IM forms on $T^*[1]A$.

Using the Theorem and the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\hat{\psi}} & \text{Maps}(T[1]\Sigma, \text{Maps}(I, T^*[1]A)) \\
\downarrow{\hat{\tau}} & & \downarrow \\
\mathcal{F}_G = \text{Maps}(T[1]\Sigma, T^*[1]G(A)) & &
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Using the Theorem and the diagram

$$\mathcal{C} \xrightarrow{\hat{\psi}} \text{Maps}(T[1]\Sigma, \text{Maps}(I, T^*[1]A))$$

$$\hat{\tau}$$

$$\mathcal{F}_G = \text{Maps}(T[1]\Sigma, T^*[1]G(A))$$

**Main Lie-Theoretic result**

$$\hat{\psi}^* S_A = \hat{\tau}^* S_G \text{ where } S_G = \int_{T[1]\Sigma} i_d^* \text{ev}^* \Lambda_{can} + \text{ev}^* \pi_G$$
Final step: Putting everything together

It is easy to recognize that

\[
\begin{align*}
\mathcal{F}_G &= \text{Maps}(T[1]\Sigma, T^*[1]G(A)) \\
S_G &= \int_{T[1]\Sigma} i_dev^*\Lambda + ev^*\pi_G
\end{align*}
\]

Give the Poisson $\sigma$-model with source $\Sigma$ and target $(G(A), \pi_G)$. 

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\end{align*}
\]

Give the Poisson $\sigma$-model with source $\Sigma$ and target $(G(A), \pi_G)$.

If the observable is also homotopy invariant we have the following relation between the path integrals:

\[
\int_{\mathcal{L}_K} (r^* \mathcal{O}) e^S = \int_{\mathcal{K}} \mathcal{O} \delta_{\mathcal{C}} e^{S_A} = \int_{\mathcal{K}_{\text{red}}} \mathcal{O}_1 e^{S_G}
\]
Final remark: Formulation with sprays

One can gauge $A$-homotopies by using a Lie algebroid spray: $V \in \mathfrak{X}(A)$ that is linear and $dp(V_a) = \rho(a)$, $a \in A$.

The Lie algebroid spray identify some $A$-paths with a neighbourhood of the units in $G(A)$. 
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The Lie algebroid spray identify some $A$-paths with a neighbourhood of the units in $G(A)$.

In particular, if $E = TM \oplus T^*M$ and $A = \text{graph}(\pi)$ for $\pi \in \mathfrak{X}^2(M)$ Poisson, the multiplicative symplectic form around the units of $G(T^*M)$ yields

$$\omega_\pi = \int_0^1 \varphi_t^* \omega_{\text{can}} dt$$

where $\varphi : I \times T^*M \to T^*M$ is the flow of the chosen Poisson spray. So we recover the Crainic, Marcut formula.
Future projects

- Perturbative computation of the path integral
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- Can we include $\Sigma$ with boundary in our computation?
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- Can we include $\Sigma$ with boundary in our computation?
- Is some version of this result related to Morita equivalence?
Thanks !!