Multisymplectic (co-)momentum geometry

Leonid Ryvkin

University of Duisburg-Essen

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References

The talk is based on :

- **L.R. and T. Wurzbacher** Existence and unicity of co-moments in multisymplectic geometry, Journal of Differential Geometry and Applications, 2015.
- **L.R. T. Wurzbacher and M. Zambon** Conserved Quantities on Multisymplectic Manifolds, Journal of the Australian Mathematical Society, 2020 (2016).
- **A. Miti and L.R.** Multisymplectic actions of compact Lie groups on spheres, accepted to the Journal of Symplectic Geometry, 2019.
- **L. Mammadova and L.R.** On the extension problem for weak moment maps, arXiv :2001.00264, 2020.

And :

- **M. Callies, Y. Fregier, C. L. Rogers, M. Zambon** Homotopy moment maps, Advances in Mathematics, 2016.
- **Y. Fregier, C. Laurent-Gengoux, M. Zambon** A cohomological framework for homotopy moment maps, Journal of Geometry and Physics, 2015.
- **J. Hermann** Weak Moment Maps in Multisymplectic Geometry, Journal of Geometry and Physics, 2018.

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Reminder : Symplectic manifolds

Definition

Symplectic manifold : (M, ω), such that $\omega \in \Omega^2(M)$ is **closed** and **non-degenerate**.

Non-degeneracy : The map $\iota_\bullet \omega : TM \to \Lambda^k T^*M, \quad v \mapsto \iota_v \omega$ is injective.

Examples :

- T^*Q with $\omega = d\theta$, $\theta_\eta(v) = \eta(\pi_* v)$
- 2-dimensional manifolds with volume forms
- coadjoint orbits

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Reminder : Symplectic moment maps

Let (M,ω) be symplectic and $\vartheta : M \times G \to M$ a Lie group action that preserves ω.

Definition

Moment map : $f \in C^\infty(M,\mathfrak{g}^*)$, such that

1. $df^\xi(\cdot) = \omega(v_\xi,\cdot)$ for all $\xi \in \mathfrak{g}$.
2. $f(\vartheta_g(x)) = Ad^*_g(f(x))$ for all $g \in G$, $x \in M$.

Assume G is connected, then from a dual perspective,

$$f : \mathfrak{g} \to C^\infty(M)$$

1. that lifts the infinitesimal action $v : \mathfrak{g} \to \mathfrak{X}(M)$.
2. and is a Lie algebra homomorphism.

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Examples :

- $k = 1$: ω is symplectic.
- $k = dim(M) - 1$: ω is a volume form.
- $k = 2$: Let G be a semsisimple Lie group and $\langle \cdot, \cdot \rangle$ its Killing form. Then $\langle [\cdot, \cdot], \cdot \rangle$ extends to a biinvariant (multisymplectic) form ω.

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Multisymplectic manifolds (2)

More examples :

- **Multi-Phase spaces :** Let Q be a manifold. Consider $M = \Lambda^k T^*Q$. The canonical form $\theta \in \Omega^k(M)$ is defined by

$$\theta_\eta(v_1, ..., v_k) = \eta(\pi_*(v_1), ...\pi_*(v_k)).$$

  The form $\omega = d\theta$ is multisymplectic.

- **... with magnetic term :** Let $\mu \in \Omega^{k+1}_{cl}(Q)$. Then the following form is also multisymplectic :

$$\omega + \pi^*\mu$$

- **Sphere** $M = S^6$, standard metric g and almost-complex structure J. Then $\eta = g(J\cdot, \cdot) \in \Omega^2(S^6)$ is not closed. However,

$$\omega = d\eta \in \Omega^3(S^6)$$

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No Darboux theorem, not necessarily transitive Diffeomorphisms.

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The L_∞ algebra of observables (1)

Let (M, ω) be multisymplectic.

$$\Omega^{k-1}_{Ham}(M,\omega) := \{\alpha \mid d\alpha = -\iota_{X_\alpha}\omega \ \text{ for some } X_\alpha \in \mathfrak{X}(M)\} \subset \Omega^{k-1}(M)$$

$$\begin{array}{ccccccccc} L_{k-1} & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & L_1 & \longrightarrow & L_0 \\ \| & & & & & & \| & & \| \\ C^\infty(M) & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \Omega^{k-2}(M) & \longrightarrow & \Omega^{k-1}_{Ham}(M,\omega) \end{array}$$

$$\{\cdots\}_i : \bigwedge^i L_0 \to L_{i-2}$$

$$\{\alpha_1, \dots, \alpha_i\}_i = -(-1)^{\frac{i(i+1)}{2}} \omega(X_{\alpha_1}, \dots, X_{\alpha_i}, \cdot, \cdots, \cdot)$$

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The L_∞ algebra of observables (2)

Definition/ Proposition (Rogers2012)

The L_∞-algebra of multisymplectic observables : $L_\infty(M,\omega)$

1. vector spaces L_i for $i = 0, ..., k-1$
2. differential $l_1 = \{\cdot\}_1 := d$
3. higher brakets $l_i = \{\cdots\}_i$ for $i = 2, ..., k+1$

is an L_∞-algebra.

$$0 = \sum_{\substack{i+j=n+1 \\ \sigma \in ush(i,n-i)}} (-1)^{i(j+1)} sgn(\sigma)\epsilon(\sigma;x)\{\{x_{\sigma(1)}, ..., x_{\sigma(i)}\}_i, x_{\sigma(i+1)} \cdots, x_{\sigma(n)}\}_j$$

Moreover, $L_\infty(M,\omega) \to \mathfrak{X}(M)$ is a L_∞-homomorphism.

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Why is the observable algebra simple?

$$C^\infty(M) \longrightarrow \dots \longrightarrow \Omega^{k-2}(M) \longrightarrow \Omega^{k-1}_{Ham}(M,\omega) \dashrightarrow \mathfrak{X}_{Ham}(M,\omega)$$

Theorem (BFLS)

Let $L_\bullet = (\dots \to L_1 \to L_0 \to \mathcal{F})$ be a resolution of the Lie algebra $\mathcal{F}$.

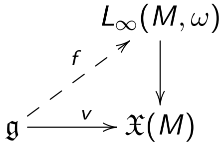
- A skew-symmetric $l_2 : L_0 \times L_0 \to L_0$ covering the Lie bracket on $\mathcal{F}$ can be extended to an L_∞-algebra structure on $L_\bullet$.
- If l_2 is zero on boundaries, then the structure can be chosen such that l_i are non-zero only on L_0.

Homotopy comoments (1)

Let (M,ω) multisymplectic and $v:\mathfrak{g}\to\mathfrak{X}(M,\omega)$ an action preserving ω.

Definition FRZ-2013

A *(homotopy) comoment* is an L_∞ homomorphism $f:\mathfrak{g}\to L_\infty(M,\omega)$ projecting to v.



I.e. it is given by $f_i:\bigwedge^i\mathfrak{g}\to L_{i-1}$ for $i\in\{1,...,k\}$, such that

1. $df_1(\xi)=-\iota_{v_\xi}\omega$
2. $\sum_{k<l}-(-1)^{k+l}f_{i-1}([\xi_k,\xi_l]\wedge\xi_1...\wedge\hat{\xi}_k\wedge...\hat{\xi}_l\wedge...\xi_i)=df_i(\xi_1\wedge...\wedge\xi_i)-(-1)^{\frac{i(i-1)}{2}}\omega(v_{\xi_1},...,v_{\xi_i},\cdots)$

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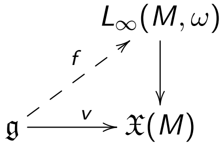

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Homotopy comoments (2)

Theorem (RW15 and FLZ15)

Let (M, ω) be multisymplectic and $v : \mathfrak{g} \to \mathfrak{X}(M, \omega)$.

- The map $\xi_1 \wedge \ldots \wedge \xi_i \mapsto \pm\omega(v_{\xi_1}, \ldots, v_{\xi_i}, \cdots)$ defines classes $c_i \in H^i(\mathfrak{g}) \otimes H^{k+1-i}_{dR}(M)$
- A homotopy comoment f for the action exists if and only if $c_i = 0$ for all $i \in \{1, \ldots, k+1\}$

Symplectic case : The obstructions to the existence of a comoment live in $H^1(\mathfrak{g}) \otimes H^1_{dR}(M)$ and $H^2(\mathfrak{g}) \otimes H^0_{dR}(M)$.

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Proof (1)

$$dl_{i+1} = \delta l_i$$

$$
\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\uparrow{\scriptstyle \delta\otimes id} & & \uparrow{\scriptstyle \delta\otimes id} & & \uparrow{\scriptstyle \delta\otimes id} & & \uparrow{\scriptstyle \delta\otimes id} & & \\
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proof (2)

$f_i : \bigwedge^i \mathfrak{g} \to L_{i-1}$ for $i \in \{1, ..., k\}$, such that

1. $df_1(\xi) = -\iota_{v_\xi}\omega$
2. $f_k \circ \partial_{CE} + df_{k+1} = -f_1^* l_k$

$$
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Homotopy comoments (3)

Theorem (MR19)

Let (M,ω) as above, G compact, $\vartheta : M \times G \to M$ preserves ω. A comoment exists if and only if

$$[\vartheta^*\omega - \pi^*\omega] = 0 \in H^{k+1}_{dR}(M \times G)$$

Proof :

$$\Omega(M)^{\vartheta} \dashrightarrow \Omega(M \times G)^{id \times r} \longleftrightarrow \Omega(M) \otimes \Omega(G)^{r} \longleftrightarrow \Omega(M) \otimes \Lambda\mathfrak{g}^*$$

Homotopy comoments (4)

Corollaries :

- If ω has an (invariant) potential, then there is a comoment.
- If ω can be extended to an equivariant cohomology class $\tilde{\omega} \in H_G^{k+1}(M)$, then there is a comoment.
- If G_1 has a comoment on (M_1, ω_1) and G_2 has a comoment on (M_2, ω_2), then $G_1 \times G_2$ has a comoment on $(M_1 \times M_2, \pi_1^*\omega_1 + \pi_2^*\omega_2)$
- ... and on $(M_1 \times M_2, \pi_1^*\omega_1 \wedge \pi_2^*\omega_2)$. [SZ16]

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- ... and on $(M_1 \times M_2, \pi_1^*\omega_1 \wedge \pi_2^*\omega_2)$. [SZ16]

Homotopy comoments (4)

Corollaries :

- If ω has an (invariant) potential, then there is a comoment.
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Conserved Quantities (1)

Let (M, ω) be n-plectic and $H \in \Omega^{n-1}_{Ham}(M, \omega)$.

Definition (RWZ)

A differential form $\alpha \in L_\infty(M, \omega)$ is called

- conserved quantity if $L_{X_H}\alpha$ is exact.
- strictly conserved quantity if $L_{X_H}\alpha$ is 0.
- locally conserved quantity if $L_{X_H}\alpha$ is closed.

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Let $\alpha_1, ..., \alpha_k$ be locally conserved. Then $\{\alpha_1, ..., \alpha_k\}_k$ is strictly conserved.

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Conserved Quantities (2)

Theorem

Let H be $\mathfrak{g}$-invariant and f a comoment. Then $f_k(P_{k,\mathfrak{g}})$ are globally conserved.

	H locally $\mathfrak{g}$-pres.	H globally $\mathfrak{g}$-pres.	H strictly $\mathfrak{g}$-pres.
$f_k(Z_k(\mathfrak{g}))$	locally cons.	locally cons.	globally cons.
$f_k(B_k(\mathfrak{g}))$	globally cons.	globally cons.	globally cons.

Let Σ be an d-dimensional closed manifold, $\sigma_0 : \Sigma \to M$ smooth and $\sigma_t = \phi_t^{X_H} \circ \sigma_0$, where ϕ^{X_H} is the flow of X_H. Then, for a conserved quantity $\alpha \in \Omega^d(M)$, the value of the following integral is independent of t :

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Comoments for spheres (1)

Theorem (MR19)

Let $M = S^n$, ω a volume form and G compact, acting effectively, preserving the volume. The action admits a homotopy comoment map if and only if

- n is even,
- or the action is not transitive.

Examples :

- $SO(2n+1) \circlearrowleft S^{2n}$: has a comoment
- $SO(2n) \circlearrowleft S^{2n-1}$: has no comoment
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Idea of proof :

- The obstructions live in :
  $H^{n-1}(S^n) \otimes H^1(G)$
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- intransitive case : There exists an orbit $p \cdot G \subset S^n$ of $dim < n$.

- transitive case : classification
  - $SO(n)/SO(n-1) = S^{n-1}$
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Weak comoment maps (1)

Let (M, ω) and G as above. $P_{\mathfrak{g}}^k = ker(\Lambda^k \mathfrak{g} \to \Lambda^{k-1}\mathfrak{g})$

Definition (H2018)

A *weak comoment map* is a collection $f_i : P_{\mathfrak{g}}^i \to L_{i-1}$ for $i \in \{1, ..., k\}$, such that

1. $df_1(\xi) = -\iota_{v_\xi}\omega$
2. $0 = df_i(\xi_1 \wedge ... \wedge \xi_i) - (-1)^{\frac{i(i-1)}{2}} \omega(v_{\xi_1}, ..., v_{\xi_i}, \cdots)$

- If a homotopy comoment exists, it exists.
- If $H^1_{dR}(M) = ... = H^k_{dR}(M) = 0$, then it exists.
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Theorem (MR2020)

A weak comoment f for the action exists if and only if $c_i = 0$ for all $i \in \{1, ..., k\}$.

Given a weak comoment, can it be extended ?

Theorem (MR2020)

Let $\widehat{f}$ be a weak moment map and $c_{n+1} = 0$. Then there exists $\gamma \in \bigoplus_{k=1}^{n} \delta(\Lambda^k \mathfrak{g}^*) \otimes H^{n-k}(M)$, such that :

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Thank you for your attention !