Subspaces with or without a common complement

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January 17, 2025

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1. Introduction

The results that will be presented here are taken from two recent unpublished papers:

- E.A. and E. Chiumiento, *Subspaces with or without a common complement*, arXiv 2412.18113, and
- E.A. and E. Chiumiento, *A note on common complements*, arXiv 2412.19316.

These manuscripts are reflections on the paper by M. Lauzon and S. Treil (*Common complements of two subspaces of a Hilbert space*, J. Funct. Anal. **212** (2004), no. 2, 500–512). These authors characterize pairs of (closed) subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$, which have a common complement $\mathcal{Z}$. That is, for which there exists a closed subspace $\mathcal{Z} \subset \mathcal{H}$ such that

$$\mathcal{S} \dot{+} \mathcal{Z} = \mathcal{T} \dot{+} \mathcal{Z} = \mathcal{H},$$

where the symbol $\dot{+}$ stands for direct sum.

Denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators in a complex separable Hilbert space $\mathcal{H}$, $\mathcal{P}(\mathcal{H})$ the subset of orthogonal projections, and $Gr(\mathcal{H})$ the Grassmann manifold of $\mathcal{H}$, $Gl(\mathcal{H})$ the group of invertible operators in $\mathcal{H}$, and $\mathcal{U}(\mathcal{H})$ the unitary group. Throughout, we identify subspaces in $\mathcal{S} \in Gr(\mathcal{H})$ with their corresponding orthogonal projections in $P_{\mathcal{S}} \in \mathcal{P}(\mathcal{H})$.

Our objects of study will be the complementary sets

$$\boldsymbol{\Delta} = \{(P_{\mathcal{S}}, P_{\mathcal{T}}) : \mathcal{S} \text{ and } \mathcal{T} \text{ have a common complement}\},$$

and

$$\boldsymbol{\Gamma} = \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \setminus \boldsymbol{\Delta}.$$

The space $\mathcal{P}(\mathcal{H})$ is a complemented C^∞ submanifold of $\mathcal{B}(\mathcal{H})$ (see [4], [8]). Therefore it is natural to ask about the geometric structure of the sets $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$. By an elementary argument, or using results by J. Giol [6] or D. Buckholtz [3], it can be shown that $\boldsymbol{\Delta}$ is an open subset of $\mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H})$. We shall see that its complement $\boldsymbol{\Gamma}$ is a (non complemented) closed C^∞ submanifold of $\mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H})$. Thus, both spaces have differentiable structure.

2. The space Δ

Let us state the following result from [7] by M. Lauzon and S. Treil.

Denote by $G = P_{\mathcal{T}}\big|_{\mathcal{S}}$, regarded as an operator $G : \mathcal{S} \to \mathcal{T}$ (so that $G^* : \mathcal{T} \to \mathcal{S}$ is $P_{\mathcal{S}}\big|_{\mathcal{T}}$). Denote by $\mathbf{E}$ the projection valued spectral measure of G^*G. Note that

$$N(G) = \mathcal{S} \cap \mathcal{T}^\perp \quad \text{and} \quad N(G^*) = \mathcal{S}^\perp \cap \mathcal{T}.$$

Theorem (Lauzon-Treil 2004)
$\mathcal{S}$ and $\mathcal{T}$ have a common complement if and only if

$$\dim N(G) + \dim \mathbf{E}(0, 1-\epsilon)\mathcal{S} = \dim N(G^*) + \dim \mathbf{E}(0, 1-\epsilon)\mathcal{S}$$

for some $\epsilon > 0$ (equivalently, for all sufficiently small $\epsilon > 0$). This characterization also holds for non separable Hilbert spaces (Thm. 0.1 [7]).

As a straightforward consequence of the previous statement, Lauzon and Treil oberved that

Remark (Lauzon-Treil 2005)

the subspaces $\mathcal{S}$ and $\mathcal{T}$ do not have a common complement in a separable Hilbert space $\mathcal{H}$ if and only if $\dim \mathcal{S} \cap \mathcal{T}^{\perp} \neq \dim \mathcal{S}^{\perp} \cap \mathcal{T}$ and the operator $(1 - G^*G)\big|_{N(G)^{\perp}}$ is compact in $N(G)^{\perp}$ (Rem. 0.5 [7]).

Later on J. Giol proved the following equivalence (see Prop. 6.2. [6]):

Theorem (Giol 2005)

i) $\mathcal{S}$ and $\mathcal{T}$ are subspaces with a common complement.

ii) There exists $P \in \mathcal{P}(\mathcal{H})$ such that $\|P_{\mathcal{S}} - P\| < 1$ and $\|P - P_{\mathcal{T}}\| < 1$.

In particular, note that if $\|P_{\mathcal{S}} - P_{\mathcal{T}}\| < 1$, then $(P_{\mathcal{S}}, P_{\mathcal{T}}) \in \mathbf{\Delta}$.

The Grassmann manifold $Gr(\mathcal{H})$ of $\mathcal{H}$ is defined as the set of all the closed subspaces of $\mathcal{H}$. We identify the Grassmann manifold with the manifold of all orthogonal projections in $\mathcal{H}$ given by

$$\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}) : P = P^2 = P^*\}.$$

the connected components of $\mathcal{P}(\mathcal{H})$ are parametrized by the rank and the co-rank. We denote by $\mathcal{P}_{i,j}$ the connected component of $\mathcal{P}(\mathcal{H})$ consisting of projections with rank i and corank j, where the indices satisfy $0 \leq i, j \leq \infty$ and $i + j = \infty$ (usual convention if both are infinite).
Our main interest is the component $\mathcal{P}_{\infty,\infty}$.

The unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ acts on $\mathcal{P}(\mathcal{H})$: $U \cdot P = UPU^*$. The orbits of this action are the connected components $\mathcal{P}_{i,j}$. For a given $P \in \mathcal{P}_{i,j}$, the map π_P induced by the action,

$$\pi_P : \mathcal{U}(\mathcal{H}) \to \mathcal{P}_{i,j}, \ \pi_p(U) = UPU^*$$

is a fibre bundle. The fibre $\pi_P^{-1}(P)$ over P identifies with the product $\mathcal{U}(R(P)) \times \mathcal{U}(N(P))$. Thus the homotopy type of $\mathcal{P}_{i,j}$ is determined by the homotopy of $\mathcal{U}(\mathcal{H})$, $\mathcal{U}(i)$ and $\mathcal{U}(j)$. In particular, by Kuiper's theorem, $\mathcal{P}_{\infty,\infty}$ is contractible.

If $i = k < \infty$ or $j = l < \infty$, then $\mathcal{P}_{i,j} \times \mathcal{P}_{k,l} \subseteq \mathbf{\Delta}$. Indeed, take $(P, Q) \in \mathcal{P}_{i,j} \times \mathcal{P}_{k,l}$ with $i = k$. Since $T := Q|_{R(P)} : R(P) \to R(Q)$ is an operator defined in finite-dimensional spaces, we have $k = \dim N(T) + \dim R(T) = \dim N(T^*) + \dim R(T^*)$. From $\dim R(T) = \dim N(T)^{\perp} = \dim R(T^*)$, it follows that $\dim R(P) \cap N(Q) = \dim N(T) = \dim N(T^*) = \dim R(Q) \cap N(P)$.
The case where $j = l < \infty$ follows similarly.
On the other hand, assume now that $i \neq k$ or $j \neq l$, and take $(P, Q) \in \mathcal{P}_{i,j} \times \mathcal{P}_{k,l}$ with $i \neq k$. Then $R(P)$ and $R(Q)$ cannot be isomorphic, and therefore $(P, Q) \in \mathbf{\Gamma}$. Similarly for the case where $(P, Q) \in \mathcal{P}_{i,j} \times \mathcal{P}_{k,l}$ with $j \neq l$.
From these facts, we obtain that

$$\mathbf{\Delta}_{ij} := \mathbf{\Delta} \cap (\mathcal{P}_{i,j} \times \mathcal{P}_{i,j}) = \mathcal{P}_{i,j} \times \mathcal{P}_{i,j},$$

whenever $i < \infty$ or $j < \infty$, are the only connected components of $\mathbf{\Delta}$ with finite dimensional rank or corank.

To analize $\boldsymbol{\Delta}_\infty$, we need lo look briefly into the geometry of $\mathcal{P}(\mathcal{H})$, as studied by G. Corach, H. Porta and L. Recht ([8], [4]). Specifically, that if $P, Q \in \mathcal{P}(\mathcal{H})$ satisfy that $\|P - Q\| < 1$, then there exists a unique geodesic $P(t) = e^{itX} P e^{-itX}$ with $X^* = X$, $PXP = P^\perp X P^\perp = 0$ and $\|X\| < \pi/2$, such that ($P(0) = P$ and) $P(1) = Q$.

Lemma

Let P, Q be orthogonal projections such that $\|P - Q\| < 1$, and let $P(t)$ be the unique minimal geodesic of $\mathcal{P}(\mathcal{H})$ such that $P(0) = P$ and $P(1) = Q$. Then for all $t \in [0, 1]$ we have that $\|P - P(t)\| < 1$.

Using this Lemma we get that

Theorem

The subset $\boldsymbol{\Delta}_\infty$ of $\boldsymbol{\Delta}$, consisting of pairs of pojections in $\mathcal{P}_{\infty,\infty}$ with a common complement, is arcwise connected. Therefore $\boldsymbol{\Delta}_\infty$ is the connected component of such pairs.

Proof:

Let $(P_{\mathcal{S}}, P_{\mathcal{T}}) \in \mathbf{\Delta}_{\infty}$. We proceed in steps. First we show that there is a continuous path inside $\mathbf{\Delta}$ connecting $(P_{\mathcal{S}}, P_{\mathcal{T}})$ with a pair $(P_{\mathcal{S}}, E)$ such that $\|P_{\mathcal{S}} - E\| < 1$. Indeed, the theorem by Giol says that there exists $E \in \mathcal{P}(\mathcal{H})$ such that $\|P_{\mathcal{S}} - E\| < 1$ and $\|P_{\mathcal{T}} - E\| < 1$. Let $E(t)$ be the minimal geodesic of $\mathcal{P}(\mathcal{H})$ with $E(0) = E$ and $E(1) = P_{\mathcal{T}}$. Then the curve $(P_{\mathcal{S}}, E(t))$ remains inside $\mathbf{\Delta}$ for $t \in [0, 1]$. This follows again using the result by Giol, for we have the intermediate projection E satisfying $\|P_{\mathcal{S}} - E\| < 1$ and $\|E(t) - E\| < 1$ (by the above Lemma).

Next, we find a continuous path inside $\mathbf{\Delta}$ connecting $(P_{\mathcal{S}}, E)$ with $(P_{\mathcal{S}}, P_{\mathcal{S}})$. Let $P(t)$ be the minimal geodesic joining $P(0) = P_{\mathcal{S}}$ and $P(1) = E$. Then the curve $(P_{\mathcal{S}}, P(t))$ remains inside $\mathbf{\Delta}$ for $t \in [0, 1]$, since, again by the Lemma we know that $\|P_{\mathcal{S}} - P(t)\| < 1$.

The proof finishes by showing that any two pairs $(P_{\mathcal{S}}, P_{\mathcal{S}})$ and $(P_{\mathcal{S}'}, P_{\mathcal{S}'})$ with $\mathcal{S}, \mathcal{S}'$ infinite and co-infinite, can be joined by a continuous path inside $\mathbf{\Delta}$. If $\mathcal{S}$ and $\mathcal{T}$ have a common complement $\mathcal{Z}$ and U is a unitary operator, then $U\mathcal{S}$ and $U\mathcal{T}$ also have a common complement (namely $U\mathcal{Z}$). Since $P_{\mathcal{S}}, P_{\mathcal{S}'} \in \mathcal{P}_{\infty,\infty}$, there exists a continuous path of unitaries $U(t)$ such that $U(0) = 1$ and $U(1)\mathcal{S} = \mathcal{S}'$. Then $(U(t)P_{\mathcal{S}}U^*(t), U(t)P_{\mathcal{S}}U^*(t))$ is a continuous curve in $\mathbf{\Delta}$ wich joins $(P_{\mathcal{S}}, P_{\mathcal{S}})$ and $(P_{\mathcal{S}'}, P_{\mathcal{S}'})$. $\square$

We also have

Theorem

$\mathbf{\Delta}_\infty$ *is dense in* $\mathcal{P}_{\infty,\infty} \times \mathcal{P}_{\infty,\infty}$.

Pairs $(P_{\mathcal{S}}, P_{\mathcal{T}})$ which do not belong to $\mathbf{\Gamma}$ satisfy that certain operator must be compac. Namely:
$(1 - G^*G)\big|_{N(G)^\perp}$ is compact in $N(G)^\perp$, where $G = P_{\mathcal{T}}\big|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}$.

The proof consists then, essentially, in approximating compact operators with non compact ones.

In order to further study the topology of $\mathbf{\Delta}$ we have to introduce more notation and ideas. Given a fixed subspace $\mathcal{Z} \subset \mathcal{H}$, denote by

$$Gr^{\mathcal{Z}} := \{\mathcal{S} \in Gr(\mathcal{H}) : \mathcal{S} \dot{+} \mathcal{Z} = \mathcal{H}\}.$$

D. Buckholtz proved [3] that $\mathcal{S} \in Gr^{\mathcal{Z}}$ iff $\|P_{\mathcal{S}} + P_{\mathcal{Z}} - 1\| < 1$. Therefore it is clear that $Gr^{\mathcal{Z}}$ is open in $Gr(\mathcal{H})$.

Moreover, elements in $Gr^{\mathcal{Z}}$ correspond naturally with graphs of bounded linear operators $\mathcal{Z}^{\perp} \to \mathcal{Z}$ (inducing the usual atlas for the classical Grassmann manifold):

- to a bounded linear operator $B : \mathcal{Z}^{\perp} \to \mathcal{Z}$ corresponds the closed subspace $\mathcal{S} = \mathrm{Graph}_B = \{z' + Bz' : z' \in \mathcal{Z}^{\perp}\}$;
- to a closed subspace $\mathcal{S} \in Gr^{\mathcal{Z}}$ corresponds the operator $B = -P_{\mathcal{Z}\|\mathcal{S}}\big|_{\mathcal{Z}^{\perp}} : \mathcal{Z}^{\perp} \to \mathcal{Z}$.

These correspondences are continuous, and reciprocal (here $P_{\mathcal{Z}\|\mathcal{S}}$ denotes the idempotent wih range $\mathcal{Z}$ and nullspace $\mathcal{S}$ induced by the decomposition $\mathcal{S} \dot{+} \mathcal{Z} = \mathcal{H}$).
In particular, it follows that $Gr^{\mathcal{Z}} \simeq \mathcal{B}(\mathcal{Z}^{\perp}, \mathcal{Z})$ is contractible.

$Gr^{\mathcal{Z}}$ is a homogeneous space of the Banach-Lie group

$$Gl^{\mathcal{Z}}(\mathcal{H}) := \{G \in Gl(\mathcal{H}) : G(\mathcal{Z}) = \mathcal{Z}\}.$$

This group acts on $Gr^{\mathcal{Z}}$: if $\mathcal{S}\dot{+}\mathcal{Z} = \mathcal{H}$, and $G \in Gl^{\mathcal{Z}}(\mathcal{H})$, then also $G(\mathcal{S})\dot{+}\mathcal{Z} = \mathcal{H}$, i.e., $G(\mathcal{S}) \in Gr^{\mathcal{Z}}$. We define the space

$$\mathcal{E} := \bigsqcup_{\mathcal{Z}\in Gr(\mathcal{H})} Gl^{\mathcal{Z}} \times Gl^{\mathcal{Z}}$$

$$= \{(\mathcal{Z}, G, K) \in Gr(\mathcal{H}) \times Gl(\mathcal{H}) \times Gl(\mathcal{H}) : G(\mathcal{Z}) = K(\mathcal{Z}) = \mathcal{Z}\}.$$

The set $\mathcal{E}$ can be endowed with a manifold structure by using the same ideas of the frame bundle construction in classical differential geometry.

Note the fact that $(G(\mathcal{Z}^{\perp}), K(\mathcal{Z}^{\perp})) \in \Delta$, for every $\mathcal{Z} \in Gr(\mathcal{H})$.

This leads us to define the following map

$$\mathfrak{p} : \mathcal{E} \to \Delta, \;\; \mathfrak{p}(Z, G, K) = (G(\mathcal{Z}^\perp), K(\mathcal{Z}^\perp)).$$

Theorem

The map $\mathfrak{p}$ is a real analytic fibre bundle.

The next step is to identify the fibers

$$\mathfrak{p}^{-1}(\mathcal{S},\mathcal{T})$$

of $\mathfrak{p}$:

Proposition

Take $(\mathcal{S},\mathcal{T}) \in \Delta_{ij}$ and two subspaces $\mathcal{H}_+$, $\mathcal{H}_-$ such that $\dim \mathcal{H}_- = i$, $\dim \mathcal{H}_+ = j$ and $\mathcal{H}_+ \oplus \mathcal{H}_- = \mathcal{H}$. Then $\mathfrak{p}^{-1}(\mathcal{S},\mathcal{T})$ is a closed submanifold of $\mathcal{E}$, and there is a diffeomorphism

$$\mathfrak{p}^{-1}(\mathcal{S},\mathcal{T}) \simeq Gr^{\mathcal{S}} \times (Gl^{\mathcal{H}_+} \cap Gl^{\mathcal{H}_-})^2.$$

The group $Gl^{\mathcal{H}_+} \cap Gl^{\mathcal{H}_-}$ consists of invertible operators which are *diagonal* in the decomposition $\mathcal{H}^+ \oplus \mathcal{H}^- = \mathcal{H}$. It follows that the homotopy type of the fiber $\mathfrak{p}^{-1}(\mathcal{S}, \mathcal{T})$ can be described in terms of the dimensions and co-dimensions of $\mathcal{S}$ and $\mathcal{T}$ (i.e., of $Gl(i), Gl(j)$).

In particular, if $P_{\mathcal{S}}, P_{\mathcal{T}} \in \mathcal{P}_{\infty,\infty}$, then $\mathfrak{p}^{-1}(\mathcal{S}, \mathcal{T})$ is contractible.

The space $\mathcal{E}$ is the total space of a more natural bundle, namely

$$\pi : \mathcal{E} \to Gr(\mathcal{H}), \;\; \pi(\mathcal{Z}, G, K) = \mathcal{Z},$$

whose fibers are

$$\pi^{-1}(\mathcal{Z}) \simeq Gl^{\mathcal{Z}} \times Gl^{\mathcal{Z}}.$$

Again, if $P_{\mathcal{Z}} \in \mathcal{P}_{\infty,\infty}$, this fiber is contractible. Let $\mathcal{E}_\infty$ denote the connected component of $\mathcal{E}$, corresponding to the infinite and co infinite component of $Gr(\mathcal{H})$. The image of π restricted to $\mathcal{E}_\infty$ is clearly $\mathcal{P}_{\infty,\infty}$, also a contractible space.

Corollary

$\mathcal{E}_\infty$ is contractible.

Then we have

Theorem

$\mathbf{\Delta}_\infty$ *is contractible.*

Proof.

The bundle $\mathfrak{p} : \mathcal{E}_\infty \to \mathbf{\Delta}_\infty$ has contractible fibers, and contractible total space. Thus $\mathbf{\Delta}_\infty$ has trivial homotopy groups, and is a C^∞ manifold modelled in a Banach space. □

3. The space $\mathbf{\Gamma}$:

We consider now

$$\mathbf{\Gamma} := \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \setminus \mathbf{\Delta}.$$

Recall that

$$\mathbf{\Gamma}_{ijkl} := \mathbf{\Gamma} \cap (\mathcal{P}_{i,j} \times \mathcal{P}_{k,l}) = \mathcal{P}_{i,j} \times \mathcal{P}_{k,l},$$

whenever $i \neq k$ or $j \neq l$, are the only connected components of $\mathbf{\Gamma}$ with finite dimensional rank or corank. Hence we are left to understand the structure of pairs in $\mathcal{P}_{\infty,\infty} \times \mathcal{P}_{\infty,\infty}$ without a common complement.

Recall also the Remark by Lauzon and Treil, which can be rephrased

$$(P_{\mathcal{S}}, P_{\mathcal{T}}) \in \mathbf{\Gamma} \iff \begin{cases} \textbf{a}) \ P_{\mathcal{S}} P_{\mathcal{T}}^{\perp} \text{ is compact in } (\mathcal{S} \cap \mathcal{T}^{\perp})^{\perp} \\ \textbf{b}) \ dim \mathcal{S} \cap \mathcal{T}^{\perp} \neq \dim \mathcal{S}^{\perp} \cap \mathcal{T}. \end{cases}$$

At this point it is useful to recall the five space decomposition of the Hilbert space in the presence of two (fixed) subspaces $\mathcal{S}, \mathcal{T}$. Namely, the subspaces

$$\mathcal{S} \cap \mathcal{T}, \ \mathcal{S}^\perp \cap \mathcal{T}^\perp, \ \mathcal{S} \cap \mathcal{T}^\perp \text{ and } \ \mathcal{S}^\perp \cap \mathcal{T}^\perp$$

reduce the projections $P_\mathcal{S}, P_\mathcal{T}$, and therefore also the orthogonal of the sum of these (usually called the *generic part* of $\mathcal{H}$), reduces these projections. We shall denote $\mathcal{H}_0$ this generic part, and by $P_{\mathcal{S}_0}$ and $P_{\mathcal{T}_0}$ the reductions to $\mathcal{H}_0$.

It can be proved that the condition **a)** above

$$P_{\mathcal{S}} P_{\mathcal{T}}^{\perp} \text{ is compact in } (\mathcal{S} \cap \mathcal{T}^{\perp})^{\perp}, \tag{1}$$

can be replaced by the condition

$$P_{\mathcal{S}_0} - P_{\mathcal{T}_0} \text{ is compact;} \tag{2}$$

or by the condition

$$P_{\mathcal{S}_0} P_{\mathcal{T}_0}^{\perp} \text{ is compact;} \tag{3}$$

or also by

$$\text{either } P_{\mathcal{S}} P_{\mathcal{T}}^{\perp} \text{ or } P_{\mathcal{T}} P_{\mathcal{S}}^{\perp} \text{ is compact.} \tag{4}$$

In a previous paper [1], the first named author and G. Corach studied pairs of projections (P, Q) satisfying that PQ is compact (we called them *essentially orthogonal* projections):

$$\mathcal{C}(\mathcal{H}) = \{(P, Q) \in \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) : PQ \in \mathcal{K}(\mathcal{H})\}.$$

Let us briefly describe some observations made there.
Given a $\mathcal{L}$ is a Hilbert space, we denote by $\mathcal{K}(\mathcal{L}) \subset \mathcal{B}(\mathcal{L})$ the ideal of compact operators in $\mathcal{L}$, and by

$$\pi_{\mathcal{L}} : \mathcal{B}(\mathcal{L}) \to \mathcal{B}(\mathcal{L})/\mathcal{K}(\mathcal{L}) := \mathbf{C}(\mathcal{L})$$

the $*$-epimorphism onto de Calkin algebra $\mathbf{C}(\mathcal{L})$.

The fact that PQ is compact means that $\pi(P)\pi(Q) = 0$, i.e., $\pi(P)$ and $\pi(Q)$ are mutually orthogonal (and non trivial, different from 0 or 1, because $P, Q \in \mathcal{P}_{\infty,\infty}$) in $\mathbf{C}(\mathcal{H})$.
The projections $\pi(P)$ and $\pi(Q)$ can be written as 2×2 matrices in terms of $\pi(P)$ as

$$\pi(P) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \pi(Q) = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

We can distinguish two classes:

$$\mathcal{C}_1 := \{(P, Q) : q = 1 \text{ in } \mathbf{C}(R(P)^\perp)\},$$

and

$$\mathcal{C}_\infty := \{(P, Q) : q \text{ is a proper projection } (\neq 0, 1) \text{ in } \mathbf{C}(R(P)^\perp)\}.$$

In the first class, the fact that $q = 1$ means that the operator

$$Q\big|_{R(P)^\perp} : R(P)^\perp \to R(Q)$$

is a Fredholm operator, and has therefore an index, denoted index$(P^\perp, Q)$. This index for pair of projections was studied by several authors, let us recall the paper [2] by J. Avron, R. Seiler and B. Simon.

We recall some results from [1]:

1. The connected components of $\mathcal{C}$ are
$$\mathcal{C}_\infty \quad \text{and} \quad \mathcal{C}_1^n = \{(P,Q) \in \mathcal{C}_1 : \text{index}(P^\perp, Q) = n\}.$$
2. the set $\mathcal{C}$ is a C^∞ (non complemented) submanifold of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$.
3. $(P,Q) \in \mathcal{C}_1$ if and only if $\dim N(P) \cap N(Q) < \infty$.

Putting these facts together in our context, we have that the set $\mathbf{\Gamma}$ can be parted in two broad (disjoint) classes

$$\mathbf{\Gamma} = \mathbf{\Gamma}_1 \cup \mathbf{\Gamma}_\infty.$$

of pairs in $\mathbf{\Gamma}$ such that, respectively $(P_{\mathcal{S}}, P_{\mathcal{T}}^{\perp}) \in \mathcal{C}_1$, or $(P_{\mathcal{S}}, P_{\mathcal{T}}^{\perp}) \in \mathcal{C}_\infty$.

The mentioned index is computed in this setting by $\dim \mathcal{S} \cap \mathcal{T}^{\perp} - \dim \mathcal{S}^{\perp} \cap \mathcal{T}$. The second of the two conditions (condition **b)** above) for a pair to belong to Γ means that this index must be different from zero.

Concerning the class $\boldsymbol{\Gamma}_1$ we have:

Theorem

$$\boldsymbol{\Gamma}_1 = \{(P_{\mathcal{S}}, P_{\mathcal{T}}) \in \Gamma : \dim \mathcal{S} \cap \mathcal{T}^{\perp} < \infty \text{ and } \dim \mathcal{S}^{\perp} \cap \mathcal{T} < \infty\}.$$

The connected components of $\boldsymbol{\Gamma}_1$ *are*

$$\boldsymbol{\Gamma}_1^n := \{(P_{\mathcal{S}}, P_{\mathcal{T}}) : \mathit{index}(P_{\mathcal{S}}, P_{\mathcal{T}}^{\perp}) = n\}, \text{ for } n \in \mathbb{Z} \setminus \{0\}.$$

The set $\boldsymbol{\Gamma}_1$ *is a non complemented* C^{∞} *submanifold of* $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$.

For the class $\boldsymbol{\Gamma}_\infty$ we have

$$\boldsymbol{\Gamma}_\infty := \{(P_{\mathcal{S}}, P_{\mathcal{T}}) \in \boldsymbol{\Gamma} : \dim \mathcal{S} \cap \mathcal{T}^\perp = +\infty \text{ or } \dim \mathcal{S}^\perp \cap \mathcal{T} = +\infty\}.$$

Clearly only one of the two dimensions can be infinite. Then this set parts into two disjoint subsets

$$\boldsymbol{\Gamma}_\infty = \boldsymbol{\Gamma}_\infty^l \cup \boldsymbol{\Gamma}_\infty^r,$$

where

$$\boldsymbol{\Gamma}_\infty^l := \{(P_{\mathcal{S}}, P_{\mathcal{T}}) : \dim \mathcal{S} \cap \mathcal{T}^\perp < \infty \text{ (and } \dim \mathcal{S}^\perp \cap \mathcal{T} = +\infty)\},$$

and

$$\boldsymbol{\Gamma}_\infty^r := \{(P_{\mathcal{S}}, P_{\mathcal{T}}) : \dim \mathcal{S}^\perp \cap \mathcal{T} < \infty \text{ (and } \dim \mathcal{S} \cap \mathcal{T}^\perp = +\infty)\}$$

And we have

Theorem

Both sets $\mathbf{\Gamma}^l_\infty$, $\mathbf{\Gamma}^r_\infty$ are C^∞ (non complemented) submanifolds of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$, which are diffeomorphic to $\mathcal{C}_\infty$. They are the connected components of $\mathbf{\Gamma}_\infty$.

4. Examples

1. The Hilbert space is $L^2 := L^2(\mathbb{T}, \frac{dt}{2\pi})$, and denote by $H^2 \subset L^2$ the Hardy space, H^∞ the algebra of bounded analytic functions in the disk $\mathbb{D}$, and C the continuous functions in $\mathbb{T}$. The *Sarason algebra* is defined as

$$H^\infty + C = \{f + g : f \in H^\infty, g \in C\}.$$

We write $(H^\infty + C)^\times$ for the invertible functions of the algebra $H^\infty + C$. Denote by hf the harmonic extension of f to $\mathbb{D}$. One has that $f \in (H^\infty + C)^\times$ if and only if there exist $\delta, \epsilon > 0$ such that $|(hf)(re^{it})| \geq \epsilon$ for $1 - \delta < r < 1$. For $f \in (H^\infty + C)^\times$, one can define an index $ind(f)$ as minus the winding number with respect to the origin of the curve $(hf)(re^{it})$ for $1 - \delta < r < 1$. This index is stable under small perturbations and it is an homomorphism of $(H^\infty + C)^\times$ onto the group of integers.

We consider subspaces of the form

$$\mathcal{S} = fH^2, \quad \mathcal{T} = gH^2, \quad f, g \in (H^\infty + C)^\times.$$

We obtain that the subspaces fH^2, gH^2 admit a common complement if and only if $ind(f) = ind(g)$.
If $ind(f) \neq ind(g)$, we have that $(fH^2, gH^2) \in \mathbf{\Gamma}_1^n$, where $n = ind(f) - ind(g)$.

2. Let $I, J \subset \mathbb{R}^n$ be measurable sets with finite and positive Lebesgue measure. Consider $\mathcal{H} = L^2(\mathbb{R}^n)$ with Lebesgue measure and the projections P_I onto the elements of $L^2(\mathbb{R}^n)$ supported in I and Q_J onto the elements whose Fourier-Plancherel transform is supported in J.

The following facts are known:

- $R(P_I) \cap R(Q_J) = R(P_I) \cap N(Q_J) = N(P_I) \cap R(Q_J) = \{0\}$ and $N(P_I) \cap N(Q_J)$ is infinite dimensional.
- $P_I Q_J P_I$ is compact, in fact, nuclear.

A reference for these facts is the survey paper by G.B. Folland and A. Sitaram [5].

Therefore we have the following:

1. $\mathcal{S}_I = \{f \in L^2(\mathbb{R}^n) : \sup(f) \subset I\}$ and $\mathcal{T}_J = \{g \in L^2(\mathbb{R}^n) : \sup(\hat{g}) \subset J\}$ have a common complement (and belong to $\mathbf{\Delta}_\infty$).
2. $\mathcal{S}_I$ and $\mathcal{T}_J^\perp = \mathcal{T}_{J^c}$ do not have a common complement. The role of $\mathcal{T}$ is reversed: now $\mathcal{S}_I \cap (\mathcal{T}_J^\perp)^\perp = R(P_I) \cap R(Q_J) = \{0\}$, but $\mathcal{S}_I^\perp \cap \mathcal{T}_J^\perp$ is infinite dimensional. Moreover

$$1 - P_{\mathcal{S}_I}\big|_{\mathcal{T}_J^\perp} = P_I - P_I Q_J^\perp P_I = P_I Q_J P_I$$

is compact. Hence $(P_{\mathcal{S}_I}, P_{\mathcal{T}_J^\perp}) \in \mathbf{\Gamma}_\infty^I$.

References

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