Last time: $Gr_{k,n}^{\geq 0} := \{V \in Gr_{k,n} : p_I \geq 0 \ \forall I\} = \bigsqcup_{\substack{\mathcal{P} \text{ positroid} \\ \text{type}(k,n)}} C_{\mathcal{P}}$ ← positroid cells

- Can also label $C_{\mathcal{P}}$ by a plabic graph G; bdry measurement gives way to parametrize C_G using G.

- Even tho C_G gives us a geom. space for ea. term in BCFW expression, these C_G are too far apart ⇝ maybe shld "compress" $Gr_{k,n}^{\geq 0}$

§2: Amplituhedron

Defn: Fix k,m,n w/ $k+m \leq n$. Fix $Z \in Mat_{n,k+m}(\mathbb{R})$ w/ pos. max'l minors. The amplituhedron map is

$$\tilde{Z} : Gr_{k,n}^{\geq 0} \longrightarrow Gr_{k,k+m}$$
$$[A] \longmapsto [A \cdot Z]$$

{ map is linear on Plücker coords

The amplituhedron $\mathcal{A}_{n,k,m}^{Z} := \tilde{Z}(Gr_{k,n}^{\geq 0})$.

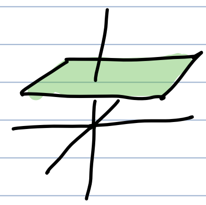
e.g. $k+m=n$, $\mathcal{A}_{n,k,m}^{Z} \simeq Gr_{k,n}^{\geq 0}$

e.g. $k=1$, $m=2$

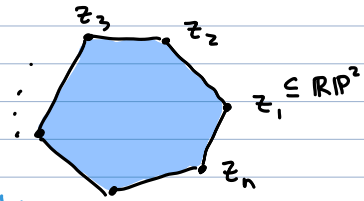
$$Z = \underbrace{\begin{bmatrix} -Z_1- \\ \vdots \\ -Z_n- \end{bmatrix}}_{3}$$

max minors $Z > 0$

$$\tilde{Z} : \overset{\text{simplex}}{Gr_{1,n}^{\geq 0}} \longrightarrow Gr_{1,3} \simeq \mathbb{RP}^2 :$$
$$[x_1 \ldots x_n] \mapsto x_1 Z_1 + \cdots + x_n Z_n$$
$$x_i \geq 0$$



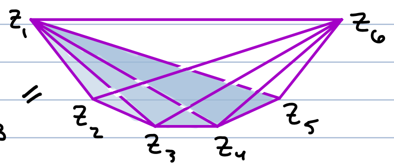
$\mathcal{A}_{n,1,2}^{Z} =$



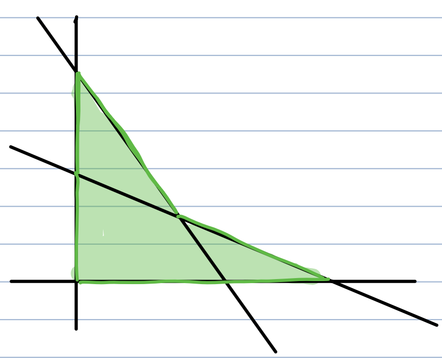
$k=1$, $m \leq n-1$ arbitrary: $\tilde{Z} : \overset{\text{simplex}}{Gr_{1,n}^{\geq 0}} \to Gr_{1,m+1}(\mathbb{R}) \simeq \mathbb{RP}^m$

$\mathcal{A}_{n,1,m}^{Z}$ = cyclic polytope w/ vt $Z_1, Z_2, \ldots, Z_n$ [Sturmfels '88]

$\mathcal{A}_{6,1,3}^{Z} =$



- $m=1$ [Karp-Williams] $\mathcal{A}_{n,k,1}^{Z} \subseteq Gr_{k,k+1} \simeq \mathbb{RP}^k$ is the bdd clx of cyclic hyperplane arrange. (read normals off of Z)



$\mathcal{A}_{4,2,1}$

Remaining $\mathcal{A}_{n,k,m}^{Z}$ are curvy!

$\dim \mathcal{A}_{n,k,m}^{Z} = km = \dim Gr_{k,k+m}$.

$\overline{\tilde{Z}(C_G)} =: Z_G$ positroid cell images

What we want from $\mathcal{A}^Z_{n,k,4}$: should be covered by some pieces reflecting BCFW formula
(motivated by physics)

pieces are full-dim, no overlap

"volumes" of Z_G shld be terms in BCFW formula.

Mathematical formalization:

<u>Defn:</u> $Z_G := \tilde{Z}(\overline{C_G})$ is a <u>tile</u> for $\mathcal{A}^Z_{n,k,m}$ if $\tilde{Z}|_{S_G}$ injective & $\dim Z_G = km$.
$Z^\circ_G := \tilde{Z}(C_G)$ is <u>open tile</u>.

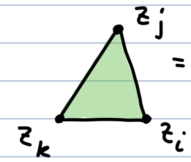
A <u>tiling</u> of $\mathcal{A}^Z_{n,k,m}$ is a decomp. $\mathcal{A}^Z_{n,k,m} = \bigcup_{G \in \mathcal{C}} Z_G$ s.t. Z_G are tiles
& $Z^\circ_G \cap Z^\circ_{G'} = \emptyset$ for $G, G' \in \mathcal{C}$.

e.g. $k+m=n$, $\mathcal{A}^Z_{n,k,m} \simeq Gr^{\geq 0}_{k,n}$, only tile is $\mathcal{A}^Z_{n,k,m}$, have 1 tiling.

2-dim'l cell

e.g. $k=1, m=2$: $C_G = \{[0 \ldots : x_i : \;\; \ldots : x_j : \ldots : x_k : 0 \ldots]\}$

$\tilde{Z}|_{C_G}: [\ldots x_i \ldots x_j \ldots x_k] \longmapsto x_i Z_i + x_j Z_j + x_k Z_k$ is injective!



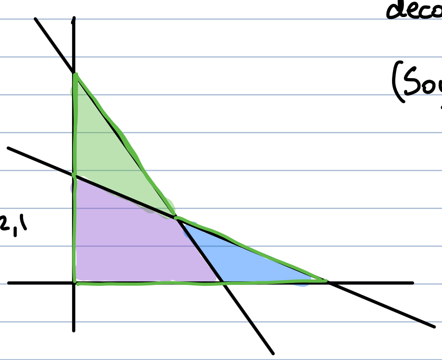
$= \tilde{Z}(\overline{C_G})$

- Tiles = triangles, tilings = triangulations of n-gon.

- Same for $k=1$ more generally. $\mathcal{A}^Z_{n,1,m}$ = cyclic polytope, tiles = simplices, tilings = triang.
[Oppermann–Thomas]
- <u>$m=1$</u> [Karp–Williams]: ea. bdd region of hplane arrangement is a tile. ⇝ natural decomp of bdd clx is tiling.

(Some unions of tiles are also tiles, so also have coarser tilings. This is weird odd-m behavior)

$\mathcal{A}_{4,2,1}$



§3: Big questions:

<u>Q1:</u> How nice is $\mathcal{A}^Z_{n,k,m}$?

<u>Conj:</u> homeo to closed ball

<u>Known:</u> $k=1$; $m=1$ [Karp–Williams]; $m=4$ [Even-Zohar–Lakrec–Tessler]; $n=k+m+1$ [Blagojević–Galashin–Palić–Ziegler]; for cyclically symmetric Z [Galashin–Karp–Lam]

<u>Conj:</u> [Karp–Williams–Zhang] $\mathcal{A}^Z_{n,k,m}$ has cell decomp $\bigsqcup Z^\circ_G$ w/ $M(k, n-k-m, \lceil \frac{m}{2} \rceil)$ cells of top dimension.

MacMahon's #; counts plane partitions in certain box.

<u>Known:</u> $m=1$ [KW]

<u>Conj:</u> [Lam] $\mathcal{A}^Z_{n,k,2}$ w/ its semi-algebraic stratification is a regular CW-clx & is a positive geometry.

[Arkani-Hamed–Bai–Lam, Brown–Dupont]

<u>Thm.</u> [Ranestad–Sinn–Telen] $\mathcal{A}^Z_{n,2,2}$ is a positive geometry.

Conj: $\mathcal{A}^{Z}_{n,k,4}$ is a positive geometry & canonical form is scattering amplitude.

recursive structure. Comes w/ ! "canonical" diff. form Ω w/ logarithmic singularities on ea. facet, & residue $\mathrm{Res}_{\text{facet}}\,\Omega$ = canonical form of facet.

- $\mathcal{A}^{Z}_{n,k,m}$ & Z_G are semialgebraic sets in $\mathbb{RP}^{\binom{k+m}{k}-1}$. Nice descriptions?

Q2: What about these tilings?

The physics connection:

[Even-Zohar-Lakrec-Tessler] For arbitrary n, k, & ~~a particular~~ any BCFW expression
+ Parisi, S, Williams

$(\text{n-pt helicity } k)$ amplitude $= G_1 + \cdots + G_r$, $\bigcup_{i=1}^{r} Z_{G_i}$ is a tiling of $\mathcal{A}^{Z}_{n,k,4}$.

& the rat'l fcn F_{G_i} has poles on the facets of Z_{G_i}.

More mathematically:

- Characterize tilings? Done in $k=1$ [Rambau], $m=2$ [Parisi-S-Williams], probably do-able in $m=1$ using [Karp-Williams]. For $m=4$, this could give new ways to compute amplitude (conjecturally)

Conj: [Galashin-Lam] For m even, every tiling of $\mathcal{A}^{Z}_{n,k,m}$ has $M(k, n-k-m, \frac{m}{2})$ tiles.

Known: $k=1$ [Rambau], $m=2$ [Parisi-S-Tessler-Williams].

- $m \geq 6$, k relatively large: may not have tilings, but may have "r-fold tilings" consisting of $r\,M(k, n-k-m, \frac{m}{2})$ full dim'l Z_G's.

Q3: Many gen'lizations — relax conditions on Z [Mandelstam-Pavlov-Pratt] $m=1$; consider images of arbitrary C_G; clx version (see Lam's CDM notes); tropical version [Akhmedova-Tessler]; loop amplituhedra; momentum amplituhedra [Damgaard-Ferro-Lukowski-Parisi], BCFW tiling for $m=4$ by [Galashin]