$\mathscr{B}$-free systems and the existence of natural density

Aurelia Dymek / Stanisław Kasjan / Joanna Kułaga-Przymus (Nicolaus Copernicus University in Toruń)

Vienna 25.04.2024

Number theory 1930's

For $\mathscr{B} \subseteq \mathbb{N}$: $\mathcal{M}_\mathscr{B} := \bigcup_{b\in\mathscr{B}} b\mathbb{Z}$ (**set of multiples**), $\mathcal{F}_\mathscr{B} := \mathbb{Z} \setminus \mathcal{M}_\mathscr{B}$ ($\mathscr{B}$-**free set**).

Remark:

- If $M \subseteq \mathbb{Z}$ is closed under taking multiples then $M = \mathcal{M}_M$.
- We say that $\mathscr{B}$ is **primitive** if $b|b'$ for $b, b' \in \mathscr{B} \implies b = b'$.
- $\mathscr{B}^{prim} := \mathscr{B} \setminus \bigcup_{k\geq 2} k\mathscr{B}$. Then $\mathcal{M}_\mathscr{B} = \mathcal{M}_{\mathscr{B}^{prim}}$ and $\mathscr{B}^{prim}$ is primitive.

If $1 \in \mathscr{B}$ then $\mathscr{B}^{prim} = \{1\}$ and $\mathcal{F}_\mathscr{B} = \emptyset$.

For $\mathscr{B} = \mathbb{P}^2$ = squares of primes, $\mathcal{F}_\mathscr{B}$ = **square-free numbers** and $\mathbf{1}_{\mathcal{F}_\mathscr{B}} = \mu^2$.
($\mu\colon \mathbb{Z} \to \{-1, 0, 1\}$ is the symmetric extension of the Möbius function: $\mu(0) = 0$, $\mu(p_1 \cdot \ldots \cdot p_k) = (-1)^k$, $\mu(p^2 n) = 0$.)

Abundant numbers: $n \in \mathbf{A}$ if sum its proper divisors exceeds n. We have $\mathbf{A} = \mathcal{M}_\mathbf{A}$ and $\mathcal{F}_\mathbf{A} = \mathbf{P} \cup \mathbf{D}$ (perfect + deficient).

Number theory 1930's: density questions

Davenport / Erdös / Chowla (independently) 1930's: $d(\mathbf{A}) = \lim_{n\to\infty} \frac{1}{n}|\mathbf{A} \cap [1,n]|$ exists (question by Bessel-Hagen 1929).

Does $d(\mathcal{M}_{\mathscr{B}})$ always exist?

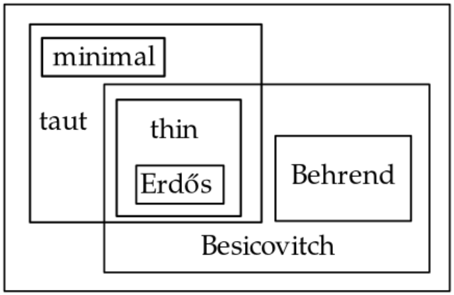
- **Besicovitch 1934**: For $\mathscr{G} := (\bigcup_{k\geq 1}[T_k, 2T_k))^{prim}$, where T_k is a rapidly increasing sequence, we have $\overline{d}(\mathcal{M}_{\mathscr{G}}) \geq \frac{1}{2}$ and $\underline{d}(\mathcal{M}_{\mathscr{G}}) < \varepsilon$.
- **Davenport-Erdős 1936**: $\delta(\mathcal{M}_{\mathscr{B}}) = \lim_{n\to\infty} \frac{1}{\log n} \sum_{k\leq n, k\in\mathcal{M}_{\mathscr{B}}} \frac{1}{k} = \underline{d}(\mathcal{M}_{\mathscr{B}}) = \lim_{K\to\infty} d(\mathcal{M}_{\mathscr{B}\cap[1,K]})$.
- **Erdős 1948**: $d(\mathcal{M}_{\mathscr{B}})$ exists iff

$$\lim_{\varepsilon\to 0} \limsup_{x\to\infty} x^{-1} \sum_{x^{1-\varepsilon}<a\leq x, a\in\mathscr{B}} |[1,x] \cap a\mathbb{Z} \cap \mathcal{F}_{\mathscr{B}\cap[1,a)}| = 0.$$

Set $\mathscr{B}$ is...

- **Besicovitch**: if $d(\mathcal{M}_{\mathscr{B}})$ exists,
- **thin**: if $\sum_{b \in \mathscr{B}} 1/b < \infty$,
- **Erdős**: if $|\mathscr{B}| = \infty$, pairwise coprime and thin, e.g. $\mathscr{B} = \mathbb{P}^2$,
- **Behrend**: if $d(\mathcal{M}_{\mathscr{B}}) = 1$ (e.g. $\mathscr{B} \subseteq \mathbb{P}$ satisfying $\sum_{p \in \mathscr{B}} \frac{1}{p} = \infty$),
- **taut**: if $\delta(\mathcal{M}_{\mathscr{B} \setminus \{b\}}) < \delta(\mathcal{M}_{\mathscr{B}})$ for $b \in \mathscr{B} \iff c\mathcal{A} \not\subseteq \mathscr{B}$ for $\mathcal{A}$ Behrend.,
- **minimal**: if for any $n \in \mathbb{Z}$, there exists $s \in \mathbb{N}$ with $n + s\mathbb{Z} \subseteq \mathcal{M}_{\mathscr{B}}$ or $n + s\mathbb{Z} \subseteq \mathcal{F}_{\mathscr{B}}$.

For primitive sets $\mathscr{B}$, we have the following relations:



$\mathscr{B}$-free subshifts and its friends

Given $\mathscr{B} \subseteq \mathbb{N}$, let $\eta := \mathbf{1}_{\mathcal{F}_{\mathscr{B}}} \in \{0,1\}^{\mathbb{Z}}$ and $X_\eta := \overline{\{\sigma^n \eta : n \in \mathbb{Z}\}}$ (the $\mathscr{B}$-**free subshift**).

If $\mathscr{B}$ is Erdős, then X_η is **hereditary**, i.e. if $y \in \{0,1\}^{\mathbb{Z}}$, $x \in X_\eta$ and $y \leq x$ then $y \in X_\eta$.

- $\widetilde{X_\eta} =$ the hereditary closure of X_η (the smallest hereditary subshift containing X_η).
- $X_{\mathscr{B}} = \mathscr{B}$-**admissible subshift** $= \{x \in \{0,1\}^{\mathbb{Z}} : |\mathrm{supp}\, x \bmod b| < b \text{ for } b \in \mathscr{B}\}$

Notice that $X_\eta \subseteq X_{\mathscr{B}}$ as $\eta \in X_{\mathscr{B}}$. Since $X_{\mathscr{B}}$ is hereditary, we get $X_\eta \subseteq \widetilde{X_\eta} \subseteq X_{\mathscr{B}}$.

If $\mathscr{B}$ is Erdős then $X_\eta = \widetilde{X}_\eta = X_{\mathscr{B}}$ (el Abdalaoui, Lemańczyk, de la Rue 2015).

Dynamical properties of $X_\eta \subseteq \widetilde{X}_\eta \subseteq X_{\mathscr{B}} \leftrightarrow$ number theoretic properties of $\mathcal{M}_{\mathscr{B}}$ and $\mathcal{F}_{\mathscr{B}}$.

E.g. (Kasjan, Lemańczyk, Alterman 2023):

- $\mathscr{B}$ is Erdős $\iff X_\eta = X_{\mathscr{B}}$ and $h_{top}(X_\eta) > 0$,
- $\mathscr{B}$ is Behrend $(d(\mathcal{M}_{\mathscr{B}}) = 1) \iff X_\eta$ is proximal and $h_{top}(X_\eta) = 0$.

Mirsky measure

If $\mathscr{B}$ is Erdős then $\eta = \mathbf{1}_{\mathcal{F}_{\mathscr{B}}}$ is a generic point: $\frac{1}{N}\sum_{n\le N}\delta_{\sigma^n\eta} \to \nu_\eta \rightsquigarrow$ **Mirsky measure** ($\eta \in \{0,1\}^{\mathbb{Z}}$, so we look at frequencies of blocks). It is ergodic and has full support (el Abdalaoui, Lemańczyk, de la Rue 2015).

In general, η might only be quasi-generic: if (N_i) is such that $\frac{1}{N_i}|\mathcal{M}_{\mathscr{B}} \cap [1, N_i]| \to \underline{d}(\mathcal{M}_{\mathscr{B}})$ then $\frac{1}{N_i}\sum_{n\le N_i}\delta_{\sigma^n\eta} \to \nu_\eta$.

Algebraic definition of ν_η:

- $G := \prod_{b\in\mathscr{B}} \mathbb{Z}/b\mathbb{Z} \supseteq H := \overline{\{(n,n,\dots) : n \in \mathbb{Z}\}} = \overline{\{R^n(0,0,\dots) : n\in\mathbb{Z}\}}$, where $Rg = g + (1,1,\dots)$ for $g \in G$
- $W := \{h \in H : h_b \not\equiv 0 \bmod b \text{ for each } b \in \mathscr{B}\}$
- $\varphi\colon H \to \{0,1\}^{\mathbb{Z}}$ given by $\varphi(h)(n) = 1 \iff R^n h \in W$
- we have $\varphi \circ R = \sigma \circ \varphi$
- $\nu_\eta = \varphi_*(m_H)$

Tautification $\mathscr{B}'$ of $\mathscr{B}$

$\mathscr{B}$ is taut if $\delta(\mathcal{M}_{\mathscr{B}\setminus\{b\}}) < \delta(\mathcal{M}_{\mathscr{B}})$.

Theorem (Dymek, Kasjan, KP, Lemańczyk 2018 and 2023): For any $\mathscr{B}$, there exists a (unique!) taut set $\mathscr{B}'$ such that $\mathcal{M}_{\mathscr{B}'} \supset \mathcal{M}_{\mathscr{B}}$ and $\delta(\mathcal{M}_{\mathscr{B}'}) = \delta(\mathcal{M}_{\mathscr{B}})$.

Rmk: The conditions on $\mathcal{M}_{\mathscr{B}'}$ can be replaced by $\nu_\eta = \nu_{\eta'}$.

To be more precise:

$\mathscr{B}' = (\mathscr{B} \cup C)^{prim}$, where $C = \{c \in \mathbb{N} : c\mathcal{A} \subseteq \mathscr{B}$ for some Behrend set $\mathcal{A}\}$.

Moreover, $\mathscr{B}$ is taut $\iff$ $\mathscr{B}$ is primitive and $C = \emptyset$.

Minimisation $\mathscr{B}^*$ of $\mathscr{B}$

Tautification: $\mathscr{B}' = (\mathscr{B} \cup C)^{prim}$, $C = \{c \in \mathbb{N} : c\mathcal{A} \subseteq \mathscr{B}$ for some Behrend set $\mathcal{A}\}$.

$\mathscr{B}$ is taut $\iff$ $\mathscr{B}$ is primitive and $C = \emptyset$.

Minimisation: $\mathscr{B}$ is minimal if η is a Toeplitz sequence.

$\mathscr{B}^* := (\mathscr{B} \cup D)^{prim}$, $D = \{d \in \mathbb{N} : d\mathcal{A} \subseteq \mathscr{B}$ for some infinite pairwise coprime set $\mathcal{A}\}$.

We have $C \subseteq D$, as any Behrend set contains an infinite pairwise coprime subset.

Moreover (Dymek, Kasjan, Keller, KP, Lemańczyk 2018-2023), if $\mathscr{B}$ is primitive then

$\mathscr{B}$ is minimal $\iff$ $D = \emptyset$ $\iff$ X_η is minimal.

Theorem (Dymek, Kasjan, Keller, KP, Lemańczyk 2018-2023): Each X_η is essentially minimal. Moreover, $X_{\eta^*} \subseteq X_\eta$ is the unique minimal subset of X_η.

Tautification $\mathscr{B}'$ and minimisation $\mathscr{B}^*$ of $\mathscr{B}$

We have the following (joint) strenghtening of $X_{\eta^*} \subseteq X_\eta$ and $\mathcal{M}_{\mathscr{B}^*} \supset \mathcal{M}_{\mathscr{B}'} \supset \mathcal{M}_{\mathscr{B}}$:

Theorem (Dymek, Kasjan, KP, M.D. Lemańczyk 2021-2023): $X_{\eta^*} \subseteq X_{\eta'} \subseteq X_\eta$.

Why "minimal"?

Theorem (Dymek, Kasjan, KP, Sell 2024) Suppose that $\mathscr{C}$ is taut. Then $X_{\eta_{\mathscr{C}}} \subset X_\eta$ $\iff$ $\mathcal{M}_{\mathscr{B}^*} \supset \mathcal{M}_{\mathscr{C}} \supset \mathcal{M}_{\mathscr{B}}$. (In fact, this implies $X_{\eta^*} \subset X_{\eta_{\mathscr{C}}} \subset X_\eta$.)

Partial order on taut sets: $\mathscr{C}_1 \prec \mathscr{C}_2 \iff X_{\eta_{\mathscr{C}_1}} \subset X_{\eta_{\mathscr{C}_2}}$.

- $\{\mathscr{B} - \text{minimal}\}$ = minimal elements of $\prec$
- $\mathscr{B}^*$ = the smallest element of

$$Taut(\mathscr{B}) = \{\mathscr{C} \subseteq \mathbb{N} : \mathscr{C} \text{ is taut such that } X_{\eta_{\mathscr{C}}} \subset X_\eta\}.$$

- $\mathscr{B}'$ = the largest element of $Taut(\mathscr{B})$.

Motivation

1. Prop (Dymek, Kasjan, KP, Lemańczyk 2015): $d(\mathcal{M}_{\mathscr{B}'})$ exists $\implies$ $d(\mathcal{M}_{\mathscr{B}})$ exists.

This result was wrongly quoted ($\Longleftarrow$ instead of $\implies$) in [Bergelson, KP, Lemańczyk, Richter 2019]...

Question 1: Can all triples $ijk \in \{0,1\}^3$ with $ij \neq 01$ encoding the information whether $d(\mathcal{M}_{\mathscr{B}}), d(\mathcal{M}_{\mathscr{B}'}), d(\mathcal{M}_{\mathscr{B}^*})$ exist occur? **Yes.**

2. Conjecture (Keller 2021) on $\mathcal{M}(X_\eta, \sigma)$.

- Proved (Dymek, KP, Sell 2024).
- Original plan: prove Keller's conjecture first for taut sets $\mathscr{B}$ and then pass from $\mathscr{B}$ to $\mathscr{B}'$ by proving that $d^*(\mathcal{M}_{\mathscr{B}'} \setminus \mathcal{M}_{\mathscr{B}}) = 0$.

Question 2: Is $d^*(\mathcal{M}_{\mathscr{B}'} \setminus \mathcal{M}_{\mathscr{B}}) = 0$ always true? **No.**

Main results (Dymek, Kasjan, KP 2024)

Thm A: For any taut set $\mathscr{C}$ there exists a Besicovitch set $\mathscr{B}$ such that $\mathscr{B}' = \mathscr{C}$.

Thm B: For any minimal set $\mathscr{D}$ there exists a Besicovitch taut set $\mathscr{B}$ such that $\mathscr{B}^* = \mathscr{D}$.

Prop C: For any taut non-Besicovitch set $\mathscr{C}$ there exists a set $\mathscr{B}$ such that $\mathscr{B}' = \mathscr{C}$ and $\overline{d}(\mathcal{M}_{\mathscr{B}'} \setminus \mathcal{M}_{\mathscr{B}}) > 0$.

Rmk: Prop C is a consequence of Thm A, but also has a separate simpler proof.

Prop D: There exists a set $\mathscr{B}$ such that both $\mathscr{B}, \mathscr{B}'$ are Besicovitch and $d^*(\mathcal{M}_{\mathscr{B}'} \setminus \mathcal{M}_{\mathscr{B}}) > 0$.

How to get there: unions of rescaled patterns

Recall that for $\mathscr{B}$, we have $\mathscr{B}' = (\mathscr{B} \cup C)^{prim}$ and $\mathscr{B}^* = (\mathscr{B} \cup D)^{prim}$, where both C and D are of the form $E = \{e : e\mathscr{A} \subseteq \mathscr{B}$ for some set $\mathscr{A}$ satisfying an extra condition$\}$.

We consider sets of the form $\mathscr{B} = \bigcup_{i \geq 1} \mathscr{E}_i \mathscr{A}_i$ ($\mathscr{E}_i$ = scales, $\mathscr{A}_i$ = patterns), where

- $\mathscr{A}_i \subseteq \mathcal{P}_i \cap [K_i, \infty)$, where $\mathcal{P}_i$ = primes in $2^{i+1}\mathbb{Z} + 2^i + 1$,
- K_i is large,
- $\mathscr{E} = \bigcup_{i \geq 1} \mathscr{E}_i$ is taut or minimal.

Additional properties of scales and patterns $\implies$ properties of $\mathscr{B}, \mathscr{B}', \mathscr{B}^*$.

<u>Prop</u> [controlling tautification and minimisation]:

- If $\mathscr{E}$ is taut and $\mathscr{A}_i$ are Behrend sets (e.g., $\mathscr{A}_i$ is the set of all primes in $[K_i, \infty)$) then $\mathscr{E} = \mathscr{B}'$.
- If $\mathscr{E}$ is minimal and $\mathscr{A}_i$ are infinite pairwise coprime sets then $\mathscr{E} = \mathscr{B}^*$.

2. Upper (Banach) density of $\mathcal{M}_{\mathscr{B}'} \setminus \mathcal{M}_{\mathscr{B}}$.

Prop (refinement of Prop C, uses Davenport-Erdős theorem):

- If $\mathscr{E}$ is non-Besicovitch then, for K_i sufficiently large, if $\mathscr{B} = \bigcup_{i\geq 1} \mathscr{E}_i \mathscr{A}_i$, where $\mathscr{E}_i = \mathscr{E} \cap [1, K_i)$ and $\emptyset \neq \mathscr{A}_i \subset [K_i, \infty)$ then $\overline{d}(\mathcal{M}_{\mathscr{E}} \setminus \mathcal{M}_{\mathscr{B}}) > 0$.

If we additionally assume that $\mathscr{E}$ is taut and $\mathscr{A}_i$ are Behrend sets then $\mathscr{B}' = \mathscr{E}$.

Rk: If both $\mathscr{B}$ and $\mathscr{B}'$ are Besicovtich then $\overline{d}(\mathcal{M}_{\mathscr{B}'} \setminus \mathcal{M}_{\mathscr{B}}) = d(\mathcal{M}_{\mathscr{B}'}) - d(\mathcal{M}_{\mathscr{B}}) = 0$.

Thm (refinement of Prop D, uses ideas of Besicovitch):

- There exists a thin (hence Besicovitch) primitive set $\mathscr{E} = \bigcup_{i\geq 1} \mathscr{E}_i$ such that for K_i large enough and $\mathscr{B} = \bigcup_{i\geq 1} \mathscr{E}_i \mathscr{A}_i$ with $\emptyset \neq \mathscr{A}_i \subseteq [K_i, \infty)$ we have $d^*(\mathcal{M}_{\mathscr{E}} \setminus \mathcal{M}_{\mathscr{B}}) > 0$.

If we additionally assume that $\mathscr{A}_i$ are Behrend sets then $\mathscr{B}' = \mathscr{E}$.

1. Realizable triples $ijk \in \{0,1\}^3$, $ij \neq 01$

- $ijk = 111$: If $\mathscr{B}$ is Erdős then $\mathscr{B} = \mathscr{B}'$ and $\mathscr{B}^* = \{1\}$ are Besicovitch.
- $ijk = 001$: If $\mathscr{G}$ is in the class of examples by Besicovitch, i.e. $\mathscr{G} = (\bigcup_{k\geq 1}[T_k, 2T_k))^{prim}$, then (by Bertrand's postulate) $\mathscr{G}$ contains infinitely many primes, so $1 \in D$ and thus $\mathscr{B}^* = (\mathscr{B} \cup D)^{prim} = \{1\}$. It suffices to take $\mathscr{B} = \mathscr{G}$ or $\mathscr{B} = \mathscr{G}'$.
- $ijk = 000$: Keller 2022 gave examples of non-Besicovitch minimal sets $\mathscr{B}$. (here $\mathscr{B} = \mathscr{B}' = \mathscr{B}^*$).

1. Realizable triples $ijk \in \{0,1\}^3$, $ij \neq 01$

- $ijk = 100$: Let $\mathscr{E}$ be a non-Besicovitch minimal set. By Thm A, there exists a Besicovitch set $\mathscr{B}$ such that $\mathscr{B}' = \mathscr{E}$. Thus, $\mathscr{B}$ is Besicovitch with $\mathscr{B}' = \mathscr{B}^*$ non-Besicovitch. (Indeed, $\mathscr{B}^* = (\mathscr{B}')^* = \mathscr{E}^* = \mathscr{E}$).
- $ijk = 110$: Let $\mathscr{E}$ be a non-Besicovitch minimal set. By Thm B, there exists a Besicovitch taut set $\mathscr{B}$ such that $\mathscr{B}^* = \mathscr{E}$. Thus, $\mathscr{B} = \mathscr{B}'$ is Besicovitch with $\mathscr{B}^*$ non-Besicovitch.
- $ijk = 101$: Start with Besicovitch's example $\mathscr{E}$. By Thm A, there exists a Besicovitch set $\mathscr{B}$ such that $\mathscr{B}' = \mathscr{E}'$. Then $\mathscr{B}^* = (\mathscr{B}')^* = (\mathscr{E}')^* = \mathscr{E}^* = \{1\}$.

1. Realizable triples $ijk \in \{0,1\}^3$, $ij \neq 01$ (more details)

Thm: For any $\mathscr{E} = \{e_i\}$, there exist K_i such that if $\emptyset \neq \mathscr{A}_i \subseteq \mathcal{P}_i \cap [K_i, \infty)$, where $\mathcal{P}_i$ stands for the set of primes in $2^{i+1}\mathbb{Z} + 2^i + 1$, then $\mathscr{B} = \bigcup_{i \geq 1} e_i \mathscr{A}_i$ is Besicovitch.

If additionally $\mathscr{E}$ is primitive then K_i can be chosen so that $\mathscr{B}$ is primitive.

Rk: The proof uses the following version of Mertens' theorem:

$$\sum_{\mathbb{P} \ni p \leq x, p \equiv l \bmod k} \frac{1}{p} = \frac{1}{\varphi(k)} \ln \ln x + B_{k,l} + \mathrm{O}\left(\frac{1}{\ln x}\right),$$

where φ is the Euler totient function.

Cor (refinement of Thm A): If additionally $\mathscr{A}_i$ are Behrend sets then $\mathscr{B}' = \mathscr{E}$.

Cor (refinement of Thm B): If additionally each $\mathscr{A}_i$ is Erdős then $\mathscr{B}^* = \mathscr{E}$. Moreover, if additionally $\bigcup_{i \geq 1} \mathscr{A}_i$ is thin then $\mathscr{B}$ is taut (such assumptions on $\mathscr{A}_i's$ are satisfied, e.g., by sufficiently scarce subsets of $\mathcal{P}_i \cap [K_i, \infty)$).

Proof of Theorem. . .

Goal: For any $\mathscr{E} = \{e_i\}$, there exist K_i such that if $\emptyset \neq \mathscr{A}_i \subseteq \mathcal{P}_i \cap [K_i, \infty)$, where $\mathcal{P}_i$ stands for the set of primes in $2^{i+1}\mathbb{Z} + 2^i + 1$, then $\mathscr{B} = \bigcup_{i\geq 1} e_i\mathscr{A}_i$ is Besicovitch.

Recall: $d(\mathcal{M}_{\mathscr{B}})$ exists iff
$\lim_{\varepsilon\to 0} \limsup_{x\to\infty} x^{-1} \sum_{x^{1-\varepsilon}<a\leq x, a\in\mathscr{B}} |[1,x] \cap a\mathbb{Z} \cap \mathcal{F}_{\mathscr{B}\cap[1,a)}| = 0.$

Fix $\mathscr{E} = \{e_i\}$, let $\emptyset \neq \mathscr{A}_i \subseteq \mathcal{P}_i \cap [K_i, \infty)$. Let $\mathcal{M}(x, a, \mathscr{B}) = [1,x] \cap a\mathbb{Z} \cap \mathcal{F}_{\mathscr{B}\cap[1,a)}$.
Then

$$\sum_{x^{1-\varepsilon}<a\leq x, a\in\mathscr{B}} |[1,x] \cap a\mathbb{Z} \cap \mathcal{F}_{\mathscr{B}\cap[1,a)}| \leq \sum_{i=1}^{\infty} \sum_{x^{1-\varepsilon}<e_i a\leq x, a\in\mathscr{A}_i} |[1,x] \cap e_i a\mathbb{Z}|$$
$$= \sum_{i=1}^{i_0} \sum_{x^{1-\varepsilon}<e_i a\leq x, a\in\mathscr{A}_i} |[1,x] \cap e_i a\mathbb{Z}| + \sum_{i=i_0+1}^{\infty} \sum_{x^{1-\varepsilon}<e_i a\leq x, a\in\mathscr{A}_i} |[1,x] \cap e_i a\mathbb{Z}|.$$

Let $\frac{1}{2^{i_0+1}} < \varepsilon \leq \frac{1}{2^{i_0}}$.

Proof of Theorem...

Estimating the first sum $\sum_{i=1}^{i_0}$

We have

$$x^{-1} \sum_{x^{1-\varepsilon}<e_i a \leq x, a \in \mathscr{A}_i} |[1,x] \cap e_i a\mathbb{Z}| \leq \sum_{e_i a \in e_i \mathscr{A}_i \cap [x^{1-\varepsilon},x]} \frac{1}{e_i a} \leq \sum_{e_i a \in e_i \mathscr{A}_i \cap [x^{1-\varepsilon},x]} \frac{1}{a} \leq g_{i,\varepsilon}(x),$$

where $g_{i,\varepsilon}(x) := \sum_{p \in \mathcal{P} \cap (2^{i+1}\mathbb{Z}+2^i+1) \cap [\frac{x^{1-\varepsilon}}{e_i}, \frac{x}{e_i}]} \frac{1}{p}$ for every $\varepsilon > 0$ and $x \geq 1$ (we use $\mathscr{A}_i \subseteq \mathcal{P} \cap (2^{i+1}\mathbb{Z} + 2^i + 1)$).

By Merten's theorem for APs (...), $g_{i,\varepsilon}(x) \leq \frac{1}{2^i}\left(\ln\ln\frac{x}{e_i} - \ln\ln\frac{x^{1-\varepsilon}}{e_i}\right) + \mathrm{O}\left(\frac{1}{\ln x}\right)$ (the constant in $\mathrm{O}(\frac{1}{\ln x})$ depends on i and ε, but not on the choice of K_i).

This implies (...) $g_{i,\varepsilon}(x) \leq \frac{1}{2^{i-1}}\varepsilon(1+\mathrm{o}(1))$.

Thus, $\limsup_{x\to\infty} \sum_{i=1}^{i_0} g_{i,\varepsilon}(x) \leq \sum_{i=1}^{i_0} \frac{1}{2^{i-1}}\varepsilon \leq 2\varepsilon$.

Proof of Theorem. . .

Estimating the second sum $\sum_{i=i_0+1}^{\infty}$

Notice that $\left[\frac{x^{1-\varepsilon}}{e_i}, \frac{x}{e_i}\right] \subseteq \bigcup_{j=0}^{\ell} \left[\frac{x^{(1-\varepsilon')^{j+1}}}{e_i}, \frac{x^{(1-\varepsilon')^{j}}}{e_i}\right]$ whenever $\ell + 1 \geq \log_{1-\varepsilon'}(1-\varepsilon)$.

One can show that if K_i is large then $g_{i,1/2^i}(x) \leq 1/4^{i-1}$ for every x.

Thus,

$\sum_{i=i_0+1}^{\infty} g_{i,\varepsilon} \leq \sum_{i=i_0+1}^{\infty} \sum_{j=0}^{\lceil \log_{1-1/2^i}(1-\varepsilon) \rceil} g_{i,\frac{1}{2^i}}(x^{(1-1/2^i)^j})$ (by the inclusion in the union)
$\leq \sum_{i=i_0+1}^{\infty} (\log_{1-1/2^i}(1-\varepsilon)+1)\frac{1}{4^{i-1}}$ (by the choice of K_i)
$\leq \sum_{i=i_0+1}^{\infty} 2^{i+2}\varepsilon \frac{1}{4^{i-1}}$ (by the choice of i_0)
$\leq 8\varepsilon$.

It follows that the quantity that we want to estimate is bounded by 10ε.

Thank you!