New algorithm for Feynman integral reduction and ε-factorised differential equations

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ε-FACTORISED DIFFERENTIAL EQUATIONS WITHOUT RELYING ON SPECIFIC GEOMETRY

Precision predictions for high-energy experiments rely on accurately evaluating multi-loop, multi-scale Feynman integrals in dimensional regularisation. The method of differential equations [1] is by now the standard tool for this task, but its full power is realised only when the system can be brought into an ε-factorised form [2]. **We present an algorithmic framework that systematically constructs ε-factorised differential equations for arbitrary integral families, independent of their underlying geometry.**
We work in the setting of twisted cohomology and study the space of differential forms associated with a given family of Feynman integrals in the Baikov representation. Our approach consists of two steps. First, we introduce a particular ordering for the Laporta algorithm that orders Feynman integrals within a sector according to their geometric properties. We observe that this order relation yields a basis whose differential equations are in a Laurent polynomial form in the dimensional regulator ε. In the second step, we systematically construct transformation matrices such that the resulting system is in the ε-factorised form.

SETUP

- Feynman integrals in a loop-by-loop Baikov representation [3] on the maximal cut

$$\int_{C_{maxcut}} \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^{N} \sigma_j} \sim \int d^n z \prod_{i\in I_{all}} [p_i(z)]^{\alpha_i} \qquad \alpha_i = \frac{1}{2}\left(a_i + b_i\varepsilon\right),\ a_i, b_i \in \mathbb{Z} \qquad I_{all} = I_{even} \cup I_{odd}$$

- We are interested in the “minimal” case where $a_i \in \{-1,0\}$:

$$I = C_B \int \frac{d^n z}{(2\pi i)^n} \prod_{i\in I_{odd}} [p_i(z,x)]^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{i\in I_{even}} [p_i(z,x)]^{\frac{1}{2}b_i\varepsilon}$$

- Main objects: **differential forms**

$$\Psi_{\mu_0,\ldots,\mu_{N_D}}[Q] = C\,U(z)\,\hat{\Phi}_{\mu_0,\ldots,\mu_{N_D}}[Q]\,\eta$$

$\frac{Q}{\prod_i P_i^{\mu_i}},\ \mu_i \in \mathbb{N}_0$

$\sum_{j=0}^{N_V} (-1)^j z_j\, dz_0 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n$

Prefactors depending on ε and kinematics x

multivalued function: *twist*

LINEAR RELATIONS

Differential forms satisfy three types of linear relations:

1) Integration-by-parts identities

$$\frac{1}{\varepsilon}\Psi_{\mu_0\ldots\mu_i\ldots\mu_{N_D}}\left[\partial_{z_j}Q_+\right] + \sum_{i\in I_{all}} \Psi_{\mu_0\ldots(\mu_i+1)\ldots\mu_{N_D}}\left[Q_+\left(\partial_{z_j}P_i\right)\right] = 0$$

2) Distribution identities

$$\Psi_{\mu_0\ldots\mu_{N_D}}\left[Q_1+Q_2\right] = \Psi_{\mu_0\ldots\mu_{N_D}}\left[Q_1\right] + \Psi_{\mu_0\ldots\mu_{N_D}}\left[Q_2\right]$$

3) Cancellation identities

$$\Psi_{\mu_0\ldots(\mu_j+1)\ldots\mu_{N_D}}\left[P_jQ\right] = \frac{1}{\varepsilon}\left(\frac{a_j}{2} - \mu_j + \frac{b_j}{2}\varepsilon\right)\Psi_{\mu_0\ldots\mu_j\ldots\mu_{N_D}}\left[Q\right]$$

ORDERING CRITERIA FOR THE LAPORTA ALGORITHM

We use the Laporta algorithm [4] with the **order relation** $(a, w, o, |\mu|, \ldots)$ to find a basis of differential forms. These integers are defined as:

- a - localisation level
- w - number of consecutive non-zero residues* r + number of Baikov variables n
- o - pole order*
- $|\mu| = \mu_0 + \ldots\mu_{N_D}$

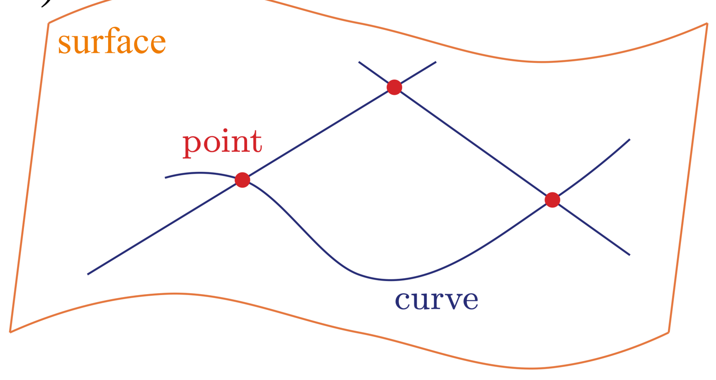
*computed after setting $\varepsilon \to 0$

HOW TO ASSIGN a?

Even and odd polynomials play a different role.

- If an even polynomial is present in the denominator, we may take a residue and reduce to a simpler problem of dimension $(n-1)$
- Odd polynomials define a geometry

$$y^2 - \prod_{j\in I_{odd}} P_j(z) = 0$$



- In general, on the maximal cut, we have a mixed geometry:
  - Inside a geometry of dimension n defined by the previous equation, we can have sub-geometries of dimension $(n-1)$, which might have sub-sub-geometries of dimension $(n-2)$,...
- We give preference to master integrands that are master integrands on sub-geometries by assigning a

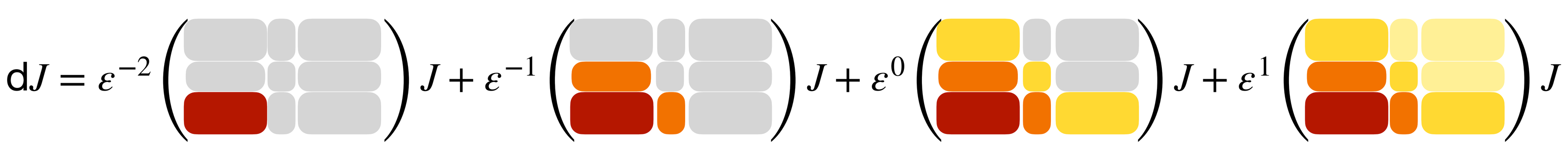
$$a = \begin{cases} -r-n, & \Psi_{\mu_0,\ldots,\mu_{N_D}} \text{ is prefered MI on } (n-r)\text{-dim sub-geometry} \\ 0, & \text{otherwise} \end{cases}$$

STEP I: INTERMEDIATE BASIS

- We observe that the basis of integrands chosen by the criteria $(a, w, o, |\mu|, \ldots)$ satisfies a differential equation in the following form

$$\mathrm{d}J = \sum_{k=k_{min}}^{1} \varepsilon^k A^{(k)}(x) J, \qquad k_{min} = -|\mu|_i - |\mu|_j \qquad (*)$$

- For example, if $k_{min} = -2$

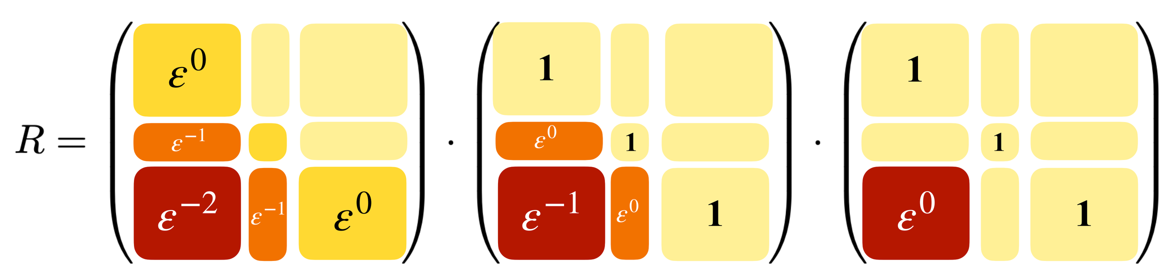


STEP II: ε-FACTORISED BASIS

- Starting from the equation (*), we prove that we can always algorithmically find a transformation matrix R leading to a basis $K = R^{-1}J$ such that

$$\mathrm{d}K = \varepsilon\tilde{A}(x)K$$

- For example,



Generic functions in x, but fixed ε-dependence

WHAT'S NEW?

(1) Trivialised ε-dependence of IBP identities

(2) Order relation inspired by geometry $\to$ differential equation is in a Laurent polynomial form

(3) An algorithm to transform into ε-factorised DE

WHAT'S NEXT?

- Open-source implementation of the proposed algorithm
- Improve efficiency by incorporating symmetries
- Test on more examples

[1] A. V. Kotikov, *Phys. Lett. B* **254**, 158 (1991); A. V. Kotikov, *Phys. Lett. B* **267**, 123 (1991); E. Remiddi, *Nuovo Cim. A* **110**, 1435 (1997); T. Gehrmann and E. Remiddi, *Nucl. Phys. B* **580**, 485 (2000)
[2] J. M. Henn, *Phys.Rev.Lett.* **110** (2013) 25160
[3] H. Frellesvig and C. G. Papadopoulos, *JHEP* **04**, 083 (2017)
[4] S. Laporta, *Int. J. Mod. Phys. A* **15**, 5087 (2000)

Based on:

For more details, proof and examples see:

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