- ABSTRACTS -
Multivariate Approximation and Interpolation with Applications
ESI - Vienna, AUSTRIA
26-30. August 2019

Anat, Amir

High order approximation to non-smooth multivariate functions

Common approximation tools return low-order approximations in the vicinities of singularities. Most prior works solve this problem for univariate functions. In this work we introduce a method for approximating non-smooth multivariate functions of the form $f = g + r_+$ where $g, r \in C^{M+1}(\mathbb{R}^n)$ and the function r_+ is defined by

$$r_+(y) = \begin{cases} r(y), & r(y) \geq 0 \\ 0, & r(y) < 0 \end{cases}, \ \forall y \in \mathbb{R}^n .$$

Given scattered (or uniform) data points $X \subset \mathbb{R}^n$, we investigate approximation by quasi-interpolation. We design a correction term, such that the corrected approximation achieves full approximation order on the entire domain. We also show that the correction term is the solution to a Moving Least Squares (MLS) problem, and as such can both be easily computed and is smooth. Last, we prove that the suggested method includes a high-order approximation to the locations of the singularities.

Berdysheva, Elena

Metric Approximation of Set-Valued Functions of Bounded Variation

We study approximation of set-valued functions (SVFs) — functions mapping a real interval to compact sets in $\mathbb{R}^d$. In addition to the theoretical interest in this subject, it is relevant to various applications in fields where SVFs are used, such as economy, optimization, dynamical systems, geometric modeling.

The earlier works in this area are mainly concerned with approximation of set-valued functions with convex images, for which the tools of Minkowski linear combinations of sets and the Aumann integral of set-valued functions are effective. Yet these techniques posses the property of convexification: the resulting approximation is always a function with convex images, even if the function to be approximated is not. For example, R.A. Vitale studied an adaptation of the classical Bernstein polynomial operator based on Minkowski linear combinations; the limit SVF in this approach consists of convex hulls of the values of the original function. Clearly, such methods are useless for the approximation of SFVs with general, not necessarily convex images.

N. Dyn, E. Farkhi and A. Mokhov developed in a series of work a new approach that is free of convexification – the so-called metric linear combinations and the metric integral. They introduced adaptations of classical approximation operators based on these tools and investigated them for continuous SFVs.

Here, we study approximation of SFVs that are not necessarily continuous. We consider SVFs of bounded variation in the Hausdorff metric. In particular, we adapt to SVFs local operators such as the symmetric Schoenberg spline operator, the Bernstein polynomial operator and the Steklov function. Error bounds, obtained in the averaged Hausdorff metric, provide rates of approximation similar to those for real-valued functions of bounded variation.

Joint work with Nira Dyn, Elza Farkhi and Alona Mokhov (Tel Aviv University, Israel).

Binev, Peter

Near-Optimal Adaptive Approximations

Finding adaptively a piecewise polynomial approximation can be related to an approximation on a full binary (or dyadic) tree. The tree describes the process of making decisions where to subdivide further the domain or more generally how to distribute the available degrees of freedom. The problem of designing a near-optimal greedy algorithm to find a near-best approximation on a general partition is well understood. The new developments, considered in the talk, address this problem for partitions with special properties, like conformity and grading.

Bölcskei, Helmut

Fundamental limits of deep neural network learning

Deep neural networks have become state-of-the-art technology for a wide range of practical machine learning tasks such as image classification, handwritten digit recognition, speech recognition, or game intelligence. This talk develops the fundamental limits of learning in deep neural networks by characterizing what is possible if no constraints on the learning algorithm and the amount of training data are imposed.

Bos, Leonard Peter

Multivariate Polynomial Interpolation and Optimal Designs

The so-called Fekete points are those which maximize the determinant of the Vandermonde matrix, V say, associated to the interpolation problem. The product V^tV may also be regarded as (essentially) the Gram matrix of the same polynomial basis with respect to the discrete probability measure, equally weighted and supported on the set of interpolation points. Hence Fekete points are those which maximize a Gram determinant for such discrete measures. An optimal design, on the other hand, is a probabilty measure that maximizes Gram determinants for the same basis among all competing probability measures (including discrete ones). In one variable, for points in the interval $[-1, 1]$ the two concepts co-incide. However, in several variables they are generally different but there are circumstances for which they again do co-incide. In this talk I will give a survey of what is known, and present some new results on this subject.

Bownik, Marcin

Exponential frames and syndetic Riesz sequences

In this talk we shall explore some of the consequences of the solution to the Kadison-Singer problem. In the first part of the talk we present results from a joint work with Itay Londner. We show that every subset S of the torus of positive Lebesgue measure admits a Riesz sequence of exponentials $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ in $L^2(S)$ such that $\Lambda \subset \mathbb{Z}$ is a set with gaps between consecutive elements bounded by $C/|S|$. In the second part of the talk we shall explore a higher rank extension of the main result of Marcus, Spielman, and Srivastava, which was used in the solution of the Kadison-Singer problem.

Carnicer, Jesus

Extensions of planar GC sets and syzygy matrices

In contrast to the univariate problem, the existence and uniqueness of solution of multivariate Lagrange interpolation problems by polynomials of total degree not greater than n depends on the node set. A correct set for degree n is a set of nodes such that the Lagrange interpolation problem has a unique solution. No line of the plane can contain more than $n + 1$ nodes of a correct set of degree n. A maximal line of a correct set is a line containing exactly $n + 1$ nodes. Even if the set of nodes is correct, the Lagrange fundamental polynomials might be described by a complicated formula involving many terms. The geometric characterization of Chung and Yao, identifies node sets for total degree interpolation such that the Lagrange fundamental polynomials are products of linear factors. Sets satisfying the geometric characterization are usually called GC sets. The Gasca-Maeztu conjecture states that for any planar GC sets of degree n, there exists a maximal line. It has been shown that the conjecture holds for degrees not greater than 5 but it is still unsolved for general degree.

One promising approach consists of describing the syzygies of the ideal of polynomials vanishing at the nodes. In order to finding syzygy matrices of a GC set, we study the extension a GC set of degree n to a GC set of degree $n + 1$, by adding a $n + 2$ nodes on a line. We explore the different cases arising according with the number of maximal lines and describe the corresponding syzygy matrices.

Celiz, Jason M.

The reproducing kernel in spaces of variable bandwidth

Given a Sturm-Liouville operator $f \to (pf')'$ on $\mathbb{R}$ for a fixed positive function p, every spectral subspaces can be interpreted as space of functions with variable bandwidth. The local bandwidth is roughly given by $1/\sqrt{p}$. We determine the reproducing kernels in the spectral subspaces for the model case of piecewise constant p. This is a first step for the numerical investigation of sampling of functions with variable bandwidth.

Conti, Costanza

Modeling image algebraic boundaries in Bernstein form

In this work, we present an improved method for detecting the boundary of a 2D-image that is expressed as the zero locus of a bivariate polynomial, i.e., as an algebraic curve. Following [1], the image parameters, i.e., the polynomial coefficients, can be calculated by solving a set of linear equations with the coefficients being the image moments. However, as pointed out in [1], the inherent sensitivity of geometric moments to noise makes the reconstruction process numerically unstable if the polynomial is expressed in terms of standard power basis. To derive a robust method for the recovery of the algebraic curve from a heavily under-sampled version of an image, our approach represents algebraic curves in terms of non-separable bivariate Bernstein polynomials and applies polynomial reproducing refinable sampling kernels. In fact, algebraic curves are uniquely described by their coefficients and, thus, their sensitivity with respect to coefficient perturbation depends on the selected bases. Our new method is robust to noise and, while its reconstruction performance is comparable with the best regularized method in [1], it is computationally much faster and simpler to implement. We consider various numerical experiments to illustrate the performance of our algorithm in reconstructing binary images, including low to moderate noise levels and a range of sampling kernels reproducing polynomials.

[1] M. Fatemi, A. Amini, M. Vetterli, Sampling and Reconstruction of Shapes with Algebraic Boundaries, IEEE Trans. Signal Process., 64 n. 22 (2016), 5807-5018.

Dahmen, Wolfgang

Tensor Approximability and High Dimensional Diffusion Equations

The numerical solution of PDEs in a spatially high-dimensional regime, such as the electronic Schrödinger or Fokker-Planck equations, is severely hampered by the ”curse of dimensionality” which, roughly speaking, means that the computational cost required for achieving a desired target accuracy increases exponentially with respect to the spatial dimension. In this talk we explore a possible remedy by exploiting a typically hidden sparsity of solutions to such problems with respect to a problem dependent basis or dictionary. Here sparsity means, roughly speaking, that only relatively few terms from such a dictionary suffice to realize a given target accuracy and a concise mathematical formulation of this notion is a first central issue discussed in this talk. Specifically, sparsity with respect to dictionaries comprised of separable functions - rank-one tensors - is shown would break or significantly mitigate the curse of dimensionality. The main result establishes such tensor-sparsity for elliptic problems over product domains when the data are tensor-sparse, which can be viewed as a structural regularity theorem. We highlight some of the main conceptual ingredients and indicate some intrinsic limitations regarding the scope of problems.

Joint work with R. DeVore, L.Grasedyck, E. Süli.

Davydov, Oleg

Error Bounds for a Least Squares Meshless Finite Difference Method

Meshless finite difference methods discretize a differential equation $Lu = f$ on a finite set $X = \{x_1, \dots, x_N\} \subset \Omega$ with the help of local numerical differentiation formulas, and seek a discrete solution $\hat{u} \in \mathbb{R}^N$ such that $\hat{u} \approx u|_X$, obtained by solving sparse linear systems of equations. The presentation will be devoted to the convergence analysis of a least squares version of this method, giving for the first time in the meshless setting sufficient conditions on a set of discretization nodes X and sets of influence that guarantee convergence $\|\hat{u} - u|_X\| \to 0$ as $N \to \infty$. The results apply to the case when $\Omega = \mathcal{M}$ is a smooth closed manifold, numerical differentiation folmulas are obtained with the help of a reproducing kernel for a Sobolev space on $\mathcal{M}$, L belongs to a certain class $\mathcal{L}$ of elliptic differential operators, and u is sufficiently smooth. The class $\mathcal{L}$ includes in particular $L = -\Delta_{\mathcal{M}} + \omega I,\ \omega > 0$, where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on the d-dimensional sphere $\mathcal{M} = \mathcal{S}_d$ or another Riemannian manifold.

De Marchi, Stefano

Polynomial interpolation via mapped bases without resampling

In this talk we propose a new method for univariate polynomial interpolation based on *mapped bases*. As theoretically shown, constructing the interpolating function via the mapped bases, i.e. in the mapped space, turns out to be equivalent to map the nodes and then construct the approximant in the classical form without the need of resampling. In view of this, we also refer to such mapped points as "fake nodes". We present a general algorithm for constructing the mapping function. We also discuss some examples to confirm that such scheme can be applied to substantially reduce both the Runge and Gibbs phenomena.

Joint work with F. Marchetti, E. Perracchione and D. Poggiali.

Dekel, Shai

Artificial intelligence based approach to numerical PDEs

We believe that the field of numerical solutions of partial differential equations is on the verge of being revolutionized through modern machine learning in the same manner as the fields of computer vision and natural language processing. We anticipate that through training on many software simulations, solvers based on deep neural network architectures will be able to surpass in performance existing methods and even provide solutions in ill-posed scenarios where existing inverse problem methods fail. As an example, we will review the problem of the wave equation where a finite number of scattered sensors collect noisy time series data and the goal is to find the unknown location of the source or the location and geometry of unknown obstacles.

Diederichs, Benedikt

Localizing Functions and the Stability of Sparse Frequency Estimation

Frequency estimation is a cornerstone of signal processing. If a model of the underlying signal is known, more efficient methods are available, which allow to overcome resolution limits of generic approaches.

One model of interest is that of an exponential sum, representing a signal with a sparse spectrum. Non-asymptotic stability results have been actively investigated in the last few years. Here, we present a stability result that does not require discretizing the set of frequencies, but only relies on discrete samples. The result guarantees that two functions with close samples, fitting our model, indeed feature close frequencies.

The proof requires particular functions that allow to estimate objects localized in the spatial domain by something localized in the frequency domain. Similar functions are used in different areas of mathematics, for example in sphere packing or eigenvalue estimates of kernel matrices and might have more applications.

Sparse frequency estimation is prototypical for continuous sparse estimation problems: Given a continuously parameterized family, and an object composed of a few element of the family, we want to identify the object with as few measurements as possible. Extending the results presented here to other problems might be interesting.

Dyn, Nira

Non-linear Subdivision Schemes Refining Point-Normal Pairs

The talk presents non-linear subdivision schemes refining point-normal pairs (PNPs), which are generated by modifying converging linear subdivision schemes in two steps: (i) Writing the linear refinement rules in terms of repeated binary linear weighted averages. (ii) Replacing each binary aver- age by the "circle average" of two PNPs with the same weight. We discuss in this talk the construction of the circle average, its properties, and the convergence of the so modified Lane-Riesenfeld algorithm, and the 4-point scheme. The C1 smoothness of the curves generated by these modified schemes is analyzed by an extended "proximity" tool.

This is a joint work with Evgeny Lipovetsky.

Floater, Michael

Supersmoothness of multivariate splines

Polynomial splines over simplicial meshes in R^n (triangulations in 2D, tetrahedral meshes in 3D, and so on) sometimes have extra orders of smoothness at a vertex. This property is known as supersmoothness, and plays a role both in the construction of splines and in the finite element method. Supersmoothness depends both on the number of simplices that meet at the vertex and their geometric configuration.

In this talk we review what is known about supersmoothness of polynomial splines and then discuss the more general setting of splines whose individual pieces are any infinitely smooth functions.

This is joint work with Kaibo Hu.

Führ, Hartmut

Classification of Besov-type decomposition spaces

The class of Besov-type decomposition spaces goes back to Feichtinger and Groebner, and comprises a large variety of spaces defined in terms of coefficient decay. Among the classes covered by this construction are modulation spaces, Besov spaces (homogeneous and inhomogeneous, isotropic and anisotropic ones), general wavelet coorbit spaces as well as shearlet and curvelet smoothness spaces. The construction of decomposition spaces is based on a covering of the frequency domain, and understanding which features of the covering are decisive for certain properties of the associated scale of function spaces is one of the major challenges in this theory.

This talk focusses on classification, i.e., the question when different frequency coverings result in the same scale of spaces. It presents recent results connecting this question to notions from coarse geometry, based on previous results by Feichtinger, Groebner, Voigtlaender. We also give an overview of recent classification results for shearlet coorbit spaces and anisotropic Besov spaces.

Based on joint results with Jahangir Cheshmavar (Payame Noor University, Iran) and Rene Koch (RWTH Aachen).

Kozynenko, Aleksandr

Piecewise constant approximation for multivariate functions

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be a cube and $1 \leqslant p \leqslant \infty$. For a function $f \in L_p(\Omega)$, we define its *best N-term approximation* by $\sigma_N(f, \mathcal{D}_S)_p = \inf\limits_{g \in \mathrm{span}\{g_1,\ldots,g_N\}} \Big\| f - g \Big\|_{L_p(\Omega)}$, where $g_i \in \mathcal{D}_S$, for $i = 1, \ldots, N$, and $\mathcal{D}_S$ denotes the *dictionary* of characteristic functions of arbitrary simplexes in Ω. It is easy to see that $\sigma_N(f, \mathcal{D}_S)_\infty \geqslant \frac{const}{N}$ for any continuous function f.

We call a partition $\triangle$ of Ω *convex* if every cell $\omega \in \triangle$ is convex. For $N \in \mathbb{N}$, denote by $\mathcal{P}_N$ the set of all convex partitions of Ω comprising at most N cells. For a partition $\triangle$ of Ω, we denote by $\mathcal{S}_0(\triangle)$ the space of piecewise constant functions $s : \Omega \to \mathbb{R}$ that are constant on every cell $\omega \in \triangle$. We define the error of the best L_p-approximation of a function $f \in L_p(\Omega)$ by piecewise constant functions on partitions from $\mathcal{P}_N$: $E_N(f)_p := \inf\limits_{\triangle \in \mathcal{P}_N} \inf\limits_{s \in \mathcal{S}_0(\triangle)} \|f - s\|_{L_p(\Omega)}$.

We have proved that piecewise constants on a convex partition which consist of N convex polyhedra provide the L_p-approximation order $O(N^{-2/(d+1)})$ for functions in Sobolev spaces $W_q^2(\Omega)$, where $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$ satisfy inequality $\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geqslant 0$. This order cannot be further improved for any function whose Hessian is positive definite at some point in Ω. This implies $\sigma_N(f, \mathcal{D}_S) = O(N^{-2/(d+1)})$ for such functions. Although still affected by the curse of dimensionality, this bound is significantly better than the standard order $O(N^{-1/d})$ expected for piecewise constants on isotropic partitions.

On the other hand, it is easy to see that piecewise constant sparse grids approximation implies $\sigma_N(f, \mathcal{D}_S)_p = O(N^{-1} \ln^{2(d-1)} N)$ for functions in Sobolev spaces $W_p^{(1,\ldots,1)}(\Omega)$ with dominating mixed derivatives. We improve this bound and show that approximation by linear combinations of tensor product Haar functions leads to $\sigma_N(f, \mathcal{D}_S)_p = O(N^{-1} \ln^{3(d-1)/2} N)$ for $2 \leqslant p < \infty$ (the case $p = 2$ was previously considered by P. Oswald). This order cannot be further improved by N-term approximation by Haar functions. Also, using a modification of the sparse grid approximation in the 2D case we obtain improved estimate $\sigma_N(f, \mathcal{D}_S)_\infty = O(N^{-1} \ln N)$ for $f \in W_\infty^3(\Omega)$.

This is the joint work with Oleg Davydov and Dmytro Skorokhodov.

Kunoth, Angela

Recent Results on Empirical Mode Decomposition Schemes

The empirical mode decomposition (EMD) was developed by Huang et al. in 1998 as an iterative method to decompose a nonlinear and nonstationary univariate function additively into multiscale components. These components called intrinsic mode functions (IMFs) are constructed such that they are approximately orthogonal to each other with respect to the L_2 inner product. Moreover, the components allow for a definition of instantaneous frequencies through complexifying each component by means of the application of the Hilbert transform. This approach via analytic signals, however, does not guarantee that the resulting frequencies of the components are always non-negative and, thus, physically meaningful, and that the amplitudes are envelopes in a strict mathematical sense.

Over the years, different approaches were formulated to overcome some of the limitations of the original EMD approach. In particular, several attempts were made to develop a solid mathematical foundation. Peng et al. introduced in 2010 the idea to design certain differential operators adapted to the signal which annihilate the signal and whose parameters can be used to properly identify an IMF. Ultimately, this idea leads to an optimization problem with or without regularization, which is typically solved based on cubic B-Splines. In addition to resumimg some of the recent results on this topic, I would like to address the potential of this approach to multivariate signals.

The results were obtained in collaboration with Boqiang Huang and Laslo Hunhold.

Levesley, Jeremy

Approximation with Gaussians at varying scales

Approximation with Gaussians has advantages and disadvantages. One of the main disadvantages is that the analytic nature of the function means that approximation problems with Gaussians are often ill-conditioned. On the other hand, since Gaussians are smooth, they carry the possibility of optimal convergence orders when the smoothness of the underlying function is unknown (as it usually is in practice). Also, in high dimensions, most functions start to look like Gaussians.

In standard radial basis function applications Gaussians of a fixed width are used, and convergence is obtained by reducing the spacing between data points while leaving the shape of the Gaussian fixed and constant. We would like to explore a different scenario, which is to use a multiscale method, where the shape depends on the scale. We will explore this for a fixed shape at each scale and also a varying shape at each scale. The shape is changed related to end-point singularities in the function in order to mimic $h - p$ adaptivity in finite elements.

We will review results in the multilevel fixed shape paradigm. Collaborators in this work are Manolis Georgoulis, Peter Dong, Fuat Usta, Fazli Subhan (all at Leicester), and Simon Hubbert at Birkbeck. We will also show that the interpolation matrices in the multiscale, multishape scenario in one dimension are numerically invertible, a new result.

Levin, David

Some remarks on Multivariate Approximation

My talk will present some ideas on scattered data multivariate approximation. I will start by suggesting a graphical visualization of the approximation power of linear bivariate approximation methods. This will lead us to the notion of quasi-interpolation and to the method of Moving Least-Squares. Approximation errors are usually larger near singularities of the approximated function and also near the boundary of the approximation domain. It turns out that by analyzing the approximation errors at the data points we can improve the approximation near singularities and near the boundary. Other interesting issues are the approximation of low dimensional manifolds, and the approximation of a functions over manifolds. I will present the idea of approximation by projection, first for the approximation of surfaces in 3D, and then for the approximation of general low dimensional manifolds in high dimension. To get the feeling of this approach, please see a short video on approximating a curve in 3D from noisy samples at: https://youtu.be/K_TzGinrexM.

Lyche, Tom

Simplex-Splines on the Clough-Tocher Split with Arbitrary Smoothness

For any positive integer r we propose a partition of unity simplex spline basis of degree $3r$ and smoothness r on the Clough-Tocher split on a triangle. This is joint work with Jean-Louis Merrien and Tomas Sauer.

Maier, Lars-Benjamin

Approximation of functions and functionals on submanifolds by ambient splines

The problem of approximating a function $f : \mathbb{M} \to \mathbb{R}$ that is defined on a manifold is a challenging task. But due to the rapid development of computer usage in industry and science, it has also gained a correspondingly rapid increase in both relevance and researcher's interest.

In the talk, a new approach to this problem is presented. The basic theory presented can afterwards, besides the initial standard problem, be applied to such various problems as sparse data interpolation, denoising of function values in scattered data sites or the solution of partial differential equations.

Mejstrik, Thomas

Modified invariant polytope algorithm

The problem of computation of the joint spectral radius of a family of matrices, i.e. the maximal exponent of the asymptotic growth rate of their products, plays an important role in the theory of refinable surfaces, subdivision schemes, wavelets, number theory, code theory, etc.. The invariant polytope algorithm by Guglielmi and Protasov was a breakthrough for this problem, allowing the exact computation for matrices of dimension up to 15-18.

We propose several modifications of the original implementation: parallelization, natural selection of new polytope's vertices, automated adding of extra vertices, norm pre-estimates, construction of simplified polytopes and an adaption to higher dimensional problems, making the original algorithm faster and applicable for higher dimensions.

Furthermore, we present a new method for finding spectral maximizing products. This method is based on Gripenberg's algorithm.

Merrien, Jean-Louis

A simplex spline basis for the Alfeld split in $\mathbb{R}^n$.

Piecewise polynomials over n-simplex have applications in several branches of the sciences ranging from finite element analysis, surfaces in computer aided design and other engineering problems. In some cases, we need smoother elements for modeling, or higher degrees to increase the approximation order.

To obtain C^1-smoothness, we begin in dimension 2 with the triangle and the Clough-Tocker element. Then, on n-simplex, $\mathcal{T}_n$, for the Alfeld split, we propose a basis of the space $\mathbb{S}_d^1$ with $d = 2n - 1$ of dimension $\binom{d+n}{n} + n\binom{d-1}{n}$. With this basis, we recover the partition of unity and Marsden identities (barycentric and cartesian).

References:

- P. Alfeld, *A trivariate Clough-Tocher scheme for tetrahedral data*, Comput. Aided Geom. Design, bf 1(1984), 69-181,
- R. W. Clough and J. L. Tocher, *Finite element stiffness matrices for analysis of plate bending*, in *Proceedings of the conference on Matrix Methods in Structural Mechanics*, Wright-Patterson A.F.B., Ohio, 1965.
- T. Lyche and JL. Merrien,*Simplex-splines on the Clough-Tocher element*, Comput. Aided Geom. Design, **65**(2018),76-92,
- A. Kolesnikov and T. Sorokina, *Multivariate C^1-continuous splines on the Alfeld split of a simplex*, Approx. Th. XIV, San Antonio, 2013, 283–294.

Mourrain, Bernard

Decomposition of moment series

We consider the problem of decomposing series as sum of polynomial-exponential series. This corresponds to the decomposition of a measure as a weighted sum of Dirac or derivatives of Dirac measures from a sequence of moments.

We recall first Prony's method for one-dimensional problems and its generalization in higher dimension. The method exploits the algebraic structure of the Artinian Gorenstein algebra associated to the moment matrix and deduce the polynomial-exponential representation from eigenvectors of multiplication operators in the Artinian Gorenstein Algebra.

We then study relaxation techniques, which transform this decomposition problem into a weaker convex optimization problem. We describe some conditions under which the relaxation is exact, i.e. the decomposition can be obtained from the solution of the relaxed problem.

We illustrate the numerical behavior of this approach on numerical experiments related to sparse interpolation and polynomial optimization problems.

Mula, Olga

Optimal reduced model algorithms for data-based state estimation

In this talk, we present an overview and some recent results on the problem of reconstructing in real time the state of a physical system from available measurement observations and the knowledge of a physical PDE model. Contrary to classical inverse problem approaches where one seeks for the parameters of the PDE that best satisfy the measurements, we use the PDE models to learn fast reconstruction mappings which satisfy certain optimality properties. The high dimensionality of the problems that arise combined with the very different nature of the potential applications (air pollution, hemodynamics, nuclear safety to name a few) demand the development of compression, optimization and learning strategies based on sound mathematical grounds. In the talk, we will present recent results on optimal affine algorithms and highlight the prominent role of reduced order modeling of PDEs. However, in its classical formulation, reduced order modeling involves the construction of linear spaces which makes it not suitable to treat hyperbolic problems. We will outline recent results on an approach involving nonlinear mappings to mitigate this obstruction.

Nouy, Anthony

Learning with tree tensor networks

We consider the approximation of high-dimensional functions in a statistical learning setting, by empirical risk minimization over model classes of functions in tree-based tensor format. After presenting some approximation results for these model classes, we present adaptive learning algorithms, with procedures for the adaptation and selection of ranks and dimension trees. Numerical examples illustrate the performance of these algorithms for regression and density estimation, and in particular their ability to recover hidden structures of high-dimensional functions.

Pena, Juan Manuel

Conditioning, Stability and Accuracy in univariate and multivariate problems of Approximation Theory and C.A.G.D.

Recent results on conditioning and accurate and stable computational methods for Approximation Theory and Computer Aided Geometric Design are discussed, for univariate as well as for multivariate problems. The relationship with numerical methods for some structured classes of matrices related to positivity is analyzed. For instance, we present some classes of matrices for which many computations can be performed with high relative accuracy. Related problems with evaluation, interpolation and least squares fitting are considered.

Plonka, Gerlind

Reconstruction of non-stationary signals by the generalized Prony method

In this talk, we reconsider the problem of parameter identification in short exponential sums which can be solved by the well-known Prony method. The exponential sum can be also interpreted as a sparse linear combination of eigenfunctions of the shift operator.

This view led to a generalization of Prony's method in Peter & Plonka (2013), where we have shown that sparse expansions of eigenfunctions of linear operators can be reconstructed completely by using only a small number of suitable sample values.

In this talk, we consider special classes of generalized shift operators and corresponding sets of eigenfunctions that admit a reconstruction of structured functions from function values.

In particular, we can show that the reconstruction of expansions of shifted Gaussians, Gabor expansions with Gaussian window functions, Gaussians with different scaling as well as non-stationary signals with special monotone phase functions can be reconstructed by the generalized Prony method.

These results have be obtained jointly with Thomas Peter (University of Vienna), Kilian Stampfer, and Inge Keller (University of Göttingen).

Prautzsch, Hartmut

Cutting edge refinement

Corner cutting refers to subdivision schemes where we iteratively cut the corners of a convex polygon or polyhedron by lines or planes, i. e., where we intersect more and more halfplanes or halfspaces. For polygons it is well-known how to cut so as to obtain limiting convex sets with C^1 boundaries and we know a number of corner cutting schemes for polygons. For polyhedra, however, no results have been published beyond a talk given by Carl de Boor at the Biri conference in 1991. Moreover no corner cutting scheme has been proposed unless we like to view Tóth's construction of the kernel of a polyhedron (which is the limit of the convex hulls of the edge midpoints of the preceeding convex hull) as such an algorithm.

The purpose of this talk is to show how the results for curves can be generalized and to present three first corner cutting schemes: the 4-8, sqrt(3) and the honeycomb cutting scheme which are related topologically to the 4-8, the sqrt(3) and the interpolatory honeycomb refinement or subdivision schemes by Velho & Zorin, Kobbelt and Dyn et al., respectively. It is shown that flattening all angles by corner cutting does not guarantee C^1 limits even if we restrict ourselves to so called local cuts that preserve all faces of a polyhedron at least partially. Instead, we need to resort to local edge cuts to obtain a C^1 guaranty. These results can be generalized to hyperedge cutting of higher dimensional polyhedra.

Protasov, Vladimir

Regularity of multivariate wavelets and synchronizing automata

Multivariate wavelets on $\mathbb{R}^d$ can be constructed with an arbitrary dilation integer expansive $d \times d$ matrix and arbitrary set of "digits" from the corresponding quotient sets. A formula for the Hölder exponent of multivariate wavelets constructed with an arbitrary dilation matrix was presented in [1]. We show that a similar techniques can be used to make the high order regularity analysis, in particular, for the multivariate B-splines. In the simplest case of multivariate Haar wavelets, this leads to computing the boundary dimension of self-similar tiles. Moreover, the same value has an interpretation in terms of the problem of synchronizing automata. A finite automata is determined by a directed multigraph with N vertices (states) and with all edges (transfers) colored with m colors so that each vertex has precisely one outgoing edge of each color. The automata is synchronizing if there exists a finite sequence of colors such that all paths following that sequence terminate at the same vertex independently of the starting vertex. The problem of synchronizing automata has been studied in great detail. It turns out that each multivariate Haar function can be naturally associated with a finite automata and the Hölder exponent is related to the length of the synchronizing sequence. We introduce a concept of synchronizing rate and show that it is actually equal to the Hölder exponent of the corresponding Haar function. Applying this result we prove that the boundary dimension of self-similar tiles, as well as the Hölder exponent of Haar functions, can be found within finite time by a combinatorial algorithm.

[1] M. Charina, V. Yu. Protasov, Regularity of anisotropic refinable functions, ACHA (2017), $https$:

$//doi.org/10.1016/j.acha$

Putinar, Mihai

Christoffel-Darboux analysis

The asymptotic behavior of the reproducing kernel, also known as the Christoffel-Darboux kernel, in a polynomial ring endowed with an inner product is an eternal theme of approximation theory, independently and efficiently promoted in classical spectral analysis. During the last decade this tool was refined and adapted to ergodic theory (via Koopman's operator) and to the detection of outliers in multivariate statistical data. An overview of such applications from the perspective of Christoffel -Darboux kernel and Christoffel function asymptotics will be presented.

Based on recent joint works with: B. Beckermann, M. Korda, J.B. Lasserre, I. Mezic, E. Pauwels and E. Saff.

Rauhut, Holger

Recovery of functions of many variables via compressive sensing

The talk will cover the use of compressive sensing techniques for the recovery of functions of many variables from sample values. We will outline applications to the numerical solution of parametric PDEs with high-dimensional parameter space and in particular discuss multilevel approximation methods.

Reif, Ulrich

Watertight Trimmed NURBS Surfaces

Trimmed NURBS are the standard for industrial surface modeling, and all common data exchange formats, like IGES or STEP, are based on them. Typically, trimming curves have so high degree and so complex knot structure that it seems to be impossible to match them properly to neighboring geometry. Thus, surfaces built from several trimmed NURBS patches are known to reveal gaps along inner boundaries, and it is a cumbersome and sometimes nontrivial task for designers to keep the magnitude of these gaps below an acceptable tolerance.

In this talk, we present a novel methodology to construct trimmed NURBS surfaces with prescribed low order boundary curves, facilitating the representation of watertight and even smooth surface models within the functionality of standard CAD systems.

Romani, Lucia

Compactly supported fundamental functions for local interpolation over arbitrary topology meshes

The aim of this work is to present a general strategy for the construction of bivariate fundamental spline (piecewise-polynomial) functions for local interpolation over arbitrary topology meshes. The design of such fundamental splines is obtained through a suitable combination of bivariate polynomial interpolants with blending functions that are either box-splines or their natural generalization via the basic limit functions of bivariate subdivision schemes. The proposed approach extends the univariate families of B1-spline and B2-spline fundamental functions (obtained in [1] and [2], respectively) to polygonal meshes with extraordinary vertices of arbitrary valence. Numerical examples of bivariate B1- and B2-spline fundamental functions arising from popoular bivariate subdivision schemes will be illustrated.

This is joint work with C.V. Beccari and G. Casciola (University of Bologna).

References

[1] Beccari, C.V., Casciola, G., Romani, L.: Construction and characterization of non-uniform local interpolating polynomial splines, Journal of Computational and Applied Mathematics 240, (2013) 5-19

[2] Antonelli, M., Beccari, C.V., Casciola, G.: A general framework for the construction of piecewise-polynomial local interpolants of minimum degree, Advances in Computational Mathematics 40(4), (2014) 945-976

Romero, Jose Luis

Sampling multivariate functions along curves

Traditionally, the cost of sampling a signal is considered to be the required number of point evaluations. Classical sampling theory formalizes this notion of acquisition cost by characterizing the possibility of recovering a function from discrete samples in terms of the spatial density of the sampling pattern. However, when functions are sampled along continuous trajectories, as, for example, in magnetic resonance imaging, other notions of acquisition cost, such as the average length traveled by a sampling sensor, are more appropriate. I will present some first steps towards a Nyquist theory of sampling along curves, including the quantification of the effects of undersampling compressible or natural signals.

Based on work with Philippe Jaming, Felipe Negreira, Karlheinz Gröchenig, Jay Unnikrishnan and Martin Vetterli.

Ron, Amos

New insights into multivariate spline functions

I will discuss in my talk the notion of 'acyclic graphs with a single source', then the notion of 'maximal parking functions', and recall a result that connects the two: given an undirected connected graph, this known result establishes a bijection between the acyclic orientations with one (fixed) source of the graph and the maximal parking functions, the latter being a set of multivariate monomials.

I will then introduce 'flow' over directed graphs. Each parking function then flows over its acyclic graph to yield a multivariate polynomial. The construction, for each given fixed, undirected, connected graph, yields a set of multivariate polynomials that solve a long standing problem in multivariate spline theory.

The talk is based on joint work with Shengnan (Sarah) Wang.

Sampoli, Lucia

Quasi-interpolation techniques for the numerical approximation of singular integrals involving a B-spline factor

We propose a new class of quadrature rules based on spline quasi-interpolation, for the approximation of singular integrals (weakly, strongly and hypersingular). This kind of integrals occur, for instance, as entries of the stiffness matrix associated with Isogeometric Boundary Element Methods (IgA-BEMs). The proposed formulas are efficient, since they combine the locality of any spline quasi-interpolation scheme with the capability to compute the modified moments for B-splines. The peculiarity of the considered singular integrals is given by the fact that the regular part of the integrand is defined as the product of a B-spline and a general function. Then exploiting a recurrence relation for B-spline products, an efficient formulation can be obtained. Convergence results of the presented quadrature rules are given, with respect to both smooth and non smooth integrands. To conclude some numerical examples and applications will be shown.

Sauer, Tomas

Generalized Convolutions

Equipping the standard convolution with a dilation factor for the shift leads to a concept that includes convolution, correlation and subdivision operators. The talk will consider such operations, the associated Toeplitz and Hankel oparators and, in particular, the annihilation and reproduction properties of such operators. Not so surprising, they are exponential polynomials and related to common or symmetric zeros of the associated symbols, which are Laurent polynomials formed by the coefficients of the generalized convolution.

Schneider, Reinhold

Reduced Isometry Property (RIP) for Variational Monte Carlo Methods

Suppose the high-dimensional PDE is cast in a variational form, in Variational Monte Carlo we replace the original objective functional by an empirical functional in a similar way as for the quadratic loss functions in regression and statistical learning. We want to provide an error analysis proofing error estimates holding with high probability. We present a new approach for the error analysis, which improves our former work using Cucker

Smale theory and is more related to results of Cohen and co-workers on least squares methods. Our analysis is not restricted to $A = I$ and orthogonal basis functions (as in Cohen et al.) but hold for elliptic operators and Riesz basis (or frames) and allows for conic but more non-linear model classes, at a price being less optimal. The central issue is an RIP expressing the V-ellipticity and guaranteeing exact reconstruction in the model class.

As an application computing an approximation a classical solution of the non-linear and high-dimensional (stationary) Hamilton Bellmann equations subordinated to an infinite horizon feedback control problem. Variational Monte Carlo method are used to solve an inhomogeneous backward Kolmogorov equation inside a policy iteration step. We use multi-polynomial ansatz-functions and HT tensor product to represent the solution and circumventing the curse of dimensions (deep neural networks and other tools from ML may be used alternatively). In our example the spatial dimension was $d = 34$. We show improvement over LQR (linear quadratic regulator) used in engineering.

Schumaker, Larry

The Immersed Penalized Boundary Method for solving PDE's

The purpose of this paper is to describe a method (which we call IPBM) for solving boundary value problems on domains with curved boundaries. The method combines two ideas from the PDE literature: a) the idea of immersing the problem in a larger and simpler domain, and b) the idea of enforcing boundary conditions by using a penalty term. The method has a number of advantages as compared to existing methods in the literature, including IGA methods. It can be used with a wide variety of spline spaces, including tensor-product splines, splines on triangulations and H-triangulations, and splines on T-meshes. The latter two are well suited for adaptive methods. The talk will include a series of examples both in 2D and 3D to illustrate the performance of the method.

Viscardi, Alberto

Regularity of Refinable Functions: Joint Spectral Radius vs. Frame Based Decompositions

There are several prominent tools for regularity analysis in the shift-invariant setting, e.g. difference operator, joint spectral radius and wavelet techniques. The computation of the joint spectral radius can be efficiently done via modified invariant polytope algorithm by Guglielmi, Mejstrik and Protasov. The corresponding transition matrices derived from multivariate subdivision schemes usually posses several invariant subspaces which may prevent the successful termination of the algorithm. We compare the modified invariant polytope algorithm with the frame based method derived from several optimal interpolatory subdivision schemes.

Voigtländer, Felix

Approximation in $L_p(\mu)$ with deep ReLU neural networks

We discuss the expressive power of neural networks which use the non-smooth ReLU activation function $\rho(x) = \max\{0, x\}$ by analyzing the approximation theoretic properties of such networks. The existing results mainly fall into two categories: approximation using ReLU networks with a fixed depth, or using ReLU networks whose depth increases with the approximation accuracy. After reviewing these findings, we show that the results concerning networks with fixed depth - which up to now only consider approximation in $L_p(\lambda)$ for the Lebesgue measure λ - can be generalized to approximation in $L_p(\mu)$, for any finite Borel measure μ. In particular, the generalized results apply in the usual setting of statistical learning theory, where one is interested in approximation in $L_2(\mathbb{P})$, with the probability measure $\mathbb{P}$ describing the distribution of the data.

Voigtländer, Felix

Invertibility of frame operators on Besov-type decomposition spaces

Many function spaces arising in harmonic analysis are so-called Besov-type spaces. The norm of such a Besov-type space $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$ is determined by a covering $\mathcal{Q} = (Q_i)_{i\in I}$ of (a subset $\mathcal{O}$ of) the frequency domain $\widehat{\mathbb{R}}^d$; precisely,

$$\|g\|_{\mathcal{D}(\mathcal{Q},L^p,\ell_w^q)} = \left\| \left(w_i \cdot \|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g}\,)\|_{L^p}\right)_{i\in I} \right\|_{\ell^q},$$

where $(\varphi_i)_{i\in I}$ is a suitable partition of unity subordinate to the covering $\mathcal{Q}$. If $\mathcal{Q}$ is a dyadic covering, then $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_w)$ is indeed a Besov space, while for the uniform covering, $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_w)$ turns out to be a modulation space.

We study conditions under which the frame operator $S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ associated to a generalized shift-invariant (GSI) system restricts to a boundedly invertible operator on such a Besov-type space $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_w)$. Our criteria become most concrete for an affinely generated covering $\mathcal{Q} = (A_i Q + b_i)_{i\in I}$ (where $A_i \in \mathrm{GL}(\mathbb{R}^d)$ and $b_i \in \widehat{\mathbb{R}}^d$) and an adapted GSI system of the form

$$(T_\gamma\, g_i)_{i\in I, \gamma\in\delta A_i^{-t}\mathbb{Z}^d} \quad \text{where} \quad g_i(x) = |\det A_i|^{1/2} \cdot e^{2\pi i b_j x} \cdot g(A_i^t x)$$

for a fixed prototype function $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In this setting, we give concrete criteria regarding the prototype function g ensuring that the frame operator S is boundedly invertible on $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_w)$, for sufficiently fine (explicitly given) sampling density $\delta > 0$. Denoting the canonical dual frame by $(\widetilde{g}_{i,\gamma})_{i\in I,\gamma\in\delta A_i^{-t}\mathbb{Z}^d}$, this ensures that the canonical frame expansions

$$f = \sum_{i\in I} \sum_{\gamma\in\delta A_i^{-t}\mathbb{Z}^d} \langle f, \widetilde{g}_{i,\gamma}\rangle\, T_\gamma\, g_i = \sum_{i\in I} \sum_{\gamma\in\delta A_i^{-t}\mathbb{Z}^d} \langle f, T_\gamma\, g_i\rangle\, \widetilde{g}_{i,\gamma}$$

extend to the Besov-type spaces. In particular, common signal-processing operations–like thresholding the dual coefficients and then reconstructing—preserve the Besov-type regularity of the functions under consideration.

The proof is based on a careful study of the Walnut-Daubechies-type representation

$$Sf = \sum_{\alpha\in\Lambda} \mathcal{F}^{-1}\big[\, T_\alpha(t_\alpha \cdot \widehat{f}\,)\big]$$

of the frame operator S. We use this representation to show that S is "diagonally dominant," in the sense that $S = T_0 + R$ and $\|T_0^{-1}\|_{\mathrm{op}} \cdot \|R\|_{\mathrm{op}} < 1$, where T_0 is the Fourier multiplier $T_0 f = \mathcal{F}^{-1}(\, t_0 \cdot \widehat{f}\,)$.

This is joint work with Jordy van Velthoven and José Luis Romero.

Zhou, Ding-Xuan

Approximation Theory of Deep Convolutional Nets

Deep learning has been widely applied and brought breakthroughs in speech recognition, computer vision, and many other domains. The involved deep neural network architectures and computational issues have been well studied in machine learning. But there lacks a theoretical foundation for understanding the approximation or generalization ability of deep learning models with network architectures such as deep convolutional neural networks (CNNs) with convolutional structures. The convolutional architecture gives essential differences between the deep CNNs and fully-connected deep neural networks, and the classical approximation theory of fully-connected networks developed around 30 years ago does not apply. This talk describes an approximation theory of deep CNNs associated with the rectified linear unit (ReLU) activation function. In particular, we show the universality of a deep CNN, meaning that it can be used to approximate any continuous function to an arbitrary accuracy when the depth of the neural network is large enough. We also give explicit rates of approximation, and show that the approximation ability of deep CNNs is at least as good as that of fully-connected multi-layer neural networks. Our quantitative estimate, given tightly in terms of the number of free parameters to be computed, verifies the efficiency of deep CNNs in dealing with large dimensional data.

List of Participants:

1. Amir Anat, Tel Aviv University, Israel
2. Elena Berdysheva, University of Giessen, Germany
3. Peter Binev, University of South Carolina, USA
4. Carl de Boor, University of Wisconsin, Madison, USA
5. Leonard Peter Bos, University of Calgary, Canada
6. Helmut Bölcskei, ETH Zürich, Switzerland
7. Marcin Bownik, University of Oregon, USA
8. Jesus Carnicer, University of Zaragoza, Spain
9. Jason M. Celiz, University of Vienna, Austria
10. Maria Charina, University of Vienna, Austria
11. Mariantonia Cotronei, Mediterranean University of Reggio Calabria, Italy
12. Costanza Conti, University of Florence, Italy
13. Wolfgang Dahmen, University of South Carolina, USA
14. Oleg Davydov, University of Giessen, Germany
15. Shai Dekel, Tel Aviv University, Israel
16. Benedikt Diederichs, Fraunhofer Research Group ”Knowledge Based Image Processing”, Passau, Germany
17. Monika Dörfler, University of Vienna, Austria
18. Nira Dyn, Tel Aviv University, Israel
19. Martin Ehler, University of Vienna, Austria
20. Hans Feichtinger, University of Vienna, Austria
21. Michael Floater, University of Oslo, Norway
22. Hartmut Führ, RWTH Aachen, Germany
23. Philipp Grohs, University of Vienna, Austria
24. Karlheinz Gröchenig, University of Vienna, Austria
25. Christopher Heil, Georgia Institute of Technology, USA,
26. Kurt Jetter, University of Hohenheim, Stuttgart, Germany
27. Bert Jüttler, Johannes Kepler Universität Linz, Austria
28. Aleksandr Kozynenko, Oles Honchar Dnipro National University, Ukraine
29. Angela Kunoth, University of Cologne, Germany
30. David Levin, Tel Aviv University, Israel
31. Jeremy Levesley, University of Leicester, England
32. Tom Lyche, University of Oslo, Norway

33. Lars-Benjamin Maier, TU Darmstadt, Germany
34. Stefano de Marchi, University of Padova, Italy
35. Thomas Mejstrik, University of Vienna, Austria
36. Jean-Louis Merrien, University of Rennes 1, France
37. Caroline Moosmüller, University of California, San Diego, USA
38. Bernard Mourrain, Inria Sophia Antipolis, France
39. Olga Mula, Paris Dauphine University, France
40. Anthony Nouy, Centrale Nantes, France
41. Peter Oswald, Jacobs University, Bremen, Germany
42. Juan Manuel Pena, Universidad of Zaragoza, Spain
43. Götz Pfander, Catholic University of Eichstätt-Ingolstadt, Germany
44. Gerlind Plonka, University of Göttingen, Germany
45. Hartmut Prautzsch, Karlsruhe Institute of Technology, Germany
46. Vladimir Protasov, Moscow State University, Russia and University of L'Aquila, Italy
47. Mihai Putinar, University of California at Santa Barbara, USA
48. Christophe Rabut, University of Toulouse, France
49. Holger Rauhut, RWTH Aachen, Germany
50. Ulrich Reif, TU Darmstadt, Germany
51. Jose Luis Romero, University of Vienna, Austria
52. Lucia Romani, University of Bologna, Italy
53. Amos Ron, University of Wisconsin-Madison, USA
54. Lucia Sampoli, University of Siena, Italy
55. Tomas Sauer, University of Passau, Germany
56. Reinhold Schneider, TU Berlin, Germany
57. Larry Schumaker, Vanderbilt University, USA
58. Joachim Stöckler, TU Dortmund University, Germany
59. Thomas Strohmer, University of California, Davis, USA
60. Alberto Viscardi, University of Bologna, Italy
61. Felix Voigtländer, Catholic University of Eichstätt-Ingolstadt, Germany
62. Johannes Wallner, TU Graz, Austria
63. Ding-Xuan Zhou, City University of Hong-Kong, China