Trace Relations and Matrix Models

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Large-N Matrix Models and Emergent Geometry

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Introduction

Consider counting the number of invariants of a system of $N \times N$ matrices, i.e. for $g \in U(N)$ invariant under conjugation:

$$X_i \to gX_ig^{-1}$$

$X \in \mathrm{Mat}(N)$ has N^2 degrees of freedom

But there are only N invariants—the N eigenvalues of X.

Eigenvalues are roots or the characteristic polynomial

$$P_N(\lambda) = Det[X - \lambda \mathbf{1}_N]$$

Hamilton-Cayley

The Hamilton-Cayley Theorem

Every finite rank square matrix, X, over a commutative ring satisfies its own characteristic equation

$$P_N(X) = 0$$

where $P_N(\lambda)$ is the characteristic polynomial of X.

$P_N(X)$ recursively

$$P_N(X) = P_{N-1}(X)X - \frac{1}{N}\mathbf{tr}(P_{N-1}(X)X).$$

with $P_1(X) = X - \mathbf{tr}(X)$.

$$\mathbf{tr}(P_N(X)) = 0 \quad \text{gives } \det(X) \text{ in terms of traces.}$$

Similarly $\mathbf{tr}(X^{N+1})$ becomes products of traces of lower powers.

2×2 matrices and 3×3 traceless matrices

For X, a generic 2×2 matrix,

$$P_2(x) = P_1(X)X - \frac{1}{N}\mathbf{tr}(P_1(X)X)\mathbf{1}_2 \qquad P_1(X) = X - \mathbf{tr}(X)$$

$$\implies P_2(X) = X^2 - X\mathbf{tr}(X) - \frac{1}{2}\left(\mathbf{tr}(X^2) - \mathbf{tr}^2(X)\right)\mathbf{1}_2$$

$$\mathbf{tr}(X^3) - \frac{3}{2}\mathbf{tr}(X)\mathbf{tr}(X^2) + \frac{1}{2}\mathbf{tr}^3(X) = 0\,.$$

For Y a generic traceless 3×3 traceless matrix

$$P_3(Y) = Y^3 - \frac{1}{2}\mathbf{tr}(Y^2)Y - \frac{1}{3}\mathbf{tr}(Y^3)$$

$$\implies \mathbf{tr}(Y^4) - \frac{1}{2}\left(\mathbf{tr}(Y^2)\right)^2 = 0\,.$$

More generally for an $N \times N$ matrix $\mathbf{tr}(X^{N+1})$ is expressible in terms of products of lower traces.

All matrix invariants are expressible in terms of the generating set $\{\mathbf{tr}(X^k)\}$ with $k \leq N$.

The algebra of GL_N invariants

The algebra of invariants of a single generic matrix X is generated by the N traces $\mathbf{tr}(X^k)$, $k = 1, \ldots, N$.

The invariants of X are, of course, the eigenvalues.

The number of invariants for a given power of the matrix is captured by a generating function (Hilbert-Poincaré series)

$$Z_N(t) = \sum_{n}^{\infty} \mathrm{dim}_n(N) t^n = \sum_{n=0}^{\infty} p_N(n) t^n$$

where dim_n is the number of invariants formed from n X's. $\mathrm{dim}_n(N) = p_N(n) = \#$ partitions of n into N or less parts.

$$Z_N(t) = \prod_{m=1}^{N} \frac{1}{1 - t^m} = 1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + \cdots .$$

Fock Space Realisation

For a single matrix the low lying states are:

$$
\begin{array}{ccccc}
|0\rangle, & & & & \\
\mathbf{tr}(a^\dagger)|0\rangle, & & & & \\
\mathbf{tr}^2(a^\dagger)|0\rangle, & \mathbf{tr}((a^\dagger)^2)|0\rangle, & & & \\
\mathbf{tr}^3(a^\dagger)|0\rangle, & \mathbf{tr}(a^\dagger)\mathbf{tr}((a^\dagger)^2)|0\rangle, & \mathbf{tr}((a^\dagger)^3)|0\rangle, & & \\
\mathbf{tr}^4(a^\dagger)|0\rangle, & \mathbf{tr}^2(a^\dagger)\mathbf{tr}((a^\dagger)^2)|0\rangle, & \mathbf{tr}((a^\dagger)^2)\mathbf{tr}((a^\dagger)^2)|0\rangle, & \mathbf{tr}(a^\dagger)\mathbf{tr}((a^\dagger)^3)|0\rangle, & \mathbf{tr}((a^\dagger)^4)|0\rangle, \\
\mathbf{tr}^5(a^\dagger)|0\rangle, & \mathbf{tr}^3(a^\dagger)\mathbf{tr}((a^\dagger)^2)|0\rangle & \cdots & \cdots &
\end{array}
$$

The partition function (Hilbert Poincaré series).

$$
Z_N(t) = \mathbf{Tr}_{Phys}(\mathrm{e}^{-\beta(\mathbf{tr}(a^\dagger a))}) = \mathbf{Tr}_{Phys}(t^{\hat{N}}) = \prod_{m=1}^{N} \frac{1}{1-t^m}..
$$

Where $t = \mathrm{e}^{-\beta}$, and *Phys* refers to $U(N)$—gauge invariant states.

$$
Z_\infty(t) = \frac{1}{\phi(t)} \qquad \phi(t) = \prod_{n=1}^{\infty}(1-t^n) \qquad \text{is the Euler function.}
$$

Two or more Matrices

What happens if we consider a pair of matrices X and Y?

For more than one matrix the invariants are no longer eigenvalues.

What can we say about the invariants of this system? A few theorems guide what to expect.

Theorem: (Nagata-Higman Theorem), Nagata (1953), Higman (1956), Dubnov and Ivanov (1943)

If the (nonunitary) algebra R is nil of bounded index $\leq N$, i.e. $r^N = 0$ for all $r \in R$, then R is nilpotent, i.e. there exists an $\mathcal{N} = \mathcal{N}(N)$ such that $r_1 \cdots r_{\mathcal{N}} = 0$ for all $r_1, \cdots, r_{\mathcal{N}} \in R$.

Theorem: (Formanek (1986), Procesi (1976& 1979), Razmyslov (1974))

Let $\mathcal{N}(N)$ be the class of nilpotency in the Nagata-Higman theorem. Then the algebra of invariants $\Omega_{nd}^{GL_N}$ is generated by the traces $\mathbf{tr}(X^{i_1} \cdots X^{i_m})$ of degree $\leq \mathcal{N}(N)$. For d sufficiently large this bound is sharp.

Razmyslov (1974) $\mathcal{N}(N) \leq N^2$; Kuzmin (1975) $\mathcal{N}(n) \geq \frac{1}{2}N(N+1)$.

Hence: $$\frac{1}{2}N(N+1) \leq \mathcal{N}(N) \leq N^2.$$

See page 8 of V. Drensky, *Computing with Matrix Invariants*, arXiv:math/0506614.

Hilbert-Poincaré series: Molien-Weyl formula

Theorem (Teranishi 1986)

The Hilbert-Poincaré series for the system on N matrices is given by the Molien-Weyl formula:

$$Z_{U(N)}(t_1, \cdots, t_d) = \frac{1}{N!} \int \prod_{l=1}^{N} \frac{dz_l}{2\pi i z_l} \Delta(z) \Delta(\frac{1}{z}) \prod_{i=1}^{d} \prod_{l,m=1}^{N} \frac{1}{1 - t_i z_l z_m^{-1}}$$

with $\Delta(z)$ the Vandermonde determinant. For small N and small d the integrals can be performed exactly and some results are known.

$Z_N(t_1, t_2)$ have been evaluated up to $N = 6$ and $Z_7(t, t)$ was evaluated in Kristensson et al arXiv:2005.06480.

The invariants of 2×2 matrices

Two matrices X and Y

$$Z_2(t_1, t_2) = \frac{1}{(1-t_1)(1-t_2)(1-t_1^2)(1-t_1 t_2)(1-t_2^2)}$$

The invariants are built from $\mathbf{tr}(X)$, $\mathbf{tr}(X^2)$, $\mathbf{tr}(Y)$, $\mathbf{tr}(Y^2)$ and $\mathbf{tr}(X.Y)$.

Three matrices X, Y and Z

$$Z_2(t_1, t_2, t_3) = \frac{1 + t_1 t_2 t_3}{\prod_{a=1}^{3}(1-t_a)\prod_{b\leq c=1}^{3}(1-t_b t_c)}$$

The term $t_1 t_2 t_3$ indicates that we need $\mathbf{tr}(X.Y.Z)$ but not higher powers—it satisfies a quadratic relation. It captures a $\mathbb{Z}_2$ invariant. The highest product appearing in the generating set is 2 consistent with the lower bound $\mathcal{N}(2) = 3$.

Schur Polynomials

The low lying states and Schur Polynomials

$$Z_N(\rho t_1, \rho t_2, \rho t_3) = 1 + s_{(1,0,0)}\rho + 2s_{(2,0,0)}\rho^2 + (2s_{(3,0,0)} + s_{(2,1,0)} + s_{(1,1,1)})\rho^3 + \cdots$$

where

$$s_{(1,0,0)} = t_1 + t_2 + t_3,\ s_{(2,0,0)} = t_1^2 + t_1 t_2 + t_2^2 + t_2 t_3 + t_3^2 + t_3 t_1$$
$$s_{(3,0,0)} = t_1^3 + t_1^2 t_2 + \cdots,\ s_{(2,1,0)} = t_1^2 t2 + t_2 t_1^2 + \cdots$$
$$s_{(1,1,1)}) = t_1 t_2 t_3$$

Traceless matrices

$$\prod_{a=1}^{3}(1 - t_a) Z_N(\rho t_1, \rho t_2, \rho t_3) = 1 + s_{(2,0,0)}\rho^2 + s_{(1,1,1)}\rho^3 + \cdots$$

The Molien-Weyl formula from Path Integrals

A Gauge Gaussian Model

$$S[X,A] = \frac{1}{2}\int_0^\beta d\tau\,\mathrm{Tr}\left\{(D_\tau X)^2 + X^2\right\} \qquad D_\tau = \partial_\tau + i[A,\cdot]\,.$$

$$Z = \int [dX][dA]\mathrm{e}^{-S[X,A]-E_0}$$

$$D_\tau X \xrightarrow{lat} \frac{U_{n,n+1}X_{n+1}U_{n+1,n} - X_n}{a}\;, U_{n,n+1} = \mathcal{P}\mathrm{e}^{i\int_{na}^{(n+1)a} d\tau\, A(\tau)}\;,$$

with $\mathcal{P}$ a path ordered product, $U_{n+1,n} = U^\dagger_{n,n+1}$.

$$S_{lat} = \sum_{n=0}^{\Lambda-1} \mathbf{tr}\left\{\frac{1}{a}(X_n^2 - X_n U_{n,n+1} X_{n+1} U^\dagger_{n,n+1}) + \frac{a}{2}X_n^2\right\}\;,$$

$Z(t) = \frac{1}{N!} \int_{-\pi}^{\pi} \frac{d\theta_1 \cdots d\theta_N}{(2\pi)^N} \mathrm{e}^{-S(\theta)}$

$$
\begin{aligned}
S(\theta) \quad = \quad & N \ln(1-t) + \frac{1}{2} \sum_{i \neq j=1}^{N} \ln \left|1 - t\mathrm{e}^{i(\theta_i - \theta_j)}\right|^2 \\
& - \frac{1}{2} \sum_{i \neq j=1}^{N} \ln \left|1 - \mathrm{e}^{i(\theta_i - \theta_j)}\right|^2
\end{aligned}
$$

The last sum is from the Vandermonde due to diagonalisation of U.

Performing the contour integrals yields

$$
Z_N(t) = \prod_{m=1}^{N} \frac{1}{1 - t^m} = \sum_{n=0}^{\infty} p_N(n) t^n
$$

Summary:

The Euclidean action with d matrices

$$S[X,A] = \frac{1}{2}\int_0^\beta d\tau \sum_{k=1}^{d} \mathrm{Tr}\left\{(D_\tau X^k)^2 + m_k^2 (X^k)^2\right\},$$

$$Z(t_1,\cdots,t_d) = \int \frac{d\theta_1\cdots d\theta_N}{(2\pi)^N N!}\mathrm{e}^{-S(\theta,d)} \qquad t_k = \mathrm{e}^{-m_k\beta}$$

Molien-Weyl (Hilbert-Poincaré series) formula=partition function

$$S(\theta,d) = \sum_{k=1}^{d}\left\{N\ln(1-t_k) + \frac{1}{2}\sum_{i\neq j=1}^{N}\ln\left|1-t_k\mathrm{e}^{i(\theta_i-\theta_j)}\right|^2\right\} - \frac{1}{2}\sum_{i\neq j=1}^{N}\ln\left|1-\mathrm{e}^{i(\theta_i-\theta_j)}\right|^2.$$

The first term arises from $i=j$ in the double sum.

For $t_1 = t_2 = \cdots = t_d = t$

$$Z_N(t, \cdots, t) = Z_N(t, d) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dz_i}{2\pi i z_i} \frac{\Delta(\{z\})\Delta(\{z^{-1}\})}{(1-t)^d \Delta(t, \{z\})^d}$$

$$= \sum_n \dim_n(N, d) t^n.$$

The dimensions $\dim_n(N, d)$ will be our principal interest.

For $\mathcal{N} = 4$ SUSY Yang Mills $\dim_n(N, d)$ count BPS states, with $d = 1$ counting $\frac{1}{2}$-BPS sector and $d = 2$ counting the $\frac{1}{4}$-BPS sector.

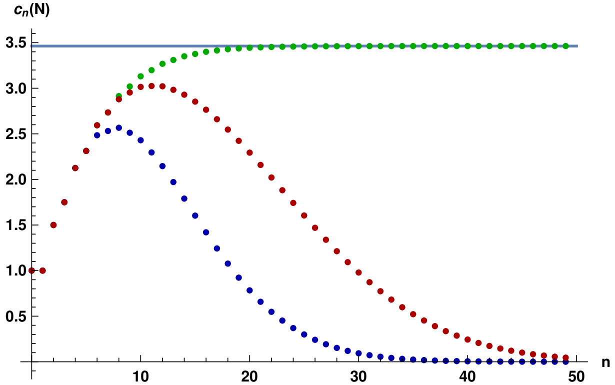
Dimensions for small N and n

$Z_N(t_1, t_2)$ $N = 2, \cdots 6$ and $Z_7(t, t)$ have been evaluated. Also, for $t_i \ll 1$, one can show (F. Dolan arXiv:0704.1038) that

$$Z_\infty(t_1, \cdots, t_d) = \prod_{n=1}^{\infty} \frac{1}{1 - \sum_{i=1}^{d} t_i^n}$$

$$Z_\infty(t, d) = \prod_{n=1}^{\infty} \frac{1}{1 - dt^n} .$$

$\frac{\dim_N(n,2)}{2^n}$ for N=5,7 and ∞ and $\Phi(\frac{1}{2})^{-1}$



Fermionic Matrix Models

We can ask the same questions for fermionic systems.

States for a single 2×2 matrix

$$|0\rangle, \quad \mathbf{tr}(b^\dagger)|0\rangle >, \mathbf{tr}((b^\dagger)^3)|0\rangle, \mathbf{tr}((b^\dagger)^3\mathbf{tr}(b^\dagger)|0\rangle$$

So the partition function is:

$$Z_2(t) = 1 + t + t^3 + t^4 = (1+t)(1+t^3)$$

$N \times N$ single matrix

$$Z_N(t) = \prod_{n=1}^{N}(1 + t^{2n-1})$$

Finite N fermionic two matrix model

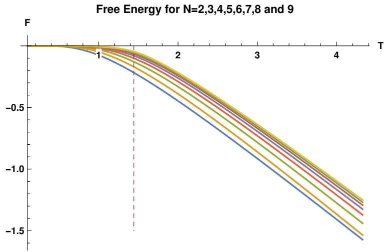
$$Z_2(t_1,t_2)=(1+t_1)(1+t_2)(1+t_1^3+t_1t_2+t_1^2t_2+t_1t_2^2+t_1^2t_2^2+t_2^3+t_1^3t_2^3)$$

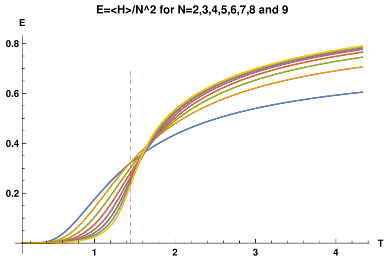
Palendromic—due to fermion hole symmetry

$$Z_2(t)=1+2t+2t^2+6t^3+10t^4+6t^5+2t^6+2t^7+t^8$$

$$Z_3(t)=1+2t+2t^2+6t^3+14t^4+26t^5+40t^6+50t^7+71t^8+88t^9$$
$$+71t^{10}+\cdots+2t^{17}+t^{18}$$

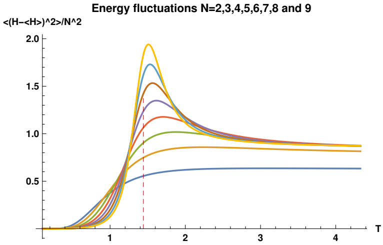
Fermionic Matrix Models:Small N Observables.

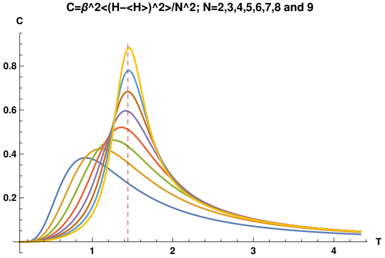




The Free Energy and Internal Energy for gauged Fermion matrix models.

Fermionic Matrix models

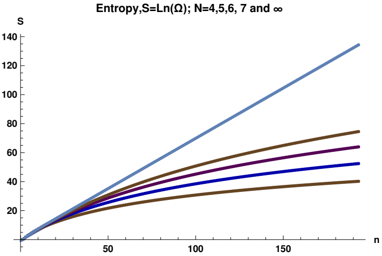


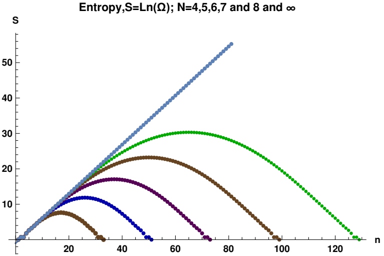


The Standard Deviation of the Energy and the Heat Capacity for gauged fermion matrix models.

The Entropy as a function of the Energy

An advantage of small N studies is that one can extract the Boltzmann entropy, $S = \ln \Omega$, from the partition function. Taking $t = \mathrm{e}^{-\beta}$ then the coefficients in a power series in t give Ω the number of states.



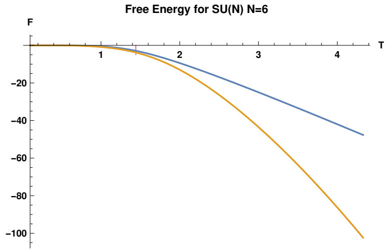


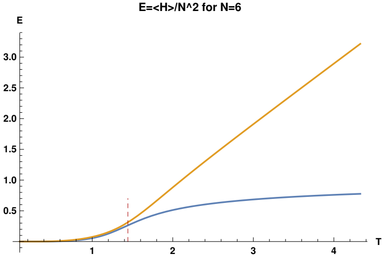
The Entropy vs Energy for pure Bosonic and Fermionic models.

Note:

$$Z_B^{SU(\infty)}(t,2) = \prod_{n=1}^{\infty} \frac{1}{1-2t^n} \qquad Z_F^{SU(\infty)}(t,2) = \prod_{n=1}^{\infty} \frac{1}{1+2(-t)^n}.$$

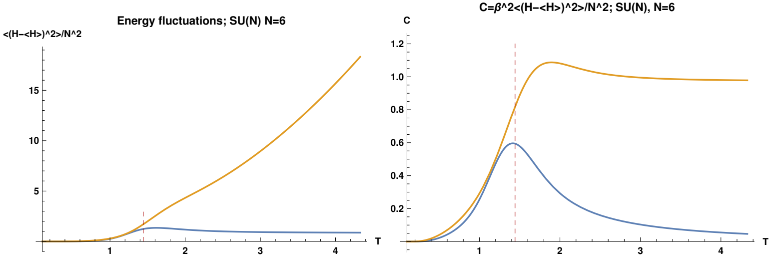
Comparing Bosonic and Fermionic Matrix models





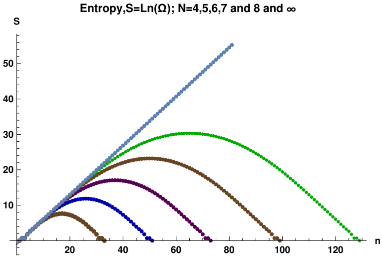
The Free Energy and Internal Energy for $N = 6$ of Bosonic and Fermionic models.

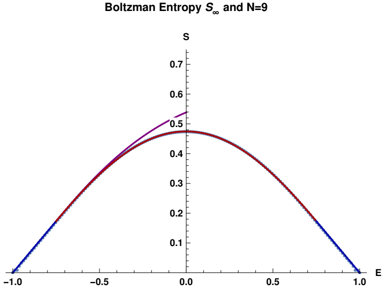
Comparing Bosonic and Fermionic Matrix models



The Standard Deviation of the Energy and the Heat Capacity for $N = 6$ of Bosonic and Fermionic models.

Limiting Entropy





Note: The diagram has been centered by restoring the zero-point energy so that $E = \frac{n-N^2}{N^2}$. The flat edges in blue have $\frac{dS(E)}{dE} = \beta_H = \ln 2$. **The flat region at low energy ($E = -1$ to $E = -\frac{3}{4}$) is universal and agrees with $N = \infty$.** Generalised Caley-Hamilton relations enter at $E = -1 + \frac{1}{N}$ but only become important at $E = -1 + \frac{1}{4}$ where they become dominant.

Large N low temperature fermionic mode, d matrices

Large N ignoring traces

$$Z_N(t) = \prod_{n=1}^{\infty} \frac{1}{1 + 2(-t)^n}$$

This reproduces the finite N fermionic coefficients up to $k = 2N - 1$

A large N analysis

Low temperature, $\beta \to \infty \implies t_k \ll 1$

Expanding the t_i logarithms one finds

$$S(\theta, d) = -N^2 \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{d} t_k^n}{n} |u_n|^2 - \frac{1}{2} \sum_{i \neq j = 1}^{N} \ln \left|1 - \mathrm{e}^{i(\theta_i - \theta_j)}\right|^2 .$$

where $u_n = \frac{1}{N} \sum_{i=1}^{N} \mathrm{e}^{in\theta_i}$. The partition function becomes

$$Z(t_1, \cdots, t_d) = \int [dU] \exp[\sum_{n=1}^{\infty} \frac{a_n}{n} \mathbf{tr}(U^n) \mathbf{tr}(U^{-n})]$$

Keeping only the $n = 1$ term gives the a_1 model

The a_1 model.

$$Z(a_1) = \int [dU] \mathrm{e}^{a_1 \mathbf{tr}(U) \mathbf{tr}(U^{-1})}$$

The Hagedorn (confining/deconfining) Phase Transition.

High Temperature (small β)

$$S[X,A] = \frac{1}{2}\int_0^\beta d\tau \, \mathrm{Tr}\left\{(D_\tau X)^2 + X^2\right\} \qquad D_\tau = \partial_\tau + i[A,\cdot]$$

for β small becomes the random matrix model

$$S[X,A] \simeq \frac{\beta}{2}\mathrm{Tr}\left\{-[A,X]^2 + X^2\right\}$$

The eigenvalues of βA, the θ_i, are distributed roughly with a Wigner semi-circle distribution.

For $\beta \to 0$

$$Z_N(t,d) \sim \beta^{(d-1)N^2} = \mathrm{e}^{(d-1)N^2 \ln(-\ln t)}$$

$$\dim_n(N,d) \sim \mathrm{e}^{N^2(d-1)\ln n}$$

The transition Point

From

$$S(\theta, d) \simeq N^2 \sum_{n=1}^{\infty} \frac{(1-a_n)}{n} |u_n|^2 \,,$$

we see that the transition occurs at $a_1 = 1$ where the coefficient of $|u_1|^2$ changes sign. For $a_1 = \sum_{i=1}^{d} t_i = d\mathrm{e}^{-\beta}$ the transition occurs at $T_H = \frac{1}{\beta_H} = \frac{1}{\ln d}$.

If we integrate over u_n (Aharoney et al arXiv:hep-th/0310285) and set $Z_\infty = 1$ for $a_n = 0$, we obtain

$$Z_\infty = \prod_{n=1}^{\infty} \frac{1}{1-a_n} = \prod_{n=1}^{\infty} \frac{1}{1-\sum_{i=1}^{d} t_i^n}$$

F. Dolan arXiv:0704.1038 obtained this for $d = 2$ by exact methods. Though the result is exact for $d = 1$ it breaks down for $a_1 \to 1$, but still allows us to count low energy states count states at large N.

Counting states at large N

$$Z_\infty(t,d) = \prod_{n=1}^{\infty} \frac{1}{1-dt^n} = \sum_{n=1}^{\infty} \mathrm{dim}_n(\infty,d)t^n$$

This is dominated by the $n=1$ term so one gets $\mathrm{dim}_n(\infty,d) \sim d^n$. A more careful estimate gives

$$\mathrm{dim}_n(\infty,d) \sim \frac{1}{\phi(\frac{1}{d})}d^n \qquad \text{with } \phi(q) \text{ the Euler function.}$$

To estimate $\mathrm{dim}_n(N,d)$ for large n and and large N we use the a_1 model.

The a_1 model in detail

$$Z(a_1) = \int [dU] \mathrm{e}^{a_1 \mathbf{tr}(U)\mathbf{tr}(U^{-1})}$$

Expanding directly in a_1 gives

$$Z(a_1) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_R [d_R(S_k)]^2 a_1^k$$

where $d_R(S_k)$ is the dimension of the representation R of the permutation group S_k.

$$d_{n_1,\cdots,n_N}(S_k) = (n_1 + \cdots n_N)! \frac{\prod_{i=1}^{N-1} \prod_{j=i+1}^{N} (n_i - i - (n_j - j))}{\prod_{i=1}^{N} (n_i + N - i)!}$$

We have

$$\frac{1}{k!} \sum_R [d_R(S_k)]^2 = 1 \quad k \leq N$$

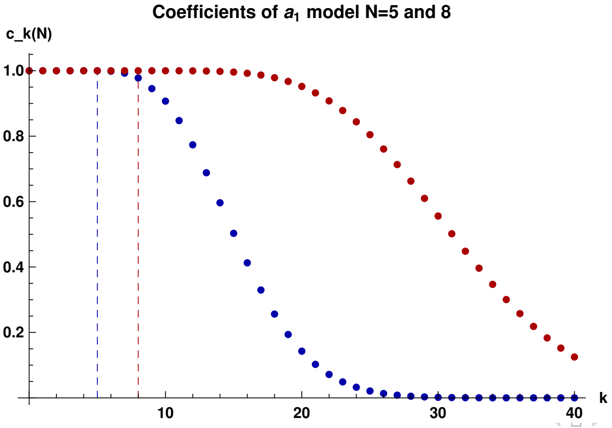
and decreases slowly above N (at least initially).

Coefficients of a_1 model

The N dependence of the a_1 model

The coefficients

$$c_k(N) = \frac{1}{k!}\sum_R [d_R(S_k)]^2$$



Solving the a_1 model for $a_1 > 1$

For large N we wish to solve

$$Z_N(\theta, d) = \sum_{k=1}^{d} \frac{1}{2} \sum_{i,j=1}^{N} \ln |1 - t_k \mathrm{e}^{i(\theta_i - \theta_j)}|^2 - \frac{1}{2} \sum_{i \neq j = 1}^{N} \ln |1 - \mathrm{e}^{i(\theta_i - \theta_j)}|^2 .$$

for $\theta_n \to \theta(n)$ with $\frac{dn}{d\theta} = \rho(\theta)$ and

$$\frac{Z(\rho)}{N^2} = \frac{1}{2} \sum_{k=1}^{d} \int \rho(\alpha) \int \rho(\beta) \ln |1 - t_k \mathrm{e}^{i(\alpha - \beta)}|^2 d\alpha d\beta - \frac{1}{2} P \int \rho(\alpha)\rho(\beta) \ln |1 - \mathrm{e}^{i(\alpha - \beta)}|^2 d\alpha d\beta .$$

The large N a_1 model is solvable

$$\frac{S_{a_1}}{N^2} = -a_1|u_1|^2 - \frac{1}{2}P\int \rho(\alpha)\rho(\beta)\ln|1-\mathrm{e}^{i(\alpha-\beta)}|d\alpha d\beta\,.$$

Has the solution

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for} \quad a_1 < 1 \\ \frac{1}{\pi\sin^2(\frac{\theta_0}{2})}\sqrt{\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})}\cos(\frac{\theta}{2}) & \text{for} \quad a_1 > 1 \end{cases} \tag{1}$$

and θ_0 is specified by

$$s^2 \equiv \sin^2(\frac{\theta_0}{2}) = 1 - \sqrt{1-\frac{1}{a_1}}. \tag{2}$$

The large N free energy as

$$-\frac{1}{N^2}\ln Z = \beta F = \begin{cases} 0 & a_1 < 1 \\ \frac{1}{2} - \frac{1}{2s^2} - \frac{1}{2}\ln s^2 & a_1 > 1 \end{cases}$$

Taking $a_1 = d\mathrm{e}^{-\beta}$ and expanding in the vicinity of the Hagadorn temperature we find with $\beta_H = \ln d$

$$-\frac{1}{N^2}\ln Z = \beta F = \begin{cases} 0 & \beta > \beta_H \\ \frac{\beta-\beta_H}{4} - \frac{1}{3}(\beta_H - \beta)^{3/2} + \cdots & \beta < \beta_H \end{cases}$$

The energy

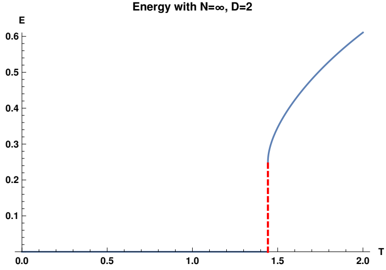
$$E = \frac{\partial(\beta F)}{\partial \beta} = \begin{cases} 0 & \beta > \beta_H \\ \frac{1}{4} + \frac{1}{2}\sqrt{\beta_H - \beta} + \cdots & \beta < \beta_H \,. \end{cases}$$

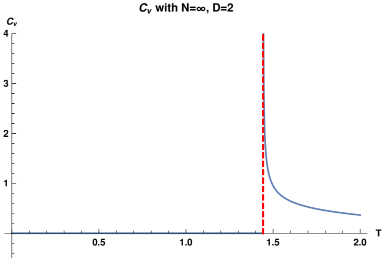
The transition occurs is $E = \frac{1}{4}$ or $n = \frac{N^2}{4}$.

The Phase Transition

The transition is NOT simply 1st order.

The transition has a divergent specific heat on either side of the transition. The stronger divergence appears to be on the low temperature side, but this is coming from subdominant contributions as the limit is approached.





Entropy from Free the Energy

$$\frac{dS}{dE} = \beta(E)$$

$$E = \frac{1}{4} + \frac{1}{2}\sqrt{\ln d - \beta} + \cdots \qquad \Longrightarrow \beta(E) = \ln d - 4(E - \frac{1}{4})^2 + \cdots$$

Inverting the expression for $E(\beta)$ and integrating gives and matching at $E = \frac{1}{4}$ gives

$$S(E) = E \ln d - \frac{4}{3}(E - \frac{1}{4})^3 + \cdots$$

The exact result from the a_1 model consistent with the eigenvalue distribution $\rho(\theta)$ is

$$S(E) = (1 + \ln d)E - \frac{1 + 2\ln 2}{4} + E \ln E - (E + \frac{1}{4})\ln(E + \frac{1}{4})^2$$

Matching across the transition

The transition occurs at $E = \frac{1}{4}$.
At low temperatures

$$Z_\infty(t,d) = \prod_{n=1}^{\infty} \frac{1}{1-dt^n} = \sum_{n=1}^{\infty} d^n t^n = \mathrm{e}^{N^2 \ln dE}$$

using $E_n = \frac{n}{N^2}$

Entropy at the transition

$$S(\frac{1}{4}) = \frac{\ln d}{4}$$

So at the transition

$$n_c = \frac{N^2}{4}$$

The transition is one from where trace relations can be ignored to where they become significant.

Implications for Matrix Traces

We have found that

- The number of states grows with energy as $\dim_n(N,d) \sim d^n = \mathrm{e}^{N^2 \ln(d) E}$ (with $E = \frac{n}{N^2}$) below the transition.
- $\dim_n(N,d) \sim \mathrm{e}^{N^2\{c + \ln(d)E - \frac{4}{3}(E-\frac{1}{4})^3 + \cdots\}}$ above.
- The low n and large n entropy match at $E = \frac{1}{4}$.

Trace relations become dominant at $n = \frac{N^2}{4}$

Rephrasing:

Main result

- For large N and all $d > 1$ trace relations "switch on" at $n = \frac{N^2}{4}$.
  This is true for bosons or fermions or a mixture of these!

$$\dim_n(N,d) = \begin{cases} c(d)d^n & \frac{N^2}{4} \geq n \gg 1 \\ c(d)d^n \mathrm{e}^{-\frac{4N^2}{3}(\frac{n}{N^2}-\frac{1}{4})^3 \cdots} & n \geq \frac{N^2}{4} \,. \end{cases}$$

  with $c(d) = \frac{1}{\phi(\frac{1}{d})}$ with $\phi(q)$ the Euler function.

- For large N trace relations become significant for traces of length $\frac{N^2}{4}$ even though we know that traces as long as $\mathcal{N}(N) = \frac{N(N+1)}{2}$ still play a role providing new states.

Conclusions

Entropy, $S(n) = \frac{1}{N^2} \ln \mathrm{dim}_n(N, d)$

- $S(n)$ has universal large N transition at $n = \frac{N^2}{4}$

Speculation

In many matrix models the Hagedorn (confining/deconfining) transition is argued to be the Hawking-Pope transition in gravitational duals. In the Hawking-Page transition a large AdS black hole becomes unstable as its mass (temperature) is decreased and there is a transition to thermal particle gas.
Entropy of black-holes universal form $S_{BH} = \frac{A}{4G}$.
Is this a coincidence?

Thanks for Your Attention!