Geometry and curvature of diffeomorphism groups

Stephen C. Preston

CUNY Brooklyn College/Graduate Center

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For motivation, consider the Euler equations for a 2D ideal incompressible fluid (derived 1755–1757, the second PDE ever, after the wave equation).

Fluid particles starting at position (x, y) in the plane follow trajectory $\gamma(t, x, y) = (X(t, x, y), Y(t, x, y))$. They are subject to a potential called the *pressure* $p(t, x, y)$, and Newton's equations say

$$X_{tt}(t, x, y) = -p_x\big(t, X(t, x, y), Y(t, x, y)\big) \qquad \Rightarrow \qquad X_{tt} = -p_x(X, Y)$$
$$Y_{tt}(t, x, y) = -p_y\big(t, X(t, x, y), Y(t, x, y)\big) \qquad \Rightarrow \qquad Y_{tt} = -p_y(X, Y).$$

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Incompressibility means that the Jacobian determinant $\det D\gamma \equiv 1$, i.e.,

$$X_x Y_y - X_y Y_x \equiv 1.$$

The natural variables here are not the X and Y, but the partial derivatives X_x, X_y, Y_x, Y_y. So let's differentiate.

Differentiating the equations of motion with respect to x and y gives

$$\partial_t^2 X_x = -p_{xx}(X,Y)X_x - p_{xy}(X,Y)Y_x \qquad \partial_t^2 X_y = -p_{xx}(X,Y)X_y - p_{xy}(\eta,\xi)\xi_y$$
$$\partial_t^2 Y_x = -p_{xy}(X,Y)X_x - p_{xy}(X,Y)Y_x \qquad \partial_t^2 Y_y = -p_{xy}(X,Y)X_y - p_{yy}(\eta,\xi)\xi_y.$$

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$$X_{tx}Y_y + X_xY_{ty} - X_{ty}Y_x - X_yY_{tx} \equiv 0.$$

X and Y still appear inside the pressure Hessian, but we could reconstruct those from the partials.

Differentiate the constraint again in time to get

$$X_{ttx}Y_y + 2X_{tx}Y_{ty} + X_xY_{tty} - X_{tty}Y_x - 2X_{ty}Y_{tx} - X_yY_{ttx} \equiv 0.$$

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$$X_{ttx} Y_y + 2X_{tx} Y_{ty} + X_x Y_{tty} - X_{tty} Y_x - 2X_{ty} Y_{tx} - X_y Y_{ttx} \equiv 0.$$

Plug in the equations of motion to get

$$\begin{aligned} &Y_y\big(p_{xx}(X,Y)X_x + p_{xy}(X,Y)Y_x\big) + X_x\big(p_{xy}(X,Y)X_y + p_{yy}(X,Y)Y_y\big) \\ &\quad - Y_x\big(p_{xx}(X,Y)X_y + p_{xy}(X,Y)Y_y\big) - X_y\big(p_{xy}(X,Y)X_x + p_{xy}(X,Y)Y_x\big) = 2X_{tx}Y_{ty} - 2X_{ty}Y_{tx}. \end{aligned}$$

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Using the incompressibility $X_x Y_y - X_y Y_x = 1$, we cancel to get

$$p_{xx}(X,Y) + p_{yy}(X,Y) = 2X_{tx}Y_{ty} - 2X_{ty}Y_{tx}.$$

To get a first-order system we set $U = X_t$ and $V = Y_t$, which turns the equations into

$$\begin{aligned} \partial_t X &= U & \partial_t Y &= V \\ \partial_t X_x &= U_x & \partial_t Y_x &= V_x \\ \partial_t X_y &= V_y & \partial_t Y_y &= V_y \end{aligned}$$

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Now replacing X_{tt} with U_t and Y_{tt} with V_t, we get

$$\begin{aligned} \partial_t U &= -p_x(X, Y) & \qquad \partial_t V &= -p_y(X, Y)\partial_t \\ U_x &= -p_{xx}(X, Y)X_x - p_{xy}(X, Y)Y_x & \qquad \partial_t U_y &= -p_{xx}(X, Y)X_y - p_{xy}(\eta, \xi)\xi_y \\ \partial_t V_x &= -p_{xy}(X, Y)X_x - p_{xy}(X, Y)Y_x & \qquad \partial_t V_y &= -p_{xy}(X, Y)X_y - p_{yy}(\eta, \xi)\xi_y. \end{aligned}$$

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Meanwhile the constraint is given by

$$p_{xx}(X,Y) + p_{yy}(X,Y) = 2(U_x V_y - U_y V_x).$$

$$\frac{\partial}{\partial t}\begin{pmatrix} X_x \\ Y_x \\ X_y \\ Y_y \\ U_x \\ V_x \\ U_y \\ V_y \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ p_{xx}(X,Y) & p_{xy}(X,Y) & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{xy}(X,Y) & p_{yy}(X,Y) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{xx}(X,Y) & p_{xy}(X,Y) & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{xy}(X,Y) & p_{yy}(X,Y) & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_x \\ Y_x \\ X_y \\ Y_y \\ U_x \\ V_x \\ U_y \\ V_y \end{pmatrix}$$

Pressure p solves Laplace equation $p_{xx}(X,Y) + p_{yy}(X,Y) = 2(U_x V_y - U_y V_x)$, where $U = X_t$, $V = Y_t$.

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Roughly speaking, if $(x, y) \mapsto (X, Y)$ is a C^1 diffeomorphism, then all the terms in this vector Ω are continuous functions of (x, y), and F is a Lipschitz function of Ω, so this is an ODE in the space $C(\mathbb{R}^2, \mathbb{R}^8)$, which can be solved by Picard iteration: $\Omega_{n+1}(t) = \Omega(0) + \int_0^t F(\Omega_n(\tau))\Omega_n(\tau)\,d\tau$.

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This is a contraction mapping on the Banach space $C([0,T], C(\mathbb{R}^2,\mathbb{R}^8))$ for T small, giving a local solution as in the Fundamental Theorem of ODEs.

It's true that this does not actually work. If q is a continuous function on $\mathbb{R}^2$ and $\Delta p = q$, then p is not quite C^2, so $\nabla^2 p$ is not quite continuous.

One really needs to work in a space of functions slightly better than continuous, like Hölder spaces C^α for $\alpha > 0$ or Sobolev spaces H^s for $s > 2$.

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However *morally* it is true that the natural configuration space of ideal fluids is (slightly better than) C^1. In such a space the Lagrangian approach (following particle paths) gives an **infinite-dimensional ordinary differential equation** for which the solution can be proved to exist. (Euler apparently thought the exact solution would be found soon afterwards in the 1750s.)

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This is essentially the Ebin-Marsden (1970) approach to proving existence of solutions, inspired by Arnold (1966). The Riemannian exponential map from the initial condition (u_0, v_0) to $\gamma(t)$ is *smooth*.

This is not what Euler actually did. Euler observed that in a river the particle paths would change their velocities a lot, but the basic velocity field often would not. So he defined the velocity field $(u(t,x,y), v(t,x,y))$ such that

$$X_t(t,x,y) = u\big(t, X(t,x,y), Y(t,x,y)\big), \qquad Y_t(t,x,y) = v\big(t, X(t,x,y), Y(t,x,y)\big).$$

One could then hope that u and v don't explicitly depend on time t.

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The chain rule gives

$$\begin{aligned} X_{tt} &= u_t(t,X,Y) + u_x(t,X,Y)X_t + u_y(t,X,Y)Y_t \\ &= u_t(t,X,Y) + u(t,X,Y)u_x(t,X,Y) + v(t,X,Y)u_y(t,X,Y) \end{aligned}$$

so that

$$u_t(t,X,Y) + u(t,X,Y)u_x(t,X,Y) + v(t,X,Y)u_y(t,X,Y) = -p_x(t,X,Y)$$

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No compositions with functions X and Y anymore!

Incompressibility is also simple:

$$u_x + v_y = 0.$$

Thus the pressure function satisfies

$$\begin{aligned} -(p_{xx} + p_{yy}) &= \partial_t(u_x + v_y) + u\partial_x(u_x + v_y) + v\partial_y(u_x + v_y) + (u_x^2 + 2v_x u_y + v_y^2) \\ &= u_x^2 + 2v_x u_y + v_y^2. \end{aligned}$$

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We've simplified a lot (and forgotten the particle paths) but we've lost something: it's not an ODE anymore for (u, v)!

$$\begin{aligned} u_t &= -uu_x - vu_y - p_x \\ v_t &= -uv_x - vv_y - p_y. \end{aligned}$$

The right side is **not** a continuous function of (u, v) in any Banach space. So Picard iteration doesn't work to prove existence.

An example where this works more cleanly: the **Camassa-Holm equation**, a PDE for $u(t,x)$ for $x \in \mathbb{R}$ meant to describe water waves that break. Let $a > 0$. Several versions of the same equation:

$$\omega_t + u\omega_x + 2u_x\omega = 0, \qquad \text{where} \qquad \omega = a^2 u - u_{xx}, \ \text{or}$$
$$a^2 u_t - u_{txx} + 3a^2 u u_x - u u_{xxx} - 2u_x u_{xx} = 0, \ \text{or}$$
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Clearly **not** an ODE for u. We can construct p, and reconstruct u from ω, using the same Green function (for each fixed time t):

$$u(x) = \frac{e^{-ax}}{2a} \int_{-\infty}^{x} e^{ay} \omega(y)\, dy + \frac{e^{ax}}{2a} \int_{x}^{\infty} e^{-ay} \omega(y)\, dy,$$
$$p(x) = \frac{e^{-ax}}{2a} \int_{-\infty}^{x} e^{ay} \left[a^2 u(y)^2 + \tfrac{1}{2} u'(y)^2\right] dy + \frac{e^{ax}}{2a} \int_{x}^{\infty} e^{-ay} \left[a^2 u(y)^2 + \tfrac{1}{2} u'(y)^2\right] dy.$$

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Note that there is a conservation law for the Sobolev H^1 energy:

$$\int_{-\infty}^{\infty} a^2 u(t,x)^2 + u_x(t,x)^2\,dx = \int_{-\infty}^{\infty} a^2 u_0(x)^2 + u_0'(x)^2\,dx.$$

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Using the formula for p we get (letting $Y = \gamma(t, y)$)

$$\begin{aligned}
\frac{\partial^2\gamma}{\partial t^2}(t, x) &= \frac{1}{2}\int_{-\infty}^{X} e^{a(Y-X)}\left[a^2 u(t, Y)^2 + \tfrac{1}{2}u_y(t, Y)^2\right] dY - \frac{1}{2}\int_{X}^{\infty} e^{a(X-Y)}\left[a^2 u(t, Y)^2 + \tfrac{1}{2}u_y(t, Y)^2\right] dY \\
&= \frac{1}{2}\int_{-\infty}^{x} e^{a(\gamma(t,y)-\gamma(t,x))}\left[a^2\gamma_t(t, y)^2 + \frac{1}{2}\frac{\gamma_{ty}(t, y)^2}{\gamma_y(t, y)^2}\right]\gamma_y(t, y)\, dy \\
&\quad - \frac{1}{2}\int_{x}^{\infty} e^{a(\gamma(t,x)-\gamma(t,y))}\left[a^2\gamma_t(t, y)^2 + \frac{1}{2}\frac{\gamma_{ty}(t, y)^2}{\gamma_y(t, y)^2}\right]\gamma_y(t, y)\, dy.
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This is an autonomous dynamical system for the variables γ, $\phi := \gamma_x$, $U := \gamma_t$, and $V := \gamma_{tx}$, all of which are in $C(\mathbb{R},\mathbb{R})$.

$$\frac{d\gamma}{dt} = U, \qquad \frac{dU}{dt} = \frac{e^{-a\gamma}}{2}\int_{-\infty}^{x} e^{a\gamma}\left[a^2U^2 + \frac{1}{2}\frac{V^2}{\phi^2}\right]\phi\,dy - \frac{e^{a\gamma}}{2}\int_{x}^{\infty} e^{-a\gamma}\left[a^2U^2 + \frac{1}{2}\frac{V^2}{\phi^2}\right]\phi\,dy$$

$$\frac{d\phi}{dt} = V, \qquad \frac{dV}{dt} = a^2U^2\phi + \frac{V^2}{\phi} - \frac{a\phi e^{-a\gamma}}{2}\int_{-\infty}^{x} e^{a\gamma}\left[a^2U^2 + \frac{1}{2}\frac{V^2}{\phi^2}\right]\phi\,dy - \frac{a\phi e^{a\gamma}}{2}\int_{x}^{\infty} e^{-a\gamma}\left[a^2U^2 + \frac{1}{2}\frac{V^2}{\phi^2}\right]\phi\,dy$$

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Looks complicated! But it's an **ODE** for $(\gamma, \phi, U, V) \in C(\mathbb{R}, \mathbb{R}^4)$, with a **smooth vector field**.

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Theorem (Misiołek 2002, Lee-P. 2017)

For any initial velocity $u_0 \in C^1_0(\mathbb{R}) \cap H^1(\mathbb{R})$ there is a time $T > 0$ and a unique solution of the system above with $\gamma(0) = \mathrm{id}$, $\phi(0) \equiv 1$, $U(0) = u_0$, $V(0) = u_0'$ with $(\gamma, U) \in C^1\big([0, T), C^1(\mathbb{R}, \mathbb{R})\big)$ and $(\phi, V) \in C^0([0, T), C^0(\mathbb{R}, \mathbb{R})$.
In particular there is a unique C^1 solution $u\colon [0, T) \times \mathbb{R} \to \mathbb{R}$ of the Camassa-Holm equation which depends continuously on the initial condition u_0, given by $u = U \circ \gamma^{-1}$.

This may not seem like such a big deal, except that every proof of local well-posedness of Camassa-Holm works in a stronger space such as Sobolev spaces $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ (Li-Olver 2000, Rodriguez-Blanco 2001) or Besov spaces $B^s_{p,r}(S^1)$ with $s > \max\{1 + 1/p, 3/2\}$ (Danchin 2001). Meanwhile $C^1 \cap H^1$ initial velocity is the optimal regularity. In the space $W^{1,\infty} \cap H^1$ there are peakon solutions, and continuous dependence on u_0 is *false* (Linares-Ponce-Sideris 2016).

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The only known way to get optimal regularity is using the Lagrangian approach as an ODE on the diffeomorphism group rather than PDE approaches. This is how geometry is especially useful: not just to reinterpret known equations in a beautiful way, but to get genuinely improved results. This is the utility of the Ebin-Marsden approach.

Alternative geometric approach:

$$\omega_t + u\omega_x + 2u_x\omega = 0, \qquad \gamma_t = u \circ \gamma, \qquad \gamma_{tx} = (u_x \circ \gamma)\gamma_x$$

imply that

$$\gamma_x(t,x)^2\omega(t,\gamma(t,x)) = \omega_0(x).$$

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Since $u = \gamma_t$ can be reconstructed from ω, we get a differential equation for γ and its spatial derivative ϕ:

$$\frac{d\gamma}{dt}(x) = \frac{e^{-a\gamma(x)}}{2a} \int_{-\infty}^{x} e^{a\gamma(y)} \frac{\omega_0(y)}{\phi(y)}\,dy + \frac{e^{a\gamma(x)}}{2a} \int_{x}^{\infty} e^{-a\gamma(y)} \frac{\omega_0(y)}{\phi(y)}\,dy$$

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Again this is a locally Lipschitz ODE for any given initial momentum $\omega_0 = a^2u_0 - u_0''$, but now it's a *first-order* equation on the diffeomorphism group. We can get a local existence result as long as $\omega_0 \in L^1(\mathbb{R})$.

In the limit as $a \to 0$ we get an equation for ϕ alone, the Hunter-Saxton equation (Lenells 2007, 2008, Bauer-Bruveris-Michor 2014, Sarria-Saxton 2015)

$$\frac{\partial \phi}{\partial t}(t,x) = -\frac{\phi(t,x)}{2}\int_{-\infty}^{x} \frac{\omega_0(y)}{\phi(t,y)}\,dy + \frac{\phi(t,x)}{2}\int_{x}^{\infty} \frac{\omega_0(y)}{\phi(t,y)}\,dy.$$

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This is essentially the Liouville equation (1853). The exact solution is

$$\gamma_x(t,x) = \phi(t,x) = \left(1 + \tfrac{1}{2} t u_0'(x)\right)^2.$$

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$$u_x(t,\gamma(t,x)) = \frac{\gamma_{tx}(t,x)}{\gamma_x(t,x)} = \frac{u_0'(x)}{1 + \frac{1}{2} t u_0'(x)} \to -\infty.$$

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Hence u remains in $\dot{H}^1$ but leaves C^1 in finite time (while γ remains C^1 for all time but leaves the diffeomorphism group).

$$\|u(t)\|_{\dot{H}^1}^2 = \int_{-\infty}^{\infty} u_x(t,X)^2\, dX = \int_{-\infty}^{\infty} \frac{\gamma_{tx}(t,x)^2}{\gamma_x(t,x)}\, dx = \int_{-\infty}^{\infty} u_0'(x)^2\, dx.$$

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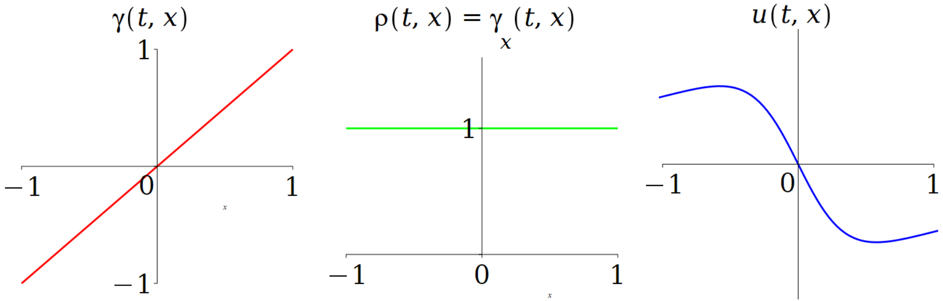
This gap between the Riemannian energy $\dot{H}^1$ and the natural diffeomorphism group topology C^1 is the source of all difficulties in this business.

We can use breakdown of the Hunter-Saxton equation to do **ODE comparison theory** for other infinite-dimensional ODEs (Bauer-P.-Valletta 2024), by showing they break down at least as fast.

Example: Solution of Hunter-Saxton equation on $\mathbb{R}$ with $u_0(x) = -\dfrac{x}{1+3x^2}$.



There is one situation where things are much easier. If instead of the H^1 Riemannian metric, we use the H^2 Riemannian metric to define the geodesic equation, then we'd get the equation

$$\omega_t + u\omega_x + 2u_x\omega = 0, \qquad \omega := (a^2 - \partial_x^2)^2 u.$$

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$$\|u(t)\|_{H^2}^2 = \int_{-\infty}^{\infty} a^4 u(t,x)^2 + 2a^2 u_x(t,x)^2 + u_{xx}(t,x)^2 \, dx = \|u_0\|_{H^2}^2.$$

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The Sobolev embedding theorem says that

$$\|u\|_{C^1(\mathbb{R})} \lesssim \|u\|_{H^s(\mathbb{R})} \qquad \text{whenever} \quad s > \tfrac{3}{2},$$

so now we have a bound on $\|u(t)\|_{C^1}$ globally. This guarantees global existence of all higher derivatives as well, if they exist initially.

Note that the H^0 inertia operator, where $\omega = u$, gives the equation

$$u_t + 3uu_x = 0.$$

At first glance this looks easy, but using $\gamma_t = u \circ \gamma$, it becomes

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which is *not* an ODE for γ.

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In summary: the equation $\omega_t + u\omega_x + 2u_x\omega = 0$ with $\omega = (a^2 - \partial_x^2)^k$ and $\gamma_t = u \circ \gamma$ is:

- not an ODE for γ when $k = 0$;
- an ODE with local but not global solutions when $k = 1$;
- an ODE with global solutions when $k = 2$.

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It also makes sense to consider noninteger values of k, using the Fourier transform.

- When $k = \frac{1}{2}$ and $a = 0$ we get a modified Constantin-Lax-Majda equation (Okamoto-Sakajo-Wunsch 2008, Bauer-Kolev-P. 2015)
- When $k = \frac{3}{2}$ and $a = 0$ we get the Euler-Weil-Petersson equation from Teichmüller theory (Gay-Balmaz-Ratiu 2015)

General framework: diffeomorphism group $G = \mathrm{Diff}(M)$, with a right-invariant Riemannian metric. Specifically $\mathrm{Diff}(M)$ or $\mathrm{Diff}_{\mathrm{vol}}(M)$ for an n-dimensional manifold M, and a Sobolev H^k metric given at the identity by a Sobolev-type operator $\Lambda = (I - \Delta)^k$ for $k \geq 0$, with $\langle u, u\rangle = \int_M u \cdot \Lambda u \, d\mathrm{vol}$.

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Geodesics are given by

$$\gamma_t = u \circ \gamma, \qquad u_t + \mathrm{ad}_u^\star u = 0,$$

where $\langle \mathrm{ad}_u^\star u, v \rangle = \langle u, \mathrm{ad}_u v \rangle = -\langle u, [u, v] \rangle$ for all $u, v \in \mathfrak{g}$. For example, if $M = \mathbb{R}$ or $M = S^1$, the Euler-Arnold equation is

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It can become an ODE for γ, either by using the second-order approach (the full geodesic equation), or using momentum transport to get a first-order equation on $\mathrm{Diff}(M)$ in terms of ω_0.

Question: For which values of k is it true that the geodesic equation is smooth (or at least locally Lipschitz) and the Riemannian exponential map $u_0 \mapsto \gamma(t)$ defined for **small** t, in some topology on G? (False when $k = 0$, Constantin-Kolev 2008, probably also for all $k < \frac{1}{2}$).

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Partial answers on $\mathrm{Diff}(M)$:

- (Misiołek-P. 2010) If $k \in \mathbb{N}$, M is compact, and $\mathrm{Diff}(M)$ has the H^s topology for $s > 2k + n/2$.
- (Escher-Kolev 2012) If $k \geq \frac{1}{2}$ and $M = S^1$ with the H^s topology for $s > \frac{3}{2}$ and $s \geq 2k$.
- (Bauer-Escher-Kolev 2015) If $k \geq \frac{1}{2}$ and $M = \mathbb{R}^n$ or $M = \mathbb{T}^n$, in H^s for $s > 1 + n/2$ and $s \geq 2k$.
- (Misiołek 2002, Lee 2017) If $k = 1$ and $M = \mathbb{R}$ or $M = S^1$, with C^1 topology.
- (Bruveris-Vialard 2016, Bauer-Bruveris-Cismas-Escher-Kolev 2020) If $k > 1 + n/2$, M compact, H^k.
- (Bauer-Bruveris-Cismas-Escher-Kolev 2020, Bauer-Harms-Michor 2020) If $k \geq \frac{1}{2}$, M compact, H^s with $s > 1 + n/2$ and $s \geq 2k$.
- (Bauer-P.-Valletta 2024) If $k = 1$ or $k = 2$ with $\Lambda = (-\Delta)^k$ for radial diffeos on $\mathbb{R}^n$ in C^1.

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- (Escher-Kolev 2012) If $k \geq \frac{1}{2}$ and $M = S^1$ with the H^s topology for $s > \frac{3}{2}$ and $s \geq 2k$.
- (Bauer-Escher-Kolev 2015) If $k \geq \frac{1}{2}$ and $M = \mathbb{R}^n$ or $M = \mathbb{T}^n$, in H^s for $s > 1 + n/2$ and $s \geq 2k$.
- (Misiołek 2002, Lee 2017) If $k = 1$ and $M = \mathbb{R}$ or $M = S^1$, with C^1 topology.
- (Bruveris-Vialard 2016, Bauer-Bruveris-Cismas-Escher-Kolev 2020) If $k > 1 + n/2$, M compact, H^k.
- (Bauer-Bruveris-Cismas-Escher-Kolev 2020, Bauer-Harms-Michor 2020) If $k \geq \frac{1}{2}$, M compact, H^s with $s > 1 + n/2$ and $s \geq 2k$.
- (Bauer-P.-Valletta 2024) If $k = 1$ or $k = 2$ with $\Lambda = (-\Delta)^k$ for radial diffeos on $\mathbb{R}^n$ in C^1.

Partial answers on $\mathrm{Diff}_{\mathrm{vol}}(M)$:

- (Ebin-Marsden 1970) If $k = 0$, M compact, in the H^s topology for $s > n/2 + 1$.
- (Shkoller 2000) If $k = 1$, M compact, H^s for $s > n/2 + 1$ (Lagrangian-averaged Euler).
- (Washabaugh 2016) If $k = -\frac{1}{2}$, $n = 2$, for $C^{1,\alpha}$ (the SQG equation).

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Partial answers on $\mathrm{Diff}(M)$:

- (Camassa-Holm 1993, McKean 1998) False for $k = 1$ with $n = 1$.
- (Castro-Córdoba 2010, Bauer-Kolev-P. 2015) False for $k = \frac{1}{2}$ with $n = 1$.
- (Li-Yu-Zhai 2013) False for $k = 1$ and $n \geq 2$ on $\mathbb{R}^n$.
- (Escher-Kolev 2014) True for $k > \frac{3}{2}$ with $n = 1$.
- (Bruveris-Vialard 2014) True for $k > 1 + n/2$ for $M = \mathbb{R}^n$ or M compact.
- (P.-Washabaugh 2018, Bauer-Kolev-P. 2020) True for $k = \frac{3}{2}$ with $n = 1$.
- (Bauer-P.-Valletta 2024) False for $k = 2$ and $n \geq 3$ on $\mathbb{R}^n$.

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Unknown cases on $\mathrm{Diff}(M)$:

- $M = \mathbb{R}$ or $M = S^1$ for $0 < k < \frac{1}{2}$, $\frac{1}{2} < k < 1$, and $1 < k < \frac{3}{2}$. (Probably false.)
- $k \geq 3$ with $k \in \mathbb{N}$ on $\mathbb{R}^n$ with $n \geq 4$. (Probably false if $k < n/2 + 1$, Bauer-P.-Valletta.)
- Noninteger k on $\mathbb{R}^n$ for any other $k < n/2 + 1$. (Probably false.)
- Critical cases $k = n/2 + 1$ for $n \geq 2$. (Probably true for $k = n = 2$, conjectured true for higher n.)

Partial answers on $\mathrm{Diff}_{\mathrm{vol}}(M)$:

- (Wolibner 1933) True for $k = 0$ and $n = 2$.
- (Shkoller 2000) True for $k = 1$ and $n = 2$.
- Probably true for all $k > 0$ if $n = 2$.
- Unknown for $k = 0$ and $n = 3$ (the most famous unsolved problem in fluids).
- Unknown also for $k = 1$ and $n = 3$ though it should be easier than $k = 0$.
- Should be true for $k \geq 3$ and $n = 3$ due to strong metric.
- Unknown for $k = -\frac{1}{2}$ and $n = 2$ (the surface quasigeostrophic equation).

Also true for symplectomorphisms with $k = 0$ in any dimension (Ebin 2012) and quantomorphisms with $k = 0$ (Ebin-P. 2015), probably false for contactomorphisms with $k = 0$ (P.-Sarria 2014).

The linearized equations:

Once we establish existence of solutions we can ask about their stability (of linear perturbations). Differentiate the equations of motion with respect to a parameter s: recall
$\gamma_t = u \circ \gamma, \qquad u_t + \mathrm{ad}_u^\star u = 0.$

The Jacobi equation for a Jacobi field $J = y \circ \gamma = \frac{\partial \gamma}{\partial s}$ is

$$y_t - \mathrm{ad}_u\, y = z; \qquad z_t + \mathrm{ad}_u^\star\, z + \mathrm{ad}_z^\star\, u = 0.$$

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We get $\dfrac{dv}{dt} = w$ and $\dfrac{d}{dt}\Big(\mathrm{Ad}_\gamma^\star\, \mathrm{Ad}_\gamma\, w\Big) + \mathrm{Ad}_\gamma^\star\, \mathrm{ad}_{\mathrm{Ad}_\gamma\, w}^\star\, u = 0.$

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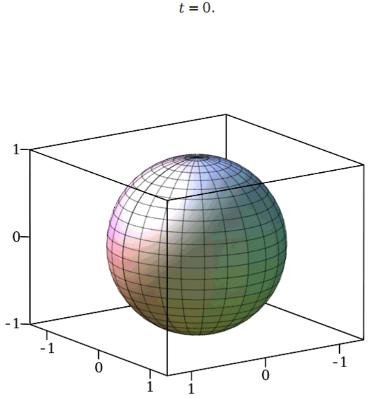
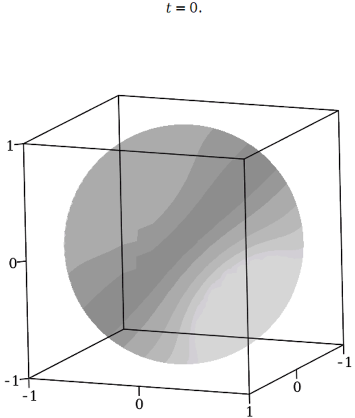
So the Jacobi equation for the left-translated Jacobi field v is

$$\frac{d}{dt}\left(\mathrm{Ad}^\star_{\gamma(t)}\, \mathrm{Ad}_{\gamma(t)}\, \frac{dv}{dt}\right) + K_{u_0}\left(\frac{dv}{dt}\right) = 0.$$

Example: rotations on the sphere

If $\eta(t)$ consists of isometries, then the stretching matrix $\Lambda(t)$ is the identity, and the left-translated Jacobi equation becomes $\frac{d^2V}{dt^2} + K_{U_0}\left(\frac{dV}{dt}\right) = 0$. Write $V(t) = \nabla^{\perp} g(t)$ for some function g, and this is $\frac{\partial^2 g}{\partial t^2} - 2\Delta^{-1}\frac{\partial}{\partial \theta}\frac{\partial g}{\partial t} = 0$. So K_{U_0} is compact and antisymmetric with eigenvalues $\frac{2in}{k(k+1)}$ for positive integers n and $k \leq n$.





If K_{u_0} is a **compact** operator, then the solution operator is **Fredholm** (Ebin-Misiołek-P. 2006, Misiołek-P. 2009).

Fredholm operators have nice properties:

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These all reflect properties of **conjugate points**, which occur at $\gamma(T)$ whenever $d\exp_{\gamma(T)}$ is noninvertible.

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It is remarkable that conjugate points only depend on the *initial velocity* u_0 and the stretching operator $\mathrm{Ad}^{\star}_{\gamma(t)}\mathrm{Ad}_{\gamma(t)}$. This applies even for nonsteady $u(t)$.

Formulas for K_{u_0} with $\Lambda = (1 - \Delta)^k$ to generate the H^k metric.

- $\mathrm{Diff}(\mathbb{R})$: $K_{u_0}(v) = \Lambda^{-1}(v\Lambda\partial_x u_0 + v_x\Lambda u_0)$
- $\mathrm{Diff}_{\mathrm{vol}}(M^2)$: $K_{u_0}(v) = \operatorname{sgrad}\Lambda^{-1}\Delta^{-1}\big(\langle v, \nabla \operatorname{curl}\Lambda u_0\rangle\big)$
- $\mathrm{Diff}_{\mathrm{vol}}(M^3)$: $K_{u_0}(v) = \operatorname{curl}\Lambda^{-1}\Delta^{-1}\big[v \cdot \nabla \operatorname{curl}\Lambda u_0 - \Lambda \operatorname{curl} u_0 \cdot \nabla v\big]$

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For fixed u_0 sufficiently smooth, the operator is compact in v if $k > k_0$, a critical value:

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- $\mathrm{Diff}_{\mathrm{vol}}(M^2)$, M^2 compact with or without boundary: $k > -\frac{1}{2}$
- $\mathrm{Diff}_{\mathrm{vol}}(M^3)$, M^3 compact without boundary: $k > 0$

For these cases we get Fredholmness. (Ebin-Misiołek-P. 2006, Misiołek-P. 2009, Benn-Misiołek-P. 2018)

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At the critical value k_0, the exponential map is smooth but not Fredholm.

In Fredholm cases, the conjugate points (if they occur) are spaced out discretely along geodesics. In non-Fredholm cases, they can cluster.

Known facts about conjugate points on $\mathrm{Diff}_{\mathrm{vol}}(M)$:

- (Ebin-Marsden 1970) If the Riemannian exponential map is smooth, then for some $T > 0$ there are no conjugate points occurring at time less than T. (By the Inverse Function Theorem.)
- (Misiołek 1993) For $\mathrm{Diff}_{\mathrm{vol}}(S^n)$, conjugate points occur along rigid rotations. However no conjugate points occur along shear flows on $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{T}^n)$.
- (Misiołek 1996, Drivas-Misiołek-Shi-Yoneda 2022, P. 2022, Le Brigant-P. 2024) For $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{T}^2)$, conjugate points occur along all steady flows with stream function $\cos mx \cos ny$ if and only if m and n are both positive integers.
- (P. 2006) For $\mathrm{Diff}_{\mathrm{vol}}(S^3)$ along the Hopf field geodesic, monoconjugate points of infinite order occur at times $T = \frac{n\pi}{m}$ for all positive integers $n \geq m$, and epiconjugate points occur at all times $T \geq \pi$.
- (Benn 2021) For $\mathrm{Diff}_{\mathrm{vol}}(S^2)$ along nonsteady Rossby-Haurwitz waves, conjugate points eventually occur along geodesics if the amplitude is sufficiently large.
- (Suri 2024) For $\mathrm{Diff}_{\mathrm{vol}}(S^2)$ conjugate points occur along almost every steady flow geodesic with stream function $P_m^\ell(\theta) \cos m\phi$ for integers $\ell > 0$ and $0 < m \leq \ell$. (Only $\ell = 2$ and $m = 1$ is unknown while for $\ell = 1$ it's false.)

On $\mathrm{Diff}(\mathbb{R})$ or $\mathrm{Diff}(S^1)$, conjugate points should be easy to find since curvature is "mostly" positive. However solutions may break down in finite time before they happen!

Known and unknown facts about conjugate points on $\mathrm{Diff}(M)$:

- (Lenells 2007, Lenells 2008) On $\mathrm{Diff}(S^1)$ with the $\dot{H}^1$ metric, all solutions break down in finite time before any conjugate points occur (even though curvature is a positive constant).
- On $\mathrm{Diff}(S^1)$ or $\mathrm{Diff}(\mathbb{R})$ with the H^1 metric (Camassa-Holm equation), some geodesics exist for all time while others end in finite time (McKean 1998). it is unknown whether any geodesic has conjugate points before breakdown.
- (Constantin-Kolev 2002) On $\mathrm{Diff}(S^1)$ with the L^2 metric, conjugate points occur along the constant-translation geodesic at times $T = \frac{\pi}{n}$ for every $n \in \mathbb{N}$. (This is how to prove the exponential map is not smooth!)
- (Bauer-Kolev-P. 2016, P.-Washabaugh 2018) On $\mathrm{Diff}(S^1)$ with the $\dot{H}^{1/2}$ metric, all smooth solutions break down in finite time. It is unknown whether any conjugate points occur before this. With the nondegenerate $\mu\dot{H}^{1/2}$ metric, the constant-translation solution has infinite-order monoconjugate points at every $T = n\pi$ for $n \in \mathbb{N}$.

What can we say about the curvature? This (indirectly) influences Jacobi fields and stability. Consider $\mathrm{Diff}_{\mathrm{vol}}(M)$ first.

- (Arnold 1966) For a fluid in $\mathbb{T}^2$, consider vector fields $u = \nabla^\perp \cos(jx + ky)$ and $v = \nabla^\perp \cos(mx + ny)$. Then $K(u, v) < 0$. Intuitively this implies weather is impossible to predict. But there are also directions of positive curvature.
- (Rouchon 1992, P. 2002) For a fluid in $\mathbb{R}^3$ and a divergence-free vector field u, we have $K(u, v) \geq 0$ for all v if and only if u is a rigid rotation.
- (Misiołek 1993, Lukatsky 1993) If u is a steady flow with constant pressure p, i.e., if $u \cdot \nabla u = 0$ with $\operatorname{div} u = 0$, then $K(u, v) \leq 0$ for all v.
- (P. 2005) If $u = phi(r)\mathbf{e}_\theta$ is a rotational vector field, then $K(u, v) \leq 0$ for every planar v.
- (P. 2006) If u is a 3D steady fluid with $u \cdot \nabla u = -\nabla p$, then $K(u, v) \leq 0$ for all v if and only if p is constant.
- (P.-Washabaugh 2016) If $u = \phi(r)\mathbf{e}_\theta$ with $\frac{d}{dr}(r\phi(r)^2) > 0$, then $K(u, v) \geq 0$ for every axisymmetric divergence-free v.

Roughly speaking, curvature wants to take both signs. But 2D fluids tend to have more negative curvature than 3D fluids.

Sign of the curvature on other diffeomorphism groups and related spaces:

- For the right-invariant L^2 metric on S^1, the curvature is $K(u,v) = \int_{S^1} (uv_x - vu_x)^2$.
- (Lenells 2008) On $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$ with the $\dot{H}^1$ metric, curvature is a positive constant.
- (Khesin-Lenells-Misiołek-P. GAFA 2013) For any compact M, on $\mathrm{Diff}(M)/\mathrm{Diff}_{\mathrm{vol}}(M)$ with the "div-div" degenerate metric, curvature is again a positive constant.
- (Khesin-Lenells-Misiołek-P. PAMQ 2013) For the H^1 metric on $\mathrm{Diff}(S^1)$ and $\mathrm{Diff}(\mathbb{T}^2)$, the curvature takes both signs.
- (Takhtajan-Teo 2006) Sectional curvature is always negative for the $\dot{H}^{3/2}$ Weil-Petersson metric on universal Teichmüller space $\mathrm{Diff}(S^1)/PSL_2(\mathbb{R})$
- Open question: is sectional curvature always positive for $\dot{H}^{1/2}$ Velling-Kirillov metric on $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$? All evidence suggests it is.
- The simplest-curvature metrics seem to be $\dot{H}^1$-type metrics on various spaces. Are there other examples where curvature has a fixed sign?

The sign of the curvature is mostly indeterminate, but what about the *size*? Is it bounded at least?

Theorem (Misiołek 1993)

On $\mathrm{Diff}_{\mathrm{vol}}(M)$ *under the* L^2 *metric, the curvature operator* $(u, v, w) \mapsto R(u, v)w$ *is bounded in* H^s *for* $s > 1 + n/2$.

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The sectional curvature is different since it's computed in the *weak* metric.

Conjecture (Bauer-Le Brigant-P.)

On $\mathrm{Diff}(S^1)$ *in the* H^k *metric, the sectional curvatures* $K(u, v)$ *for vector fields* u, v *is*

- *bounded as a function of* u *and* v *when* $k \geq 1$
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In any other situation, boundedness of curvatures seems wide open!

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Please pay them for the reception!