The BV formalism in the framework of noncommutative geometry

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Higher Structures and Field Theory

ESI, Vienna

This project has received fundings from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 753962

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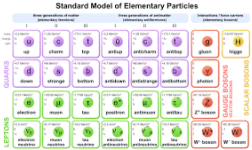
<u>Three fundamental interactions</u>



- interaction are mediated by particles ⇝ discrete nature
- framework: self-adjoint op., ... ⇝ non-commutative world

How to solve it?

Goal: noncommutative geometry was invented in order to provide a unifying background to describe objects that can be:

- continuous and discrete;
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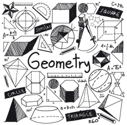
finite & infinite dim.

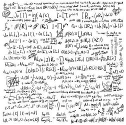
commut. & noncommutative

Noncommutative Geometry: the idea

Idea: To describe

geometric objects in algebraic terms





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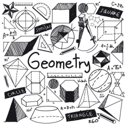

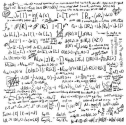

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loc. compact Hausdorff sp (+ compact) $\leftrightsquigarrow$ (Gelfand-Neimark Th. [1943]) comm. C^*-algebra (+ unit)

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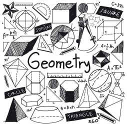

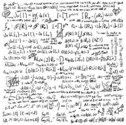

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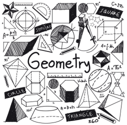

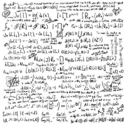

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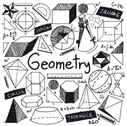

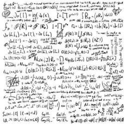

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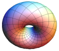
Metric/differential aspects: 

compact Riem. spin manifold $\leftrightsquigarrow$ canonical spectral triple

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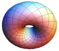

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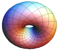
Metric/differential aspects: 

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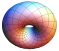
Metric/differential aspects: 

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noncomm. manifold $\leftrightsquigarrow$ noncomm. spectral triple

$(\mathcal{A}, \mathcal{H}, D)$

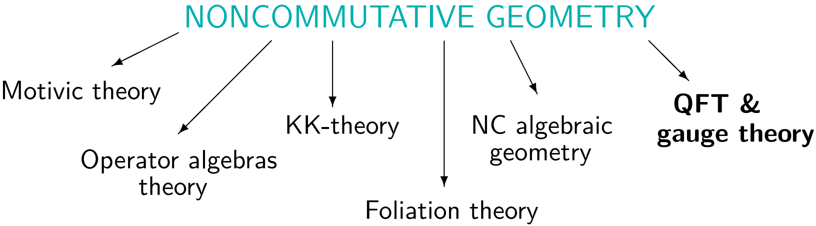
Def: *Spectral triple*

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- $\mathcal{A}$ = invol. unital alg. , faithf. repr. as op. on $\mathcal{H}$, i.e. $\mathcal{A} \cong \mathcal{B}(\mathcal{H})$
- $\mathcal{H}$ = Hilbert space
- D = self-adjoint operator

$+$ properties

Noncommutative Geometry: not only spectral triples



Noncommutative Geometry: not only spectral triples

NONCOMMUTATIVE GEOMETRY

- Motivic theory
- Operator algebras theory
- KK-theory
- Foliation theory
- NC algebraic geometry
- **QFT & gauge theory**

BUT: spectral triples play an interesting role in QFT as

spectral triple $(\mathcal{A}, \mathcal{H}, D)$ ⇒ gauge theory $(X_0, S_0, \mathcal{G})$

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spectral triple $(\mathcal{A}, \mathcal{H}, D)$ ⇒ gauge theory $(X_0, S_0, \mathcal{G})$

> Noncommutative manifolds, that is, the key geometrical object in NCG, naturally encode the concept of a gauge theory

Spectral triple & gauge theories

Def. *Gauge theory*

Given a theory (X_0, S_0) with $\mathcal{G}$ a group acting on X_0 through an action $F: \mathcal{G} \times X_0 \to X_0$, (X_0, S_0) is a *gauge theory with gauge group* $\mathcal{G}$ if it holds that

$$S_0(F(g, \varphi)) = S_0(\varphi), \qquad \forall \varphi \in X_0, \forall g \in \mathcal{G}.$$

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- $X_0 = \left\{ \varphi = \sum_j a_j [D, b_j] : \varphi^* = \varphi \right\}$
  conf. sp = inner fluctuations
- $S_0[D + \varphi] = Tr(f(D + \varphi)), \quad f \in \mathbb{R}[x]$
  action func. = spectral action
- $\mathcal{G} = \mathcal{U}(\mathcal{A})$
  gauge group = unitary el.

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F= finite noncomm. space with finite real spectral triple

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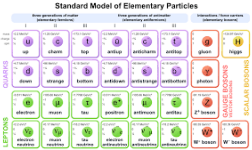

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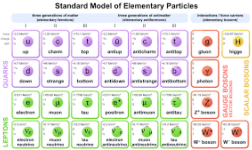
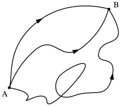


finite spectral triple = particle content

BV construction: quick overview

- *Context:* quantization of gauge theories via a path integral approach



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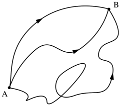

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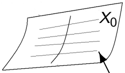

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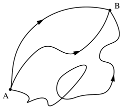

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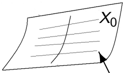

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- *Solution:* eliminate the symmetries by introducing auxiliary (non-existing) fields $\rightsquigarrow$ ghost fields Faddeev-Popov [1967]

Step 1: $(X_0, S_0) \xrightarrow[\text{BV construction}]{} (\widetilde{X}, \widetilde{S})$

The BV approach (Batalin-Vilkovisky, [1983]) fin. dim. case

initial theory — extended theory

$$(X_0, S_0) \xrightarrow{\text{BV construction}} (\widetilde{X}, \widetilde{S})$$

X_0= vector sp.
$S_0 : X_0 \to \mathbb{R}$, regular fun. ($\in \mathcal{O}_{X_0}$)

$\widetilde{X} = X_0 \cup \{$ghost/anti-ghost fields$\}$
$\widetilde{S} = S_0 +$ terms depending on gh./anti-gh.

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A ghost field φ is characterized by:

parity $\epsilon(\varphi) \in \{0, 1\}$

where:

- $\epsilon(\varphi) = 0$ bosonic/real
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ghost degree $\deg(\varphi) \in \mathbb{Z}$

such that:

$$\deg(\varphi) \equiv \epsilon(\varphi), \ mod \mathbb{Z}/\mathbb{Z}2$$

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Given a ghost field φ, the corresponding anti-gh.field φ^* has:

$$\epsilon(\varphi^*) = \epsilon(\varphi) + 1 \qquad \deg(\varphi^*) = -\deg(\varphi) - 1$$

Step 1: the extended theory $(\widetilde{X}, \widetilde{S})$ (1)

The extended conf. sp.:

$$\underbrace{\widetilde{X} = W^*[1] \oplus X_0^*[1]}_{<0} \oplus \underbrace{X_0 \oplus W}_{\geqslant 0}$$

anti-ghost fields → $W^*[1]$; anti-fields → $X_0^*[1]$; initial fields → X_0; ghost fields → W

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The extended action:

$\widetilde{S} : \widetilde{X} \to \mathbb{R}$ regular function $(\in [\mathcal{O}_{\tilde{X}}]^0)$ s.t.

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- $\{\widetilde{S}, \widetilde{S}\} = 0$ solution classical master eq.

where

$$\{\ ,\ \} : \mathcal{O}^n_{\tilde{X}} \times \mathcal{O}^m_{\tilde{X}} \to \mathcal{O}^{n+m+1}_{\tilde{X}}$$ 1-degree Poisson struct.

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$(X_0, S_0) \to (\widetilde{X}, \widetilde{S})$

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To determine the (anti)-ghost fields to introduce, one considers the Koszul-Tate resolution of the Jacobian ideal

$$J(S_0) = \langle \partial_1 S_0, \partial_2 S_0, \dots, \partial_n S_0 \rangle$$

over the ring $\mathcal{O}_{X_0}$: by introducing new variables of

alternating parity & decreasing degree

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complex of fin. gen. $\mathcal{O}_{X_0}$-modules
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$(\{A_i\}, d_i)_{i \in \mathbb{Z}_{\leqslant 0}}$

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To determine the (anti)-ghost fields to introduce, one considers the Koszul-Tate resolution of the Jacobian ideal

$$J(S_0) = \langle \partial_1 S_0, \partial_2 S_0, \ldots, \partial_n S_0 \rangle$$

over the ring $\mathcal{O}_{X_0}$: by introducing new variables of

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one constructs a free resolution of $\mathcal{O}_{X_0}/J(S_0)$ that is a differential $\mathcal{O}_{X_0}$-algebra A.

the sequence is exact:

- $H^0(A) = \mathcal{O}_{X_0}/J(S_0)$
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$$\cdots A_{-n} \xrightarrow{d_{-n}} \cdots A_{-1} = \mathcal{O}_{X_0}\langle M_1^*, \dots, M_n^* \rangle \xrightarrow{d_{-1}} A_0 \cong \mathcal{O}_{X_0} \xrightarrow{\pi} \mathcal{O}_{X_0}/J(S_0) \to 0.$$

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Step 2 & 3: the BRST cohomology and the auxiliary fields

Step 2: an extended theory $(\widetilde{X}, \widetilde{S})$, with $\{\widetilde{S}, \widetilde{S}\} = 0$ naturally induces a classical BRST coh. with:

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Step 4: given a gauge-fixing fermion $\Psi \in [\mathcal{O}_{X_0 \oplus W}]^{-1}$, we perform the gauge-fixing procedure:

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Our goals

The discovery of the BRST complex made it evident that the ghost fields are not just a tool to solve the problem of computing path integrals but they play a significant role.

However, many questions are still waiting for an answer:

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The model

spectral triple $(\mathcal{A}, \mathcal{H}, D)$ ⇒ gauge theory $(X_0, S_0, \mathcal{G})$

- $\mathcal{A}$ = unital *-alg., $\mathcal{A} \cong \mathcal{B}(\mathcal{H})$
  $= M_n(\mathbb{C})$
- $\mathcal{H}$ = Hilbert sp.
  $= \mathbb{C}^n$
- $D : \mathcal{H} \to \mathcal{H}$ = self-adj. op.
  $\in M_n(\mathbb{C})$, s.t. $D^* = D$

- $X_0 = \left\{ \varphi = \sum_j a_j [D, b_j] : \varphi^* = \varphi \right\}$
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- $S_0[D + \varphi] = Tr(f(D + \varphi)), \quad f \in \mathbb{R}[x]$
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The action S_0 plays a key role as it determines the symmetries of the theory and hence to ghost fields that have to be introduced.

For simplicity, we consider the class of models determined by the quadratic Casimir operator of $su(n)$, where $M_{n^2} \rightsquigarrow Id$ in the basis.

Step 1: The extended configuration space

To determine the (anti)-ghost fields to introduce, we consider the following Koszul-Tate resolution.

$$\cdots \to \mathcal{O}_{X_0}\langle M_i^*, C_j^*, \dots\rangle \to \cdots \xrightarrow{d_{-2}} \mathcal{O}_{X_0}\langle M_1^*, \dots, M_{n^2}^*\rangle \xrightarrow{d_{-1}} \mathcal{O}_{X_0} \xrightarrow{\pi} \frac{\mathcal{O}_{X_0}}{J(S_0)} \to 0.$$

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Consequences:

- we can only compute an approximate extended action;
- we cannot explicitly determine the classical BRST-cochain complex and the corresponding coboundary operator;
- we loose track of the type of symmetry of the model.

Step 1: The extended conf. sp. & action (2)

The extended conf. sp.

We found a method to select a finite family of (anti)-ghost generators to extend X_0, which reflects the type of invariance of the action:

$$\widetilde{X} = W^*[1] \oplus X_0^* \oplus X_0 \oplus W \qquad \text{with} \qquad W = \langle C_1, C_2, C_3 \rangle_1 \oplus \langle E \rangle_2$$

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The extended action

$$\widetilde{S} = S_0 + \textstyle\sum_{i,j,k} \epsilon_{ijk} M_i^* M_j C_k + \sum_{i,j,k} C_i^* \left[M_i E + \epsilon_{ijk} C_j C_k \right]$$

$\widetilde{S}$ is: linear in the anti-(ghost) fields; quadratic in the ghost fields

Step 1: the BV construction in NCG (1)

We applied the BV construction to gauge theories naturally induced by finite spectral triple.

Question: can the BV formalism be encoded in the NCG setting? Could the BV extended theory $(\widetilde{X}, \widetilde{S})$ be described as a new BV-spectral triple?

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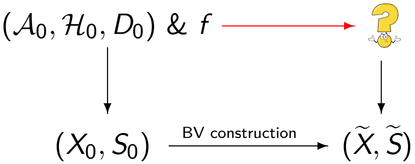

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$(\mathcal{A}_0, \mathcal{H}_0, D_0)$ & f ⟶ 

↓ ↓

(X_0, S_0) $\xrightarrow{\text{BV construction}}$ $(\widetilde{X}, \widetilde{S})$

Note:

- finite spectral triple are naturally defined over $\mathbb{C}$;
- in the extended action $\widetilde{S}$ there appear Grassmannian variables.

Intermezzo: a bit more of NCG

Problem 1: going from $\mathbb{C}$ to $\mathbb{R}$

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Def. *Real structure*

For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, a real structure J is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that:

- $J^2 = \pm Id \quad JD = \pm DJ$
- $[a, Jb^* J^{-1}] = 0, \quad [[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}.$

Then $(\mathcal{A}, \mathcal{H}, D, J)$ is called a (odd) real spectral triple

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KO-dim	1	3	5	7
$J^2 = \pm Id$	1	−1	−1	1
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- $[a, Jb^* J^{-1}] = 0, \quad [[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}.$

Then $(\mathcal{A}, \mathcal{H}, D, J)$ is called a (odd) real spectral triple

KO-dim	1	3	5	7
$J^2 = \pm Id$	1	-1	-1	1
$JD = \pm DJ$	-1	1	-1	1

We also have even spectral triples, with KO-dim. 0, 2, 4, 6 but the even case requires an extra element, that is, a grading $\gamma : \mathcal{H} \to \mathcal{H}$.

Intermezzo: a bit more of NCG (2)

Problem 2: the appearance of Grassmannian variables in $\widetilde{S}$.

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$\rightsquigarrow$ two notions of action

Spectral action:

$$S[D+\varphi] = Tr(f(D+\varphi));$$

for

- f a regular function (good decay, cut off...);
- φ a self-adjoint element, with $\varphi = \sum_j a_j[D, b_j]$, for $a_j, b_j \in \mathcal{A}$

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Fermionic action:

$$S[\psi] = \frac{1}{2}\langle (J)\psi, D\psi\rangle,$$

for

- $\langle\ ,\ \rangle$ the inner product structure on $\mathcal{H}$;
- $\psi \in \mathcal{H}_f \subseteq \mathcal{H}$

we can impose a Grassmannian nature to the elements in $\mathcal{H}_f$

Step 1: the BV construction in NCG (2)

$(\mathcal{A}_0, \mathcal{H}_0, D_0)$ & f ⟶ 

↓ ↓

(X_0, S_0) $\xrightarrow{\text{BV construction}}$ $(\widetilde{X}, \widetilde{S})$

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$$(\mathcal{A}_0, \mathcal{H}_0, D_0) \ \& \ f \xrightarrow{\text{BV construction}} (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$$

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- ghost fields: where do they come from? where are they going to be?
- extended action: how can we determine $\widetilde{S}$ starting from (D_0, f)? How can we encode it in the BV-spectral triple?

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- ghost fields $\rightsquigarrow$ related to the symmetries of $S_0 \cong (D_0, f)$
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Hence:

- anti-fields/anti-ghost fields will be encoded in D_{BV}
- ghost fields are expected to be in $\mathcal{H}_{BV,f} \subseteq \mathcal{H}_{BV}$

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$$\mathcal{H}_0 = \mathbb{C}^2 \xrightarrow{+\text{ ghost/anti-ghost fields}} \mathcal{H}_{BV} = \mathcal{Q} \oplus \mathcal{Q}^*[1]$$

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- 3 indep. ones among pairs of coord. $\rightsquigarrow$ 3 ghost in deg 1
- 1 involving all three coordinates $\rightsquigarrow$ 1 ghost in deg 2

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The BV-Hilbert space

$$\mathcal{H}_{BV} = \mathcal{Q}^*[1] \oplus \mathcal{Q} \quad \text{for} \quad \mathcal{Q} = [M_2(\mathbb{C})]_0 \oplus [M_2(\mathbb{C})]_1 \oplus [M_2(\mathbb{C})]_2$$

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The BV operator D_{BV}

$$D_{BV} = \begin{pmatrix} 0 & R \\ R^* & S \end{pmatrix}$$

for

$$R := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}Ad(C) & Ab(M) \\ \frac{1}{2}Ad(C) & -\frac{1}{2}Ad(M) & 0 \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & Ad(M^*) & Ab(C^*) \\ Ad(M^*) & Ad(C^*) & 0 \\ Ab(C^*) & 0 & 0 \end{pmatrix}$$

Step 1: the BV construction in NCG (5)

The real structure: $J_{BV} : \mathcal{H}_{BV} \to \mathcal{H}_{BV}$ with $J_{BV}(\varphi) = \varphi^\dagger$

The algebra $\mathcal{A}_{BV}$: given $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ as defined about, the maximal unital algebra completing them to a spectral triple is

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Theorem

For a finite spectral triple $(\mathcal{A}_0, \mathcal{H}_0, D_0) = (M_2(\mathbb{C}), \mathbb{C}^2, D_0)$ with induced gauge theory (X_0, S_0), the BV-spectral triple is

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In other words:

$$\widetilde{X} = (\mathcal{Q}_f + X_0) \oplus (X_0^*[1] + \mathcal{Q}_f^*[1]) \qquad \widetilde{S} = S_0 + \frac{1}{4}\langle J_{BV}(-), D_{BV}(-)\rangle$$

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$(\mathcal{A}_0, \mathcal{H}_0, D_0) \to (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \to \dots$

Step 2: the BRST cohomology in NCG

Goal: the classical BRST complex from the BV-spectral triple.

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Idea: it is related to cohom. theories appearing naturally in NCG

Riemannian diff. geom	NCG
manifold	spectral triple
differential forms	Hochschild homology
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Def: *Hochschild complex*

For a graded associative algebra $\mathcal{A}$ and a bimodule $\mathcal{M}$ over it,

$\mathcal{C}^q_H(\mathcal{A},\mathcal{M}) := Hom_{\mathbb{K}}(\mathcal{A}^{\otimes q},\mathcal{M})$ and $d_H : \mathcal{C}^q_H(\mathcal{A},\mathcal{M}) \to \mathcal{C}^{q+1}_H(\mathcal{A},\mathcal{M})$

with

$$
\begin{aligned}
d_H(\varphi)|_{(x_0\otimes\cdots\otimes x_q)} := \; & \omega_L(x_0, \varphi(x_1 \otimes \cdots \otimes x_q)) \\
& + \sum_{i=1}^{q} (-1)^{|x_1|+\cdots+|x_{i-1}|}\varphi(x_0 \otimes \cdots \otimes x_{i-1} \bigstar x_i \otimes \cdots \otimes x_q) \\
& + (-1)^{q+1}\omega_R(\varphi(x_0 \otimes \cdots \otimes x_{q-1}), x_q)
\end{aligned}
$$

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Note: the BV-spectral triple encodes only the "effective" components of the extended theory $(\widetilde{X}, \widetilde{S})$

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Effective BRST complex:

$$\mathcal{C}_f^i(\widetilde{X}, d_{\tilde{S}}) := [Sym_{\mathcal{O}_{X_0}}(\widetilde{X}_f)]^i, \quad \text{for} \quad \widetilde{X}_f := \widetilde{X} \setminus \left\{x_j^* \in [\widetilde{X}]^{-1} : \{\widetilde{S}, x_j\} = 0\right\}$$

The coboundary op. $d_{\tilde{S},f}$ is the restriction of $d_{\tilde{S}}$ to $Sym_{\mathcal{O}_{X_0}}(\widetilde{X}_f)$.

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Prop: there exists an isomorphism between classical and effective BRST coh. groups:

$$\mathcal{H}^n(\widetilde{X}, d_{\tilde{S}}) \cong \bigoplus_{a=1,\dots,\sharp J} x^*_{j_1} \cdot \dots \cdot x^*_{j_a} \cdot \mathcal{H}^{n+a}_f(\widetilde{X}, d_{\tilde{S},f}),$$

where x^*_j are the antifields corresponding to initial fields that do not enter the symmetry of the action S_0.

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To determine: $(\mathcal{A}, \bigstar)$, $(\mathcal{M}, \omega_L.\omega_R)$.

Step 2: the BRST cohomology in NCG (3)

The algebra $\mathcal{A}$

$\mathcal{A} := \mathcal{H}_{BV,f} = \mathcal{Q}_f \oplus \mathcal{Q}_f^*[1]$ for $\mathcal{Q}_f = [i\mathfrak{su}(2)]_0 \oplus [i\mathfrak{su}(2)]_1 \oplus [\mathfrak{u}(1)]_2$,

Product:

$$\alpha \star \beta := \frac{1}{4} \sum_{r\neq 0,k} \left\{ \langle J_{BV}(-), D_{BV}(-)\rangle , x_k^r \right\}\Big|_{\alpha\otimes\beta} \cdot \tau_k^r,$$

where $\{-,-\}$ is the antibracket structure induced by the pairing fields/anti-fields and τ_k^r is the dual basis.

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The module $\mathcal{M}$

$$\mathcal{M} := \langle \Omega^1(\mathcal{A}_{BV}) \rangle \cong \mathcal{O}_{X_0},$$

for

$$\Omega^1(\mathcal{A}_{BV}) = \left\{ \varphi = \sum_i a_j [D_0, b_j] : \varphi^* = \varphi, a_j, b_j \in \mathcal{A}_{BV} \right\}$$

Left/Right module structure:

$$\omega_L(\alpha, g) := \frac{1}{8} \sum_k \frac{\partial g}{\partial x_k^0} \left\{ \langle J_{BV}(-), D_{BV}(-) \rangle , x_k^0 \right\} \Big|_{\alpha} = -\omega_R(g, \alpha)$$

Step 2: the BRST cohomology in NCG (4)

BV-spectral triple

↓

Hochschild complex

$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, \underbrace{D_{BV}, J_{BV}})$

$\mathcal{A}_{BV} \to \mathcal{M}$, $\mathcal{H}_{BV} \to \mathcal{A}$, $D_{BV}, J_{BV} \to$

deg $0 : \omega_L, \omega_R$

deg $\neq 0 : \bigstar$

Step 2: the BRST cohomology in NCG (4)

BV-spectral triple → Hochschild complex

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Theorem

Given a BV spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$, let $(\mathcal{A}, \mathcal{M})$ be defined as above. Then it holds that:

$$(\mathcal{C}^\bullet_H(\mathcal{A}, \mathcal{M}), d_H) \cong (\mathcal{C}^\bullet_f(\widetilde{X}, d_{\tilde{S}}), d_{\tilde{S},f}).$$

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$$(\mathcal{A}_0, \mathcal{H}_0, D_0) \to (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \to (\mathcal{C}^\bullet_H(\mathcal{A}, \mathcal{M}), d_H) \to \dots$$

Step 3: the auxiliary fields

$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \xrightarrow{+\text{ auxiliary fields/anti-fields}} (\mathcal{A}_{tot}, \mathcal{H}_{tot}, D_{tot}, J_{tot})$$

- $\mathcal{A}_{tot} = \mathcal{A}_{BV} = \mathcal{A}_0 = M_2(\mathbb{C})$ $\rightsquigarrow$ no changes for the algebra
- $\mathcal{H}_{tot} := \mathcal{H}_{BV} \oplus \mathcal{H}_{aux}, \quad \mathcal{H}_{BV} = [M_2(\mathbb{C})]^{\oplus 6}, \quad \mathcal{H}_{aux} = [M_2(\mathbb{C})]^{\oplus 12}$

$$\mathcal{H}_{aux,f} = \mathcal{R}_f \oplus \mathcal{R}_f^*[1]$$

for

$$\mathcal{R}_f := [\mathfrak{u}(1)]_{-2} \oplus [\mathfrak{u}(1) \oplus \mathfrak{su}(2)]_{-1} \oplus [i\mathfrak{su}(2) \oplus \mathfrak{u}(1)]_0 \oplus [\mathfrak{u}(1)]_1.$$

- D_{tot}

$$D_{tot} = \begin{pmatrix} D_{BV} & 0 \\ 0 & D_{aux} \end{pmatrix} \quad \text{for} \quad D_{aux} := \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix},$$

where

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$\rightsquigarrow$ determined by the pairing between contractible pairs

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$\rightsquigarrow$ determined by the pairing between contractible pairs

Same structure as in the BV spectral triple!

Step 4 & 5: gauge fixing and g.f. BRST cohomology

Also the gauge-fixed BRST cohomology can be seen as an Hochschild complex

Theorem

Given a total spectral triple $(\mathcal{A}_{tot}, \mathcal{H}_{tot}, D_{tot}, J_{tot})$and a gauge-fixing fermion $\Psi \in [\mathcal{O}_{\mathcal{Q}_f \oplus \mathcal{R}_f}]^{-1}$, it holds that:

$$(\mathcal{C}_f^\bullet(X_{tot}|_\Psi, d_{S_{tot}}|_\Psi), d_{S_{tot}}|_\Psi) = (\mathcal{C}_H^\bullet(\mathcal{A}_\Psi, \mathcal{M}_\Psi), d_H),$$

where

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and

$$\mathcal{M}_\Psi := \left\langle \Omega^1(\mathcal{A}_0) \right\rangle$$

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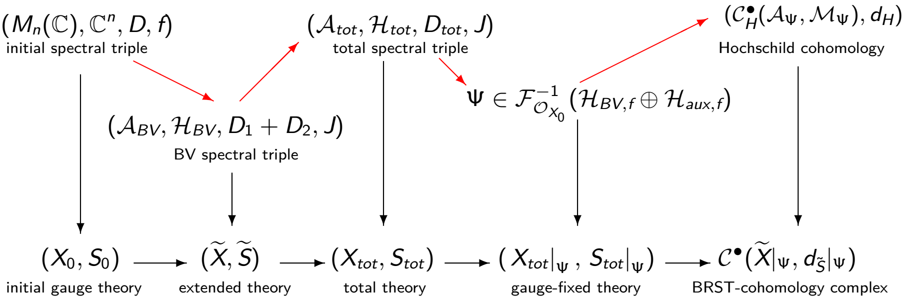
and

$$\mathcal{M}_\Psi := \left\langle \Omega^1(\mathcal{A}_0)\right\rangle$$

The construction is absolutely analogous to the one made for the classical BRST complex, we are simply imposing the gauge fixing condition

Where we are:

Noncommutative geometry



BV/BRST construction

Where we are:

Noncommutative geometry

$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$ initial spectral triple

$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_1 + D_2, J)$ BV spectral triple

$(\mathcal{A}_{tot}, \mathcal{H}_{tot}, D_{tot}, J)$ total spectral triple

$\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f})$

$(\mathcal{C}^\bullet_H(\mathcal{A}_\Psi, \mathcal{M}_\Psi), d_H)$ Hochschild cohomology

(X_0, S_0) initial gauge theory $\longrightarrow$ $(\widetilde{X}, \widetilde{S})$ extended theory $\longrightarrow$ (X_{tot}, S_{tot}) total theory $\longrightarrow$ $(X_{tot}|_\Psi, S_{tot}|_\Psi)$ gauge-fixed theory $\longrightarrow$ $\mathcal{C}^\bullet(\widetilde{X}|_\Psi, d_{\tilde{S}}|_\Psi)$ BRST-cohomology complex

BV/BRST construction

- We have considered a model $M \times F$, with $M = \{\text{point}\}$. What about $dim(M) > 0$?
- All this was classical. Can we go quantum?