Calculi, cohomology, and Hopf algebroids

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General picture: spaces, symmetries, cohomology

	Diff. Geom.	*Algebraic Geom.*	*Noncommut. Geom.*
Spaces	Manifolds	Commut. Algebras (, . . . , Schemes)	Noncommut. Algebras (, . . . , Spectral Triples)
Symmetries	Lie Groups	Algebraic Groups, Group Schemes	Quantum Groups, Hopf Algebras
Generalised Symmetries	Lie Groupoids, Pseudogroups	Groupoid Schemes	Quantum Groupoids, Hopf Algebroids

Cohomology in ☞ *for* ☟	*Differential Geometry*	*Noncommutative Geometry*
Spaces	De Rham Cohomology	Cyclic Cohomology
Symmetries	Lie Algebra Cohomology	Hopf-Cyclic Cohomology
Gen. Symmetries	Lie Algebroid Cohomology	Hopf-Cyclic Cohomology

(Co)homology theories

- The general pattern is: define two sequences

$$H^0(X, M),\ H^1(X, M), \ldots, \quad H_0(X, N),\ H_1(X, N), \ldots$$

  of abelian groups, respectively called the **(co)homology groups of** X **with coefficients in** M respectively N.
- Here X is some mathematical structure and the coefficients are the objects of categories $\mathcal{C}_X$ and $\mathcal{C}^X$ that are associated to X. Leaving M and N blank, this defines functors

$$H^m(X, -) : \mathcal{C}^X \to \mathbb{Z}\text{-}\mathbf{Mod}, \quad H_n(X, -) : \mathcal{C}_X \to \mathbb{Z}\text{-}\mathbf{Mod}.$$

- Example: X compact smooth manifold,

$$H^n(X, \mathbb{C}) := \ker d_n / \mathrm{im}\, d_{n-1},$$

  where

$$d_n : \Omega^n(X, \mathbb{C}) \to \Omega^{n+1}(X, \mathbb{C})$$

  is the exterior derivative acting on smooth complex-valued differential i-forms (recall that $d_n \circ d_{n-1} = 0$).

Examples

- Geometry and topology have created a whole zoo of (co)homology theories. Here X is a (sometimes special sort of) topological space with various $\mathcal{C}_X, \mathcal{C}^X$ involved, *e.g.*, sheaves of abelian groups on an arbitrary topological space X, or vector bundles with a flat connection on a smooth manifold, or coherent sheaves on an algebraic variety.
- In this talk, we mostly consider algebraic structures, *e.g.*,
  - X a group, $\mathcal{C}_X = \mathcal{C}^X$ its k-linear representations.
  - X a k-algebra, $\mathcal{C}_X = \mathcal{C}^X$ its bimodules with symmetric k-action.
  - X a Lie algebra, $\mathcal{C}_X = \mathcal{C}^X$ its k-linear representations.
  - X a Poisson k-algebra, $\mathcal{C}^X$ the left Poisson modules, $\mathcal{C}_X$ the right Poisson modules.

How to define (co)homology?

- One first associates to X a suitable k-algebra U:
  - To a group G its **group algebra** kG. (here $A = k$ is a field)
    $\rightsquigarrow$ group (co)homology.
  - To a Lie algebra $\mathfrak{g}$ its **universal enveloping algebra** $U(\mathfrak{g})$
    $\rightsquigarrow$ Lie algebra (co)homology.
  - Similarly (but more involved) for **Lie algebroids** (or **Lie-Rinehart algebras**, see below).
  - To an algebra A its **enveloping algebra** $A^{\mathrm{e}} := A \otimes_k A^{\mathrm{op}}$.
    $\rightsquigarrow$ Hochschild (co)homology.
  - To a Poisson algebra A the **universal enveloping algebra** of the associated Lie-Rinehart algebra $\Omega^1(A)$
    $\rightsquigarrow$ Poisson (co)homology.
- In all cases, there is a distinguished left U-module A:
  - For groups and Lie algebras the **trivial representation** $A = k$.
  - For associative respectively Poisson algebras A the A-bimodule respectively Poisson module A itself.

How to define (co)homology?

- Then $\mathcal{C}_X$ is the category **Mod**-U of right U-modules while $\mathcal{C}^X$ is the category U-**Mod** of left U-modules.
- This defines two functors:

$$H^0 := \mathrm{Hom}_U(A, -) : U\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod},$$

$$H_0 := - \otimes_U A : \mathbf{Mod}\text{-}U \to k\text{-}\mathbf{Mod}.$$

- The higher (co)homology groups are the derived functors of these (alternative definitions via the bar construction).
- To sum up, (under suitable projectivity assumptions) all these (co)homology theories can be realised as

$$H^m(X, M) := \mathrm{Ext}^m_U(A, M), \quad H_n(X, N) := \mathrm{Tor}^U_n(N, A)$$

for an augmented ring (U, A) (a ring U with a distinguished left module A) that is functorially attached to a given object X.

Main objective

- Clarify the origin and interplay of “higher structures”, that is, multiplicative structures, brackets, Lie derivatives, differentials, divergence, and dualities between such (co)homologies, which, in particular, recovers all well-known operations from differential and Poisson geometry as well as those found by Rinehart, Connes, Nest-Tamarkin-Tsygan for associative algebras.
- Insight: what one needs is that U is a *left bialgebroid* (as defined by Takeuchi) and sometimes even a *left Hopf algebroid* (as defined by Schauenburg). These are generalisations of bialgebras resp. Hopf algebras over possibly noncommutative base algebras A, and will be explained in a moment (or two).

A warm up: group (co)homology

- Let G be a group and
$$\mathbb{C}G = \{u : G \to \mathbb{C} \mid u(g) = 0 \text{ for all but finitely many } g\}$$
for $k = \mathbb{C}$ be its complex group algebra (with convolution product). Regard this as the vector space with basis $\{e_g\}_{g \in G}$ labelled by the group elements and multiplication
$$e_g e_h := e_{gh},$$
and write typical elements in the form $\sum_{g \in G} \lambda_g e_g$, where $\lambda_g e_g$ means the map $u \in kG$ such that $\lambda_g = u(g)$ and zero else.
- Left and right kG-modules can be identified via
$$e_g \triangleright m := m \triangleleft e_{g^{-1}}$$
and they are simply complex representations of G.
- Put $A := \mathbb{C}$ with the trivial action of G
$$e_g \triangleright \lambda := \lambda, \quad \lambda \in \mathbb{C}, \; g \in G,$$
and we now know what the (co)homology of a group G with coefficients in a complex representation is.

Groups, still

- Given two representations M, N of a group G we can indeed form the tensor product $M \otimes_k N$ of the underlying $\mathbb{C}$-vector spaces (if M, N have linear bases $\{m_i\}$, $\{n_j\}$, this has a basis $\{m_i \otimes_k n_j\}$).
- This becomes a representation again, just by putting

$$e_g \triangleright (m \otimes_k n) := (e_g m_i) \otimes_k (e_g n_j).$$

- Crucial observation here: There is an algebra homomorphism

$$\Delta : kG \to kG \otimes_k kG, \quad e_g \mapsto e_g \otimes_k e_g$$

and the action of some $u \in kG$ on $m \otimes_k n$ is given by acting with the left leg of $\Delta(u) \in kG \otimes_k kG$ on m and with the right on n. Sweedler had the idea to write this as

$$u \triangleright (m \otimes_k n) := u_{(1)} \triangleright m \otimes_k u_{(2)} \triangleright n.$$

Hopf algebras

- Note there is a bit more structure available here: There is a map

$$\varepsilon : kG \to k, \quad \sum_{g \in G} \lambda_g e_g \mapsto \sum_{g \in G} \lambda_g$$

  such that

$$\varepsilon(u_{(1)})u_{(2)} = u_{(1)}\varepsilon(u_{(2)}) = u \quad \forall u \in kG$$

  and we have $(\Delta \otimes_k \mathrm{id}_U) \circ \Delta = (\mathrm{id}_U \otimes_k \Delta) \circ \Delta$.
- So kG is a coalgebra and an algebra, and the two structures are compatible in the sense that Δ and ε are algebra homomorphisms, and such things are called **bialgebras**.
- Hopf algebras are just a little better and have some additional datum S which allows one to identify left and right modules over them via a map

$$S : kG \to kG, \quad e_g \mapsto e_{g^{-1}}.$$

Now towards the real thing

- Fix a commutative ground ring k and a k-algebra A, *i.e.*, a homomorphism
$$\eta_A : k \to Z(A),$$
which maps k to multiples of the unit element in A.
- The main player is a sort of algebra U over $A^e := A \otimes_k A^{\mathrm{op}}$, the enveloping algebra of A: A^e is an algebra generated as algebra by two commuting copies of A and of its opposite algebra A^{op} which is the same set as A but with the product $a \bullet b := ba$.
- A^e-modules are the same as A-bimodules (with symmetric action of k) and U a k-algebra with a k-algebra homomorphism
$$\eta \colon A^e \to U$$
that does not necessarily land in the centre, and these object are called A^{e}-*rings.*

The forgetful functors

- We consider $M \in U$-**Mod**, $N \in$ **Mod**-U as A-bimodules with

$$a \triangleright m \triangleleft b := \eta(a \otimes_k b)m, \quad a, b \in A, m \in M.$$

$$a \blacktriangleright m \blacktriangleleft b := n\eta(b \otimes_k a), \quad a, b \in A, n \in N.$$

  In particular, U itself carries two left and two right A-actions all commuting with each other.
- For the homological stuff, ${}_{\triangleright}U \in A$-**Mod** and $U_{\triangleleft} \in A^{\mathrm{op}}$-**Mod** should be projective.

Bialgebroids ($= \times_A$-bialgebras)

- Now assume U is also a coalgebra in the monoidal category A^{e}-**Mod**. That is, there are maps

$$\Delta : U \to U \otimes_A U, \quad \varepsilon : U \to A$$

satisfying the coalgebra axioms, where

$$U \otimes_A U = U \otimes_k U / \mathrm{span}_k\{u \triangleleft a \otimes_k v - u \otimes_k a \triangleright v \mid a \in A, u, v \in U\}.$$

- Recall: for $A = k$ one calls U a bialgebra if Δ and ε are algebra homomorphisms. But in general there is no natural algebra structure on $U \otimes_A U$.

Bialgebroids

- Takeuchi's solution: consider the embedding

$$\iota : U \times_A U \to U \otimes_A U,$$

where $U \times_A U$ is the centre of the A-bimodule ${}_\blacktriangleright U_\triangleleft \otimes_A {}_\triangleright U_\blacktriangleleft$:

$$U \times_A U := \Big\{ \sum_i u_i \otimes_A v_i \in U \otimes_A U \mid \sum_i a \blacktriangleright u_i \otimes_A v_i = \sum_i u_i \otimes_A v_i \blacktriangleleft a \Big\}.$$

- The product of U turns this into an A^{e}-ring, with

$$\eta_{U \times_A U} : A^e \to U \times_A U, \quad a \otimes_k b \mapsto \eta(a \otimes_k 1) \otimes_A \eta(1 \otimes_k b).$$

- Similarly, A is an algebra over k, but not an A^e-ring in general. To handle this, one needs the canonical map

$$\pi : \mathrm{End}_k(A) \to A, \quad \varphi \mapsto \varphi(1),$$

and the fact that $\mathrm{End}_k(A)$ is an A^{e}-ring, with

$$\eta_{\mathrm{End}_k(A)} : A^e \to \mathrm{End}_k(A), \quad (\eta_{\mathrm{End}_k(A)}(a \otimes b))(c) := acb.$$

Bialgebroids

- Now it makes sense to require Δ, ε to factor through ι and π:

Definition

A (left) bialgebroid is an A^{e}-ring U together with two homomorphisms

$$\hat{\Delta} : U \to U \times_A U,$$

$$\hat{\varepsilon} : U \to \mathrm{End}_k(A)$$

of A^{e}-rings such that U is a coalgebra in A^{e}-**Mod** via $\Delta = \iota \circ \hat{\Delta}$ and $\varepsilon = \pi \circ \hat{\varepsilon}$.

- Be aware that the four A-actions are not the only feature of bialgebroids that disappears for $A = k$. For example, $\varepsilon : U \to A$ is not necessarily a ring homomorphism.

The monoidal category U-**Mod**

Theorem (Schauenburg)

*The bialgebroid structures on the A^{e}-ring $\eta : A^e \to U$ correspond bijectively to monoidal structures on U-**Mod** for which the forgetful functor U-**Mod** $\to A^{\mathrm{e}}$-**Mod** induced by η is strictly monoidal.*

Given a bialgebroid structure on U, the monoidal structure on U-**Mod** is defined as for bialgebras: take $M \otimes_A N$ and define

$$u(m \otimes_A n) := u_{(1)} m \otimes_A u_{(2)} n, \quad u \in U, m \in M, n \in N.$$

The unit object in U-**Mod** is A on which U acts via $\hat{\varepsilon}$.

Right bialgebroids

- There is an analogous notion of right bialgebroid for which **Mod**-U is monoidal.
- Note: for a left bialgebroid there is no canonical monoidal structure on **Mod**-U or even only a right action of U on A.
- At first glance, this might seem of little interest but observe that while for $X = (U, A)$ we can always speak about

$$H^\bullet(X, A) = \mathrm{Ext}^\bullet_U(A, A),$$

there is in general no way of making sense of

$$H_\bullet(X, A) = \mathrm{Tor}^U_\bullet(A, A).$$

- This really becomes relevant in, for example, Poisson homology, or more general in Lie algebroid (Lie-Rinehart) homology, and in general impedes the definition of an "antipodal" map S on U.

Left Hopf algebroids

- Let U be a left bialgebroid and define the *Galois map* of U

$$\beta : {}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} U_\triangleleft \to U_\triangleleft \otimes_A {}_\triangleright U, \quad u \otimes_{A^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_A u_{(2)} v.$$

- For bialgebras over fields β is bijective if and only if U is a Hopf algebra with $\beta^{-1}(u \otimes_k v) := u_{(1)} \otimes S(u_{(2)})v$, where S is the antipode of U. This motivates:

Definition (Schauenburg)

A left bialgebroid U is called a *left Hopf algebroid* if β is a bijection.

- We adopt a Sweedler-type notation

$$u_+ \otimes_{A^{\mathrm{op}}} u_- := \beta^{-1}(u \otimes_A 1)$$

for the *translation map* $\beta^{-1}(\cdot \otimes_A 1) : U \to {}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} U_\triangleleft$.

Observe: there is no notion of antipode for left Hopf algebroids.

Examples

- Hopf algebras over k.
- Universal enveloping algebras of Lie algebroids or Lie-Rinehart algebras, as discussed below.
- The enveloping algebra $U := A^e$ of any k-algebra A is a left bialgebroid with $\eta = \mathrm{id}_{A^e}$ and

$$\Delta : A^{\mathrm{e}} \to A^{\mathrm{e}} \otimes_A A^{\mathrm{e}}, \quad a \otimes_k b \mapsto (a \otimes_k 1) \otimes_A (1 \otimes_k b),$$

$$\varepsilon : A^{\mathrm{e}} \to A, \quad a \otimes_k b \mapsto ab,$$

and a left Hopf algebroid by

$$(a \otimes_k b)_+ \otimes_{A^{\mathrm{op}}} (a \otimes_k b)_- = (a \otimes_k 1) \otimes_{A^{\mathrm{op}}} (b \otimes_k 1).$$

This might look boring but turns Hochschild theory into a subsection of bialgebroid theory, and cannot be seen as a conventional Hopf algebra, while it is a fundamental example of a non-commutative non-co-commutative bialgebroid.

The cup and cap product

- As already seen, the base algebra A of a bialgebroid (U, A) carries a left U-action and the bialgebroid structure induces the structure of a monoidal category on U-**Mod** with unit object A, *i.e.*, there exists a functor
$$\otimes : U\text{-}\mathbf{Mod} \times U\text{-}\mathbf{Mod} \to U\text{-}\mathbf{Mod}, \quad A \otimes M \simeq M \otimes A \simeq M.$$
- This induces (under some projectivity assumptions) a *cup* product
$$\smile : H^i(X, M) \times H^j(X, N) \to H^{i+j}(X, M \otimes N), \quad X = (U, A).$$
- Dually, and adding the left Hopf structure, there is a functor
$$\otimes : U\text{-}\mathbf{Mod} \times \mathbf{Mod}\text{-}U \to \mathbf{Mod}\text{-}U$$
turning **Mod**-U into a (left) module category over $(U\text{-}\mathbf{Mod}, \otimes, A)$.
- This induces for $N \in U$-**Mod** and $P \in$ **Mod**-U a *cap* product
$$\frown : H^i(X, N) \times H_j(X, P) \to H_{j-i}(X, N \otimes P).$$

However, there exists much more structure on the pair $H^i(X, N), H_j(X, P)$ or rather $\mathrm{Ext}^\bullet_U(A, N)$, $\mathrm{Tor}^U_\bullet(P, A)$, which we will discuss next.

Definition

- A **Gerstenhaber algebra** is a graded commutative k-algebra $(V^\bullet, \smile)$ along with a graded Lie bracket $\{\cdot,\cdot\}$ on the desuspension $V[1]$ for which all operators $\{\gamma,\cdot\}$ satisfy the graded Leibniz rule

$$\{\gamma, \alpha \smile \beta\} = \{\gamma, \alpha\} \smile \beta + (-1)^{(\gamma-1)\alpha} \alpha \smile \{\gamma, \beta\}.$$

- A **Gerstenhaber module** $\Omega_\bullet$ over a Gerstenhaber algebra $(V^\bullet, \smile, \{.,.\})$ is simultaneously a graded module over $(V^\bullet, \smile)$ and a graded Lie algebra module over $(V^\bullet[1], \{.,.\})$ with respective actions

$$\frown = \iota \colon V^p \otimes \Omega_n \to \Omega_{n-p}, \qquad \mathcal{L} \colon V^{p+1} \otimes_k \Omega_n \to \Omega_{n-p}$$

which for $\alpha, \beta \in V$ satisfy the mixed Leibniz rule

$$[\iota_\alpha, \mathcal{L}_\beta] = \iota_{\{\alpha,\beta\}}.$$

- A Gerstenhaber module $\mathcal{O}$ is **Batalin-Vilkovisky (BV)** if equipped with a differential $B : \Omega_n \to \Omega_{n+1}$ such that the Cartan homotopy formula

$$\mathcal{L}_\alpha = [B, \iota_\alpha]$$

holds. A pair (V, Ω) of a Gerstenhaber algebra and a BV module is also called a **(noncommutative) Cartan(-Tamarkin-Tsygan) (differential) calculus**.

Example (Classical geometric example)

- For a smooth manifold Q, consider $V^\bullet = (\mathcal{X}^\bullet(Q), [.,.]_{\rm SN})$ and $\Omega_\bullet = \Omega^\bullet(Q)$. Choosing $B = d_{deRham}$, the standard Lie derivative and contraction of a form by a multivector field, gives the well-known calculus of "fields acting on forms" with the customary formulae

$$\mathcal{L} = [\iota, d], \quad [d, \mathcal{L}] = 0, \quad [\mathcal{L}, \iota] = \iota_{[.,.]_{SN}}, \quad \mathcal{L}_{[.,.]_{SN}} = [\mathcal{L}, \mathcal{L}]$$

  from differential (or algebraic) geometry.
- The case "fields acting on fields" is obtained by $(\mathcal{X}^\bullet(Q), [.,.]_{\rm SN})$ acting on $(\mathcal{X}^\bullet(Q), d_{\rm CE})$ with $\iota_X Y := X \wedge Y$, the Lie derivative for multivector fields, and the differential $d_{\rm CE}$ from Lie algebra homology.

Example (Classical algebraic example)

The pair of Hochschild cohomology and homology

$$V^\bullet = \mathrm{Ext}^\bullet_{A^{\rm e}}(A, A) = H^\bullet(A, A), \qquad \Omega_\bullet = \mathrm{Tor}^{A^{\rm e}}_\bullet(A, A) = H_\bullet(A, A)$$

forms a calculus (Rinehart 1963, Connes, Getzler, Goodwillie, Nest-Tsygan (80/90s)). I will skip the explicit formulas of the involved operations.

The Gerstenhaber algebra

- The bar construction yields an explicit cochain complex

$$C^n(U,A) := \mathrm{Hom}_{A^{\mathrm{op}}}(U^{\otimes_{A^{\mathrm{op}}} n}, A)$$

that computes $\mathrm{Ext}^\bullet_U(A,A)$ for a left bialgebroid (U,A). This becomes an **operad with multiplication** via

$$(\varphi \circ_i \psi)(u^1, \ldots, u^{p+q-1}) := \varphi(u^1, \ldots, D_\psi(u^i, \ldots, u^{i+q-1}), \ldots, u^{p+q-1}),$$

for $\varphi \in C^p(U,A)$, $\psi \in C^q(U,A)$, and $i = 1, \ldots, p$, where

$$D_\psi : U^{\otimes_{A^{\mathrm{op}}} q} \to U, \quad (u^1, \ldots, u^q) \mapsto \psi(u^1_{(1)}, \ldots, u^q_{(1)}) \triangleright u^1_{(2)} \cdots u^q_{(2)}$$

replaces the insertion operations used by Gerstenhaber (and reflects the DG coalgebra structure of the bar resolution).

- Finally, the Gerstenhaber bracket is

$$\{\varphi, \psi\} := \varphi \bar{\circ} \psi - (-1)^{(p-1)(q-1)} \psi \bar{\circ} \varphi$$

with $\varphi \,\bar{\circ}\, \psi := \sum_{i=1}^p \pm \varphi \circ_i \psi$.

- For $P \in \mathbf{Mod}\text{-}U$, one can compute $\mathrm{Tor}^U_\bullet(M, A)$ for right A-projective U by means of the canonical chain complex

$$C_n(U, M) := M \otimes_{A^{\mathrm{op}}} U^{\otimes_{A^{\mathrm{op}}} n}, \qquad b := \sum_{i=0}^{n} (-1)^i d_i$$

with faces and degeneracies, abbreviating $(m, x) := (m, u^1, \dots, u^n)$:

$$d_i(m,x) = \begin{cases} (m, u^1, \dots, \varepsilon(u^n) \blacktriangleright u^{n-1}), & \text{if } i = 0, \\ (m, \dots, u^{n-i} u^{n-i+1}, \dots, u^n) & \text{if } 1 \leq i \leq n-1, \\ (mu^1, u^2, \dots, u^n) & \text{if } i = n, \end{cases}$$

$$s_j(m,x) = \begin{cases} (m, u^1, \dots, u^n, 1) & \text{if } j = 0, \\ (m, \dots, u^{n-j}, 1, u^{n-j+1}, \dots, u^n) & \text{if } 1 \leq j \leq n. \end{cases}$$

- An additional **left** U**-coaction** $M \to U \otimes_A M$, $m \mapsto m_{(-1)} \otimes_A m_{(0)}$ on M allows for a **cyclic operator**

$$t_n(m, x) = (m_{(0)} u^1_+, u^2_+, \dots, u^n_+, u^n_- \cdots u^1_- m_{(-1)}),$$

which in turn induces a map (see below)

$$B : C_\bullet(U, M) \to C_{\bullet+1}(U, M),$$

that generalises Connes' cyclic boundary operator.

Cyclic and saYD modules

- If there is a suitable compatibility between action and coaction on the coefficients M, the above operators lead to both a Gerstenhaber and a BV module; the easiest case if M is a **stable anti Yetter-Drinfel'd module**. In Sweedler notation this means

$$(mu)_{(-1)} \otimes_A (mu)_{(0)} = u_- m_{(-1)} u_{+(1)} \otimes_A m_{(0)} u_{+(2)}$$

  for all $u \in U, m \in M$ and

$$m_{(0)} m_{(-1)} = m.$$

- In this case, the pieces $(d_\bullet, s_\bullet, t_\bullet)$ define a **cyclic k-module** structure on $C_\bullet(U, M)$, that is, a simplicial k-module with a compatible $\mathbb{Z}_n$-action encoded in the operator t_n: the cyclic coboundary is then obtained by means of

$$B := (1 - (-1)^n t_{n+1}) s_{-1} \mathcal{N},$$

  where $\mathcal{N} := \sum_{j=0}^{n} (-1)^n t_{n+1}^j$ and $s_{-1} := t_{n+1} s_n$.

The Gerstenhaber module: illustration of the Lie derivative

Extending k to the formal power series $k[[r]]$, for any $k[[r]]$-linear map $D : C_n(U, M)[[r]] \to C_n(U, M)[[r]]$, define the operators $t^D := D\, t,\ T^D := (t^D)^{n+1}$. Apply this with D being the exp. series

$$\exp(r\varphi) := \textstyle\sum_{i \geq 0} \frac{1}{i!}(rD'_\varphi)^i.$$

for a 1-cocycle φ, where D'_φ is D_φ applied to the last element in $C_n(U, M)$. Thinking of φ as of a generalised vector field, of $\exp(r\varphi)$ as of its flow, and of

$$\Omega_\varphi := \mathrm{id} - T^{\exp(r\varphi)}$$

as of a curvature along an integral curve, define

$$\mathcal{L}_\varphi := \tfrac{d}{dr}\Omega_\varphi|_{r=0}$$

for $n > 0$. Explicitly,

$$\mathcal{L}_\varphi = \sum_{i=1}^{n} \pm t^{n-i}\, D'_\varphi\, t^{i+1} + t^n\, D'_\varphi\, t.$$

The universal Lie derivative

If φ is a general p-co*chain*, the expression for the Lie derivative assumes the truly unpleasant form

$$\mathcal{L}_\varphi := \sum_{i=1}^{n-p+1} \pm t^{n-p+1-i}\, D'_\varphi\, t^{i+p} + \sum_{i=1}^{p} \pm t^{n-p+1}\, D'_\varphi\, t^i,$$

where now D'_φ is D_φ applied on the last p components of an element in $C_n(U, M)$.

Theorem

If M is SaYD and (U, A) a left Hopf algebroid, the triple $(\mathcal{L}, \iota, \mathrm{B})$ induces the structure of a noncommutative calculus on the pair $\big(\mathrm{Ext}^\bullet_U(A, A), \mathrm{Tor}^U_\bullet(M, A)\big)$. In particular, one has the Cartan homotopy formula

$$\mathcal{L} = [\iota, \mathrm{B}]$$

on homology.

Lie-Rinehart algebras (Lie algebroids)

- Let (A, L) be a Lie-Rinehart algebra over k: L is a k-Lie algebra and a module over the commutative k-algebra A, but it also acts via derivations ∂_X on A (called the *anchor*) such that

$$[X, aY] = (\partial_X a)Y + a[X, Y], \quad \partial_{aX} = a\partial_X, \quad a \in A, X, Y \in L.$$

- Examples: $L = \mathrm{Der}_k(A, A)$. Foliations are sub Lie-Rinehart algebras. Poisson algebras $(A, \{\cdot, \cdot\})$ with $L = \Omega^1(A)$ and $[da, db] := d\{a, b\}$.
- $U := VL$ is a generalised universal algebra equipped with two maps

$$\iota_A : A \to VL, \quad \iota_L : L \to VL$$

of k-algebras resp. k-Lie algebras subject to

$$\iota_A(a)\iota_L(X) = \iota_L(aX), \quad \iota_L(X)\iota_A(a) - \iota_A(a)\iota_L(X) = \iota_A(\partial_X(a))$$

for $a \in A, X \in L$. The map ι_A is injective. If L is A-projective, then ι_L is injective as well. We'll suppress the maps in the sequel.

Lie-Rinehart algebras (Lie algebroids)

- VL carries the structure of a left bialgebroid with $\eta(- \otimes 1) = \eta(1 \otimes -)$ given by ι_A. The prescriptions

$$\Delta(X) = 1 \otimes_A X + X \otimes_A 1, \quad \Delta(a) = a \otimes_A 1$$

  can be extended by the universal property to a coproduct $\hat{\Delta} : VL \to VL \times_A VL$. The counit is similarly given by the extension of the anchor to VL.
- Easy observation: the Galois map is bijective, the translation map is given on generators as

$$a_+ \otimes_{A^{\mathrm{op}}} a_- := a \otimes_{A^{\mathrm{op}}} 1, \quad X_+ \otimes_{A^{\mathrm{op}}} X_- := X \otimes_{A^{\mathrm{op}}} 1 - 1 \otimes_{A^{\mathrm{op}}} X$$

  and then extended via universality and axioms to an inverse of the Galois map.

Application: jet spaces

The classical operators on forms & fields are obtained as follows:

- The A-linear dual $JL := \mathrm{Hom}_A(VL, A)$ of the universal enveloping algebra VL of a Lie-Rinehart algebra (A, L) is called the **jet space**. JL is a (topological) commutative left Hopf algebroid: actually all that was said before now has to be repeated for topological Hopf algebroids and all tensor products have to be completed and speak of continuous chains & cochains.
- Assume that L is fin. gen. projective over A. With $L^* := \mathrm{Hom}_A(L, A)$, one has $\mathrm{Hom}_A(\bigwedge_A^\bullet L, A) \simeq \bigwedge_A^\bullet L^*$, and chain morphisms

$$\begin{aligned} F : (C_\bullet(JL, A), b) &\to (\textstyle\bigwedge_A^\bullet L^*, 0) \\ F^* : (\textstyle\bigwedge_A^\bullet L, 0) &\to (C^\bullet(JL, A), \delta), \end{aligned}$$

  and if $\mathbb{Q} \subseteq k$, the map F has a right inverse F'.
- These morphisms may be seen as **HKR maps** for Lie algebroids.

Generalised forms and vector fields

- In the case of a Lie algebroid $E \to M$ over a smooth manifold M (*e.g.*, the tangent bundle $E = TM$), the pair

$$\left(\bigwedge\nolimits_A^\bullet L, \bigwedge\nolimits_A^\bullet L^* \right),$$

where

$$L = \Gamma^\infty(E), \quad A = C^\infty(M), \quad L^* \simeq \Gamma^\infty(E^*),$$

forms a differential calculus with respect to the well-known operators $(\mathcal{L}_X, \iota_X, d_{dR})$ of (E-)Lie derivative of an (E-)form along an (E-)vector field $X \in L$, along with (E-)contraction and the (E-)de Rham differential.
- The connection to our general operators is then, on homology,

$$\begin{aligned} d_{dR} \circ F &= F \circ B, \\ \iota_X \circ F &= F \circ \iota_{F^*X}, \\ \mathcal{L}_X \circ F &= F \circ \mathcal{L}_{F^*X}. \end{aligned}$$

More satisfactory

- You might find the last slide not very satisfactory due to the fact that I skipped all the topological details. That was on purpose.
- A slightly different approach avoids involving the jet spaces and hence any topological issues, and also drops the finiteness assumptions on L. On the other hand, it involves *contramodules, cotensor products, cohomomorphisms*, and their more exotic derived functors:

Corollary

Let (A, L) *be a Lie-Rinehart algebra, with* L *not necessarily finitely generated but projective as an* A*-module, and* M *an* A*-injective* VL*-module. Then the HKR map*

$$\mathrm{Alt}\colon \textstyle\bigwedge_A^n L \to VL^{\otimes_A n}$$

of antisymmetrisation induces an isomorphism of nc differential calculi between $\big(\bigwedge_A^\bullet L, \mathrm{Hom}_A(\bigwedge_A^\bullet L, M)\big)$ *and* $\big(\mathrm{Cotor}^\bullet_{VL}(A, A), \mathrm{Coext}^{VL}_\bullet(A, M)\big)$. *On homology,*

$$\begin{aligned}
\mathrm{Hom}_A(\mathrm{Alt}, M) \circ B &= d_{\mathrm{dR}} \circ \mathrm{Hom}_A(\mathrm{Alt}, M),\\
\mathrm{Hom}_A(\mathrm{Alt}, M) \circ \iota_{\mathrm{Alt}(-)} &= \iota_{(-)} \circ \mathrm{Hom}_A(\mathrm{Alt}, M),\\
\mathrm{Hom}_A(\mathrm{Alt}, M) \circ \mathcal{L}_{\mathrm{Alt}(-)} &= \mathcal{L}_{(-)} \circ \mathrm{Hom}_A(\mathrm{Alt}, M).
\end{aligned}$$