Group Presentation Theme K

"Beyond Plain-Vanilla Multilevel/-index Monte Carlo"

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Erwin Schrödinger Institut, Vienna

May 6th, 2022

Main Topics

- Multi-index (-level) Monte Carlo for UQ problems with a 'homotopy' parameter

$$M(u(\omega, t); t) = 0 \qquad t \in [0, 1]$$

$$\text{Find} \quad \mathbb{E}[\Phi(u(\cdot, 1))]$$

but problem hard to solve for $t = 1$; often solved via **continuation**, i.e. solving iteratively for $0 = t_0 < t_1 \ldots < t_{k-1} < t_k = 1$.

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- More generally, can have a system parameter α that controls 'difficulty' of the problem (typically linked to cost), e.g.,
  - regularisation parameter in a regularised minimisation problem
  - strength of advection or nonlinear reaction term in an advection-reaction-diffusion problem
  - loading in a nonlinear elasticity problem
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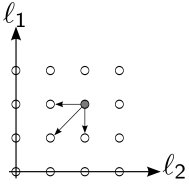
- Some other topics: e.g., MLMC for problems with stability constraints and sample-dependent mesh size restrictions.

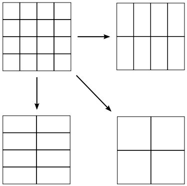
Multi Index Monte Carlo method

- Assume several discretization parameters
  (e.g. spatial mesh, time step, domain size, model, ...)
- Introduce sequences $h_0^{(i)} > h_0^{(i)} > \ldots > h_{L_i}^{(i)}$
- For $\vec{\ell} = (\ell_1, \ldots, \ell_d)$, denote $Q_{\vec{\ell}} = Q(u_{h_{\ell_1}^{(1)}, \ldots, h_{\ell_d}^{(d)}})$
- Difference operators

$$\Delta_j Q_{\vec{\ell}} = \begin{cases} Q_{\vec{\ell}} - Q_{\vec{\ell} - \vec{e}_j}, & \text{if } \ell_j > 0 \\ Q_{\vec{\ell}}, & \text{if } \ell_j = 0 \end{cases}$$

$$\Delta Q_{\vec{\ell}} = \bigotimes_{j=1}^{d} \Delta_j Q_{\vec{\ell}} = \sum_{\vec{j} \in \{0,1\}^d} (-1)^{|\vec{j}|} Q_{\vec{\ell} - \vec{j}}$$





Multi Index Monte Carlo method

Telescopic formula: given finest discretization level $\vec{L} = (L_1, \ldots, L_d)$

$$\mathbb{E}[Q_{\vec{L}}] = \sum_{\vec{\ell} \leq \vec{L}} \mathbb{E}[\Delta Q_{\vec{\ell}}]$$

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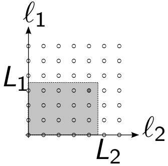
Multi Index idea: compute each expectation independently

$$\mu_{\vec{L}}^{MIMC} = \sum_{\vec{\ell} \leq \vec{L}} \frac{1}{M_{\vec{\ell}}} \sum_{i=1}^{M_{\vec{\ell}}} \Delta Q_{\vec{\ell}}^{(i,\vec{\ell})}$$



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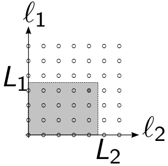
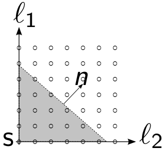
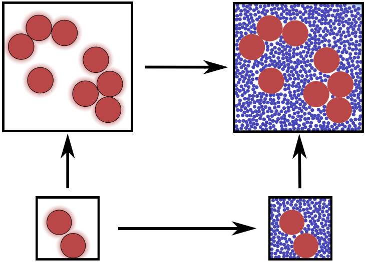


Further sparsification: often the set $\{\vec{\ell} \leq \vec{L}\}$ is not the optimal one. Optimized index sets $\mathcal{I} \subset \mathbb{N}^d$ can lead to substantial improvement

$$\mu_{\mathcal{I}}^{MIMC} = \sum_{\vec{\ell} \in \mathcal{I}} \frac{1}{M_{\vec{\ell}}} \sum_{i=1}^{M_{\vec{\ell}}} \Delta Q_{\vec{\ell}}^{(i,\vec{\ell})}$$



Particle System Example – Paul Rohrbach's Talk



- Coarse graining:

$$\pi_{1,j}(g_{1,j}) - \pi_{0,j}(g_{0,j}) < \varepsilon_{\mathsf{CG}}$$

- Domain size:

$$\pi_{i,1}(g_{i,1}) - \pi_{i,0}(g_{i,0}) < \varepsilon_{\mathsf{DS}}$$

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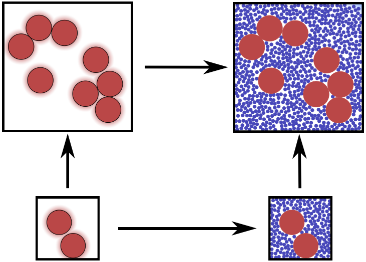

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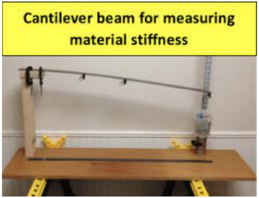
- "Double difference":

$$(\pi_{1,1}(g_{1,1}) - \pi_{0,1}(g_{0,1})) - (\pi_{1,0}(g_{1,0}) - \pi_{0,0}(g_{0,0})) \approx \varepsilon_{\mathrm{CG}}\varepsilon_{\mathrm{DS}} \approx 0$$

$$\implies \quad \pi_{1,1}(g_{1,1}) \approx \pi_{0,1}(g_{0,1}) + (\pi_{1,0}(g_{1,0}) - \pi_{0,0}(g_{0,0}))$$

Recombination technique (Bungartz, Griebel, ...)
+ possibly variance reduction

Other systems



Nonlinear Elasticity

(Dodwell, Scheichl, Seelinger,...)

- Incremental loading of beam; nonlinear solves get harder and harder
- Should have 'good decoupling' of hierarchies (mixed regularity)

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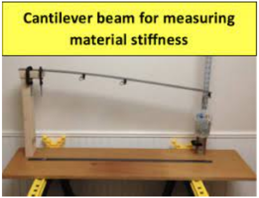

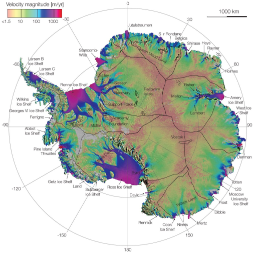

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Ice Sheet Dynamics (Cui, Peherstorfer,...)

- Simplified model

$$-\nabla \cdot (\eta(u) \nabla u) - f = 0,$$

$$\eta(u) = (2\gamma(u) + \epsilon)^{\frac{p-2}{2}}, \quad \gamma(u) = \frac{1}{2}|\nabla u|^2,$$

- Homotopy over nonlinearity p

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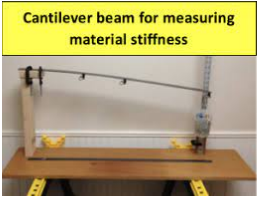

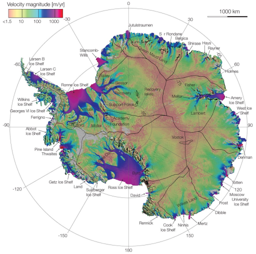

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Problem: Do we have mixed regularity (i.e. good decouplig of discretisation hierarchies)? Not sure for advection-diffusion, for example!

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- Explore other forms or regularity or other "recombination of terms" (different telescoping sums)
- Learn the best recombinations of terms from data (multifidelity / multilevel BLUE estimator)

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K. **Beyond Plain-Vanilla MLMC/MIMC** (+ Farcas, Khodabakhshi, Croci, Robbe)