Twisted Drinfeld double : from strings to the Kitaev model

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Introduction

Path integral derivation of the operators T that lift the action of a finite group G to the twisted sectors of bosonic strings on the **orbifold** $\mathcal{M}/G$ in a 3-form magnetic background H.

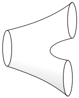
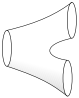
String propagator written as a sum over **worldsheets** each carrying its own magnetic contribution.

The algebra generated by the operators T is the **quasi-quantum group** $D_\omega[G]$, introduced in the context of conformal field theory by R. Dijkgraaf, V. Pasquier and P. Roche with

- a **product** is determined by the commutation with propagation

T  $=$  $T,$ (1)

- a **coproduct** follows from the commutation with the most basic interaction

T  $=$  $\Delta T.$ (2)

Magnetic amplitude for twisted sectors are also ground states of a version of the Kitaev lattice model

States and symmetries in quantum mechanics

A **quantum system** is defined by a Hilbert space $\mathcal{H}$ and observables which are Hermitian operators acting on $\mathcal{H}$. A state of the system is defined by a line s in $\mathcal{H}$ (normalized vectors defined up to a phase).

The **probability** of observing the system in the state represented by χ knowing that it is in the state represented by ψ is $\left|\langle\psi,\chi\rangle\right|^2$.

A **symmetry** is a transformation of the space of states $s \to s'$ preserving the transition probabilities, $\left|\langle\psi',\chi'\rangle\right|^2 = \left|\langle\psi,\chi\rangle\right|^2$.

Theorem (Wigner)

Each symmetry acting on states $s \to s'$ can be implemented by a unitary or antiunitary operator U on $\mathcal{H}$.

$$\psi \in s \Rightarrow \psi' = U\psi \in s'$$

and these operators are unique up to a phase.

Antiunitary symmetries : time reversal T, charge conjugation C

Projective representations

If a group G acts on the states preserving the transition probabilities, the operators U_g are only defined up to phases

$$U_g U_h = \omega_{g,h} U_{gh}$$

Projective representations are classified using group cohomology

- associativity constraint : ω is a 2-cocycle

$$U_g(U_h U_k) = (U_g U_h) U_k \quad \Leftrightarrow \quad \underbrace{\omega_{h,k}\, \omega_{gh,k}^{-1}\, \omega_{g,hk}\, \omega_{g,h}^{-1}}_{(\delta\omega)_{g,h,k}} = 1$$

- triviality : ω is a coboundary

$$\omega_{g,h} = \underbrace{\eta_h\, \eta_{gh}^{-1}\, \eta_g}_{(\delta\eta)_{g,h}} \quad \Leftrightarrow \quad V_g V_h = V_{gh} \quad \text{with} \quad V_g = \eta_g U_g$$

General group cohomology : n-cochains are functions on n copies of G with values in a abelian group carrying an action of G, $\delta^2 = 0$ with

$$\delta\omega(g_0, g_1, \dots, g_n) = g_0 \cdot \omega(g_1, \dots, g_n)$$
$$\times \prod_{i=0}^{n-1} \Big[\omega(g_0, \dots, g_i g_{i+1}, \dots)\Big]^{(-1)^{i-1}} \times \Big[\omega(g_0, g_1, \dots, g_{n-1})\Big]^{(-1)^{n-1}}$$

Magnetic amplitude for a particle

For a particle on a manifold $\mathcal{M}$ in a **magnetic background** B (closed 2-form with integral periods), wave functions are sections of a line bundle $\mathcal{L}$ over $\mathcal{M}$ with a connection ∇ of curvature B.

In the **path integral** approach, the kernel of the evolution operator is

$$K(y,x) = \int_{\substack{\varphi(a)=x \\ \varphi(b)=y}} [D\varphi]\, \mathrm{e}^{-S[\varphi]}\, \mathcal{A}[\varphi], \tag{3}$$

S classical action (not involving the magnetic field)
$\mathcal{A}[\varphi]$ **holonomy** of ∇ along the path φ.

Both $\mathcal{L}$ and $\mathcal{A}[\varphi]$ are constructed using a good open cover $\{U_i\}$

$$\begin{cases} B_i & = & \mathrm{d}A_i & \text{on} & U_i, \\ A_j - A_i & = & \mathrm{i}\,\mathrm{d}\log f_{ij} & \text{on} & U_i \cap U_j, \\ f_{jk}(f_{ik})^{-1} f_{ij} & = & 1 & \text{on} & U_i \cap U_j \cap U_k, \end{cases} \tag{4}$$

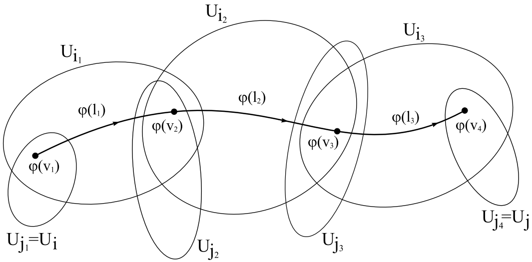
Invariance under **gauge transformations** of A_i and f_{ij}.

Explicit expression of the magnetic amplitude

Using a cover of the path, the magnetic amplitude (holonomy of the connection along the path) is

$$\mathcal{A}_{ij}[\varphi] = \exp \mathrm{i} \left\{ \sum_{l_\alpha \in l} \int_{l_\alpha} \varphi^* A_{i_\alpha} \right\} \prod_{\substack{l_\alpha \in l \\ v_\beta \in \partial l_\alpha}} f_{i_\alpha j_\beta}^{-\epsilon_{\alpha\beta}}(\varphi(v_\beta)), \qquad (5)$$

where $\epsilon_{\alpha\beta} = +1$ if l_α is arriving at v_β and -1 if it is leaving.



In accordance with its interpretation as a map from the fibre at x of $\mathcal{L}$ to that at y, it is **independent of the covering** and **gauge invariant**, except at the boundaries.

Projective group action on wave functions

Classical symmetry : Action of a (finite) group G on $\mathcal{M}$ such that S is genuinely invariant and $g^*B = B$.

Quantum symmetry : Lift of the action of G to the Hilbert space $\mathcal{H}$ of wave functions (ϕ_g isomorphism between $(g^*\mathcal{L}, g^*\nabla)$ and $(\mathcal{L}, \nabla)$)

$$T_g\psi(x) = \phi_g(x)\,\psi(x\cdot g) \tag{6}$$

The phases are determined in the path integral formalism by the **commutation** of T_g with propagation $(KT_g = T_g K)$

$$\mathcal{A}[\varphi\cdot g] = \phi_g^{-1}(y)\,\mathcal{A}[\varphi]\,\phi_g(x). \tag{7}$$

Projective representation $T_g T_h = \omega_{g,h}\,T_{gh}$ with the group 2-cocycle

$$\omega_{g,h} = \phi_h(x\cdot g)\,\phi_{gh}^{-1}(x)\phi_g(x). \tag{8}$$

The operators T_g generate the **twisted group algebra**.

The cohomology class of ω is an **obstruction** to the existence of a quantum theory on $\mathcal{M}/G$ (no invariant states in $\mathcal{H}$).

Generalization of **magnetic translations** for a particle on $\mathbb{R}^N$ in a uniform magnetic field with $G = \mathbb{Z}^N$. (twisted group algebra = **noncommutative torus**.)

Magnetic fields for closed strings

A closed string on $\mathcal{H}$ sweeping a **worldsheet** Σ couples to a 2-form magnetic potential B (Kalb-Ramond field) with 3-form field strength $H = dB$

$$\rightarrow \qquad \mathrm{e}^{\mathrm{i}\int_\Sigma B} \tag{9}$$

In general the potentials are only locally defined and correspond to a **gerb with connection** from which we compute the holonomy around Σ using a triangulation

$$\begin{cases} H_i = \mathrm{d}B_i & \text{on} \quad U_i\,, \\ B_j - B_i = \mathrm{d}B_{ij} & \text{on} \quad U_i \cap U_j\,, \\ B_{jk} - B_{ik} + B_{ij} = \mathrm{i}\,\mathrm{d}\log f_{ijk} & \text{on} \quad U_i \cap U_j \cap U_k\,, \\ f_{jkl}(f_{ikl})^{-1} f_{ijl}(f_{ijk})^{-1} = 1 & \text{on} \quad U_i \cap U_j \cap U_k \cap U_l\,, \end{cases} \tag{10}$$

with two layers of gauge transformations.

Example of **WZW** models with $\mathcal{M} = \mathrm{SU(N)}$ and $H = \frac{k}{12\pi}\mathrm{Tr}(g^{-1}dg)^3$.

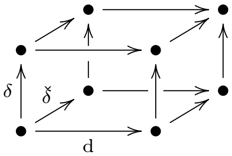
Interpretation of the holonomy around cylinders as parallel transport for a line bundle over the **loop space**.

Tricomplex with de Rham, Cech and group cohomologies

Tricomplex with cochains $C_{p,q,r}$ that are de Rham forms of degree p, defined on $(q+1)$-fold intersections of a "good invariant cover", $U_{i_0} \cap \cdots \cap U_{i_q}$ and functions of r group indices.

Three **commuting differentials**

- de Rham differential in the p direction (idlog for functions)
- Čech coboundary $\check{\delta}$ in the q direction
- group coboundary δ in the r direction



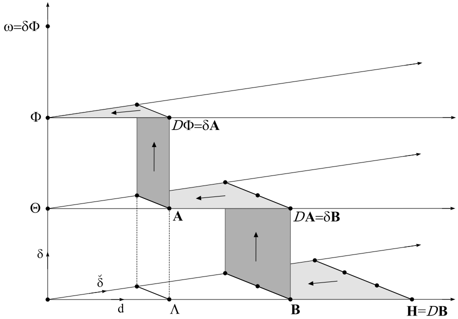
For any fixed value of r, we have a Čech-de Rham bicomplex,

$$C^{\mathrm{tot}}_{r,s} = \bigoplus_{p+q=s} C_{p,q,r}, \tag{11}$$

with the **Deligne differential** defined by $\mathcal{D} = \mp \mathrm{d} \pm \check{\delta}$ fulfilling $\mathcal{D}^2 = 0$ and $\delta \mathcal{D} = \mathcal{D} \delta$.

Symmetries of 2-form potentials

Starting with $\mathsf{H} = (H_i, 0, 0, 1) \in C^{\text{tot}}_{0,3}$ such that $\mathcal{D}\mathsf{H} = 0$ and $\delta\mathsf{H} = 0$ (globally defined closed invariant 3-form), we solve a series of cohomological equations ending in a constant 3-cocycle $\omega \in C^{\text{tot}}_{3,0}$, with gauge ambiguities in the definition of B and A.



- $g^*\mathsf{B} - \mathsf{B} = \mathcal{D}\mathsf{A}_g$
- $g^*\mathsf{A}_h - \mathsf{A}_{gh} + \mathsf{A}_g = \mathcal{D}\Phi_{g,h}$
- $g^*\Phi_{h,k}\left(\Phi_{gh,k}\right)^{-1}\Phi_{g,hk}\left(\Phi_{g,h}\right)^{-1} = \omega_{g,h,k}$

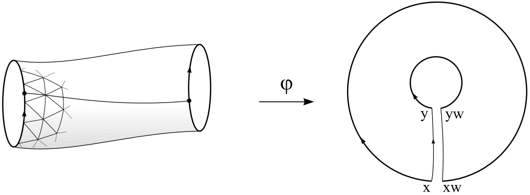
Magnetic amplitude for twisted sectors

Twisted sectors on $\mathcal{M}/G$ are strings $X : [0, 2\pi] \to \mathcal{M}$ that close up to their winding $w \in G$: $\quad X(2\pi) = X(0) \cdot w$.

Free string propagation involves a path integral with **magnetic amplitude**

$$\mathcal{A}[\varphi] = \mathrm{e}^{\mathrm{i} \int_\Sigma \mathsf{B} + \mathrm{i} \int_x^y \mathsf{A}_w} \tag{12}$$

for the cylinder with **cut** and **triangulation** embedded in $\mathcal{M}$



String wave functions Ψ = sections of a line bundle over twisted sectors
Magnetic amplitude for cylinders = parallel transport

Invariance under simultaneous **gauge transformations** of B, A, Φ and Ψ

Stringy magnetic translations and their algebra

Stringy magnetic translations $T^w_g : \mathcal{H}_{w^g} \to \mathcal{H}_w$ lift the group action to the twisted sectors commuting with propagation

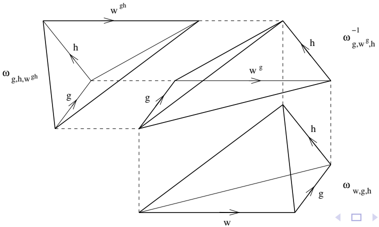
$$T^w_g \Psi(X) = \Gamma_{w,g}(x)\, \mathrm{e}^{-\mathrm{i}\int_x^{xw} A_g}\, \Psi(X\cdot g), \tag{13}$$

with $\Gamma_{w,g} = \Phi_{g,w^g} \Phi^{-1}_{w,g}$ and $w^g = g^{-1}wg$.

Projective representation on the twisted sectors identical to the multiplication law of the **quasiquantum group** $D_\omega[G]$

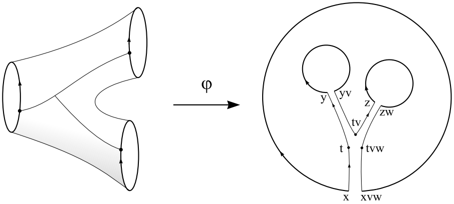
$$T^w_g\, T^v_h = \delta_{v,w^g}\, \frac{\omega_{w,g,h}\, \omega_{g,h,w^{gh}}}{\omega_{g,w^g,h}}\, T^w_{gh}. \tag{14}$$

Combinatorial interpretation in terms of tetrahedra representing the 3-cocycle (transgression $(n+1)$-cocycle $\to$ n-cocycle depending on w)



Interactions

Most basic interaction involves **pair of pants**



with magnetic amplitude contributing to the decay $\mathcal{H}_{vw} \to \mathcal{H}_v \otimes \mathcal{H}_w$

$$\mathcal{A}[\varphi] = \mathrm{e}^{\mathrm{i}\int_\Sigma \mathsf{B} + \mathrm{i}\int_x^t \mathsf{A}_{vw} + \mathrm{i}\int_t^y \mathsf{A}_v + \mathrm{i}\int_{tv}^z \mathsf{A}_w} \, \Phi_{v,w}^{-1}(t), \tag{15}$$

Φ is inserted at the splitting point to maintain gauge invariance for A.

Global anomalies for magnetic amplitudes on arbitrary surfaces that depend on ω and on a representation of $\pi_1(\Sigma)$.

Consistency condition $\omega = 1$ for the orbifold $\mathcal{M}/G$. (Analogous to the particle's case.)

Quasi-Hopf algebras

An algebra $\mathcal{A}$ is a Hopf algebra if it admits a counit $\epsilon : \mathcal{A} \to \mathbb{K}$, coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and and antipode $S : \mathcal{A} \to \mathcal{A}$ such that $m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \epsilon$.

Examples :

- group algebra $\mathbb{C}[G] = \{ \sum a(g)\, g \}$ with $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$
- functions on G with pointwise product, $\Delta f(g,h) = f(gh)$, $\epsilon(f) = 1$, $Sf(g) = f(g^{-1})$.

Modules over a Hopf algebra form a category with trivial representation (ϵ), tensor products (Δ) and duals (S).

A quasi-Hopf algebra $\mathcal{A}$ has a coproduct associative up to the *Drinfel'd associator* $\Omega \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

$$(\mathrm{id} \otimes \Delta) \circ \Delta = \Omega \, [(\Delta \otimes \mathrm{id}) \circ \Delta] \, \Omega^{-1}$$

obeying the pentagon axiom.

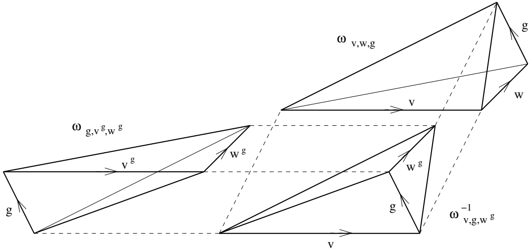
A bialgebra is quasi-cocommutative, if there is an invertible $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$
$\Delta^{\mathrm{op}}(b) = \mathcal{R}\Delta(b)\mathcal{R}^{-1}$
$\Rightarrow$ braid group action $\sigma \circ R : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_1$, $\sigma(\psi \otimes \chi) = \chi \otimes \psi$

Derivation of the coproduct

Commutation of the orbifold group action with the decay process dictates the action of T_g^w on $\mathcal{H}_u \otimes \mathcal{H}_v$ form whcih we read the **coproduct**

$$\Delta(T_g^u) = \sum_{vw=u} \frac{\omega_{v,w,g}\,\omega_{g,v^g,w^g}}{\omega_{v,g,w^g}}\, T_g^v \otimes T_g^w . \qquad (16)$$

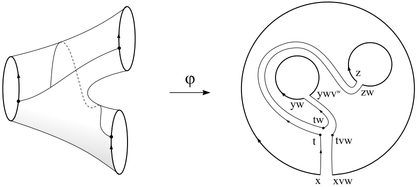
Combinatorial interpretation of the extra phase in the action on tensor products



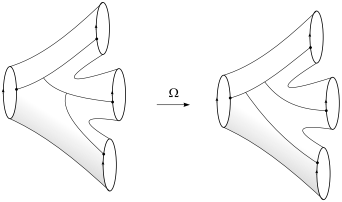
The operators T_g^w generate the **quasi-quantum group** $D_\omega[G]$ which is a quasi-triangular quasi-Hopf algebra deformation of the **quantum double** of the group algebra of G.

Action of the quasi-Hopf algebra

$\mathcal{R}$-matrix defines a **braid group** action on tensor products



Coassociativity up to the **Drinfeld associator** : states in $(\mathcal{H}_u \otimes \mathcal{H}_v) \otimes \mathcal{H}_w$ and in $\mathcal{H}_u \otimes (\mathcal{H}_v \otimes \mathcal{H}_w)$ only differ by the global phase $\omega_{u,v,w}$.



Antipode related to reversing the string **orientation** $S(T_g^w) =\propto T_{g^{-1}}^{(w^{-1})^g}$

Discrete de Rham cohomology

A n-form $\Omega(x_0, \ldots, x_n)$ is an antisymmetric function on X^{n+1} with values in an abelian group, equipped with a differential

$$d\Omega(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \Omega(\underbrace{x_0, \ldots, \check{x}_i, \ldots, x_{n+1}}_{x_i \text{ removed}})$$

Geometrical interpretation : $\Omega(x_0, \ldots, x_n)$ flux over a n-simplex with $n+1$ vertices $x_0, \ldots, x_n$.

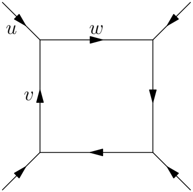
Some simple examples

$$\begin{aligned} d\Phi(x,y) &= \Phi(y) - \Phi(x) \\ dA(x,y,z) &= A(y,z) - A(x,z) + A(x,y) \\ dB(x,y,z,t) &= B(y,z,t) - B(x,z,t) + B(x,y,t) - B(x,y,z) \end{aligned}$$

This differential is nilpotent $d^2 = 0$.

Kitaev model

Kitaev model defined on a triangular graph Γ on a surface Σ with Hilbert space constructed by assigning group elements to the (oriented) edges



$$\Psi(\{g_e\}) \in \mathcal{H} = \bigotimes_{\text{edges}} \text{Fun}(G \to \mathbb{C}) \tag{17}$$

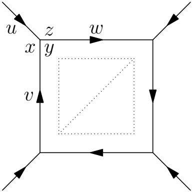
$$H = -\sum_{\text{faces } f} P_f - \sum_{\text{vertices } v} \delta_v \tag{18}$$

P_f (translation of face variables $u \to g^{-1}u, w \to wg, \ldots$) and δ_v (constraint $uvw^{-1} = 1$) are mutually commuting projectors

Ground states given by moduli space $\text{Hom}(\pi_1(\Sigma) \to G)/\text{Ad}(G)$

Twisted Kitaev model

See "Twisted Quantum Double Model of Topological Phases in Two–Dimension" Yuting Hu, Yidun Wan, Yong-Shi Wu, https://arxiv.org/abs/1211.3695



Triangulate each face of Γ and decorate vertices with variables $x_v \in X$

$$\Psi(\{g_e\}, \{x_v\}) \in \bigotimes_{\text{edges}} \text{Fun}(G \to \mathbb{C}) \bigotimes_{\text{vertices}} \text{Fun}(X \to \mathbb{C})$$

$$P_f \psi(x, w, \dots) = \psi(xg, g^{-1}w, \dots) \times \prod_{\text{vertices in } \partial f} \omega_{u,v,g} \qquad \text{for } x, w \in f$$

$$\delta_v = \prod_{\text{around } v} \delta_{x_j, x_i g_{ij}}$$

Ground states

A ground state can be constructed using the previous gerbe amplitude

$$\Psi(\{g_e\},\{x_v\}) = \prod_{\text{vertices}} \delta_{xu,y}\delta_{yv,z}\delta_{yw,z} \times$$
$$\times \prod_{\text{triangles}} \exp \mathrm{i} B(x,y,z) \prod_{\text{edges}} \exp \mathrm{i} A_w(x,y) \prod_{\text{vertices}} \Phi_{u,v}(x)$$

- lift of a single corner

$$\Phi_{g^{-1}u,u^{-1}v}(xg)\,\omega_{g,g^{-1}u,u^{-1}v} = \frac{\Phi_{g,g^{-1}v}(x)}{\Phi_{u,u^{-1}v}(x)\Phi_{g,g^{-1}u}(x)}$$

- lift of a triangle :

$$B(xg,yg,zg) = B(x,y,z) + A_g(x,y) + A_g(y,z) + A_g(z,x)$$

- lift of an edge

$$A_{g^{-1}w}(xg,yg) =$$
$$A_w(x,y) - A_g(x,y) + \log -\mathrm{i}\Phi_{g,g^{-1}w}(y) + \log -\mathrm{i}\Phi_{g,g^{-1}w}(x)$$

Conclusion and outlooks

$D_\omega[G]$ is a higher dimensional generalization of projective group representations

particles	strings
2-form B	3-form H
line bundle	gerbe
2-cocycle ω	3-cocycle ω
twisted group algebra	quasi-quantum group

In both cases, theory on $\mathcal{M}/G$ is **consistent** only if $\omega = 1$.

Alternative application : Ground state of a twisted and extended Kitaev model :

- Enlarge Hilbert space by adding "matter" degrees of freedom.
- Enforce gauge invariance and flatness constraint.
- Are there anyonic excitations ?