N-POINT VIRASORO ALGEBRAS ARE MULTI-POINT KRICHEVER–NOVIKOV TYPE ALGEBRAS

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INTRODUCTION

- the classical genus zero (two point) algebras (Witt algebra, Virasoro algebra, affine Kac-Moody algebras of untwisted type, ...) are well-established and of relevance e.g. in CFT
- but from the application there is a need for the multi-point algebras in every genus (of course including genus zero)
- higher genus and still two points this was done by Krichever and Novikov
- the multi-point theory was done to a large extend by the speaker
- importance for KZ equations for genus zero in CFT is nowadays classical
- for higher genus KZ connections in the context of $M_{g,n}$ see joint work of the speaker with Oleg Sheinman
- recently revived interest in genus zero multi-point quantum field theory (N-point Virasoro algebra)

- Goal: show that the recently discussed N-point Virasoro algebras (Cox, Jurisisch, Martins, and others) are special examples of the multi-point KN type algebras
- Gain: gives useful structural insights and an easier approach to calculations

- Something to be learned again: Generalize the situation and understand the structure better.

- Side-effect was also to remove some misconceptions about certain observed phenomena

What I will do here:

- recall the geometric setup for KN type algebras
- introduce the algebras
- almost-grading including triangular decomposition
- determine “all” central extensions

Why are central extensions so important??

In the process of quantization we are forced to pass from our algebras to their central extensions, e.g. by regularisation, subtraction of infinity, etc.

What will be the outcome for KN type, genus zero:

- all cocycle classes for vector field algebra and the differential operator algebras are geometric
- give the universal central extensions for them explicitly
- the same for the current algebra, yielding affine algebras
- Heisenberg algebra obtained by cocycles for the function algebra which are multiplicative
- give access to easy calculations of structure constants and cocycle values for these algebras
- As illustration: three point genus zero situation.

CLASSICAL ALGEBRAS

- purely algebraic terms the Virasoro algebra
  generators $\{e_n (n \in \mathbb{Z}), t\}$ and relations

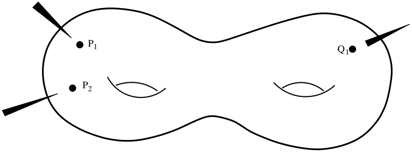
$$[e_n, e_m] = (m-n)e_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m} \cdot t, \quad [t, e_n] = 0.$$

- without central term: Witt algebra
- $\mathfrak{g}$ a finite-dimensional simple Lie algebra,
  β the Cartan–Killing form,

$$[\widehat{x \otimes z^n}, \widehat{y \otimes z^m}] := \widehat{[x, y] \otimes z^{n+m}} - \beta(x, y) \cdot n\,\delta_m^{-n} \cdot t.$$

$\widehat{\mathfrak{g}}$ is called affine Lie algebra.

GEOMETRIC SET-UP (KN TYPE ALGEBRAS)



- Σ_g be a compact Riemann surface of genus $g = g(\Sigma_g)$.
- A be a finite subset of Σ_g, $A = I \cup O$, both non-empty, disjoint, $I = (P_1, \ldots, P_K)$ in-points and $O = (Q_1, \ldots, Q_M)$ out-points
- genus zero: $A = \{P_1, P_2, \ldots, P_N\}$, P_N can be brought to ∞ by fractional linear transformation
- $P_i = a_i, \quad a_i \in \mathbb{C}, \ i = 1, \ldots, N-1, \quad P_N = \infty$
- local coordinates $z - a_i, \ i = 1, \ldots, N-1, \qquad w = 1/z$
- classical situation: $\Sigma_0 = S^2, \quad I = \{0\}, \quad O = \{\infty\}$

Geometric realizations of the KN type algebras

- $\mathcal{K}$ is the canonical bundle, i.e. local sections are the holomorphic differentials
- $\mathcal{K}^\lambda := \mathcal{K}^{\otimes \lambda}$ for $\lambda \in \mathbb{Z}$
- the sections are the forms of weight λ, e.g. $\lambda = -1$ are vector fields, $\lambda = 0$ are functions,
- for half-integer λ we need to fix a square root L of $\mathcal{K}$ (also called theta characteristics, or spin structure)
- for $g = 0$ only one square-root, the tautological bundle U
- we ignore in this presentation the half-forms (e.g. the supercase) (but see the journal article)

- $\mathcal{F}^\lambda := \mathcal{F}^\lambda(A) := \{f$ is a global meromorphic section of $\mathcal{K}^\lambda$ such that f is holomorphic over $\Sigma \setminus A\}$.
- infinite dimensional vector spaces
- meromorphic forms of weight λ
- $$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda.$$

- We define an associative structure

$$\cdot : \mathcal{F}^{\lambda} \times \mathcal{F}^{\nu} \to \mathcal{F}^{\lambda+\nu}.$$

- in local representing meromorphic functions

$$(s\,dz^{\lambda}, t\,dz^{\nu}) \mapsto s\,dz^{\lambda} \cdot t\,dz^{\nu} = s \cdot t\; dz^{\lambda+\nu}.$$

- $\mathcal{F}$ is an associative and commutative graded algebra.
- $\mathcal{F}^0 =: \mathcal{A}$ is a subalgebra and $\mathcal{F}^{\lambda}$ are modules over $\mathcal{A}$.

- Lie algebra structure:

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (s,t) \mapsto [s,t],$$

- in local representatives of the sections

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto [s\,dz^\lambda, t\,dz^\nu] := \left((-\lambda)s\frac{dt}{dz} + \nu\,t\frac{ds}{dz}\right) dz^{\lambda+\nu+1},$$

- $\mathcal{F}$ with $[.,.]$ is a Lie algebra
- $\mathcal{F}$ with respect to $\cdot$ and $[.,.]$ is a Poisson algebra
- $\mathcal{L} := \mathcal{F}^{-1}$ is a Lie subalgebra (the algebra of vector fields), and the $\mathcal{F}^\lambda$'s are Lie modules over $\mathcal{L}$.
- $\mathcal{F}^0 \oplus \mathcal{F}^{-1} = \mathcal{A} \oplus \mathcal{L} =: \mathcal{D}^1$ is also a Lie subalgebra of $\mathcal{F}$, it is the Lie algebra of differential operators of degree ≤ 1

Almost-graded structure

- The classical algebras are graded, which is important for defining representations of relevance in the application, e.g. semi-infinite forms, necessary for quantization
- In general, the KN type algebras will not have a non-trivial grading.
- But there is a replacement which still allows to construct these representations.
- Definition: Let $\mathcal{L}$ be a (Lie-) algebra such that $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ is a vector space direct sum, then $\mathcal{L}$ is called an almost-graded (Lie-) algebra if
  (i) $\dim \mathcal{L}_n < \infty$,
  (ii) There exist constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}.$$

- introduce an almost-grading for $\mathcal{F}^\lambda$ by exhibiting certain elements $f_{n,p}^\lambda \in \mathcal{F}^\lambda$, $p = 1, \ldots, K$ which constitute a basis of the subspace $\mathcal{F}_n^\lambda$ of homogeneous elements of degree n.
- the basis element $f_{n,p}^\lambda$ of degree n is of order

$$\mathrm{ord}_{P_i}(f_{n,p}^\lambda) = (n + 1 - \lambda) - \delta_i^p$$

at the point $P_i \in I$, $i = 1, \ldots, K$.
- prescription at the points in O is made in such a way that the element $f_{n,p}^\lambda$ is essentially unique
- Warning: The basis elements depend on the splitting of A into $I \cup O$.

GENUS ZERO – STANDARD SPLITTING

- standard splitting: $I = \{P_1, P_2, \ldots, P_{N-1}\}$ and $O = \{\infty\}$, we have $K = N - 1$
- it is enough to construct a basis $\{A_{n,p}\}$ of $\mathcal{A}$
- then $\mathcal{F}_n^\lambda = \mathcal{A}_{n-\lambda} dz^\lambda$, $\quad f_{n,p}^\lambda = A_{n-\lambda,p} dz^\lambda$
- $A_{n,p}(z) := (z - a_p)^n \cdot \prod_{\substack{i=1 \\ i \neq p}}^K (z - a_i)^{n+1} \cdot \alpha(p)^{n+1}$, $p = 1, \ldots, K$
- $\alpha(p)$ normalization factor such that $A_{n,p}(z) = (z - a_p)^n (1 + O(z - a_p))$
- the order at ∞ is fixed as $-(Kn + K - 1)$
- $e_{n,p} = f_{n,p}^{-1} = A_{n+1,p} \frac{d}{dz}$, $\quad p = 1, \ldots, K$

General Genus

- The above algebras are almost-graded algebras.
- the almost-grading depends on the splitting of the set A into I and O.
- $\mathcal{F}^\lambda = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}^\lambda_m, \qquad \text{with} \quad \dim \mathcal{F}^\lambda_m = K.$
- there exist R_1, R_2 (independent of n and m) such that

$$\mathcal{A}_n \cdot A_m \subseteq \bigoplus_{h=n+m}^{n+m+R_1} \mathcal{A}_h \,, \qquad [\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{L}_h \,,$$

R_1 and R_2 depends on the genus and $\#I, \#O$ (i.e. on the splitting).

- for genus zero and standard splitting

$$R_1 = \begin{cases} 0, & N = 2, \\ 1, & N > 2, \end{cases} \qquad R_2 = \begin{cases} 0, & N = 2, \\ 1, & N = 3, \\ 2, & N > 3\,. \end{cases}$$

- triangular decomposition $\mathcal{U} = \mathcal{U}_{[-]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[+]}$ with

$$\mathcal{U}_{[+]} := \bigoplus_{m>0} \mathcal{U}_m, \quad \mathcal{U}_{[0]} = \bigoplus_{m=-R_i}^{m=0} \mathcal{U}_m, \quad \mathcal{U}_{[-]} := \bigoplus_{m<-R_i} \mathcal{U}_m.$$

Here $\mathcal{U}$ is any of the above algebras $\mathcal{A}$, $\mathcal{L}$, ...

Before central extensions

- C_i be positively oriented (deformed) circles around the points P_i in I, $i = 1, \ldots, K$
- C_j^* positively oriented circles around the points Q_j in O, $j = 1, \ldots, M$.
- A cycle C_S is called a separating cycle if it is smooth, positively oriented of multiplicity one and if it separates the in-points from the out-points.
- we will integrate meromorphic differentials on Σ_g without poles in $\Sigma_g \setminus A$ over closed curves C.
- hence, C and C' are equivalent if $[C] = [C']$ in $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{Z})$.

- $[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*]$
- given such a separating cycle C_S (respectively cycle class) we define $\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega$
- This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega \; = \; \sum_{i=1}^{K} \mathrm{res}_{P_i}(\omega) \; = \; -\sum_{l=1}^{M} \mathrm{res}_{Q_l}(\omega).$$

- Krichever-Novikov duality

$$\mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} \to \mathbb{C}, \quad (f,g) \mapsto \langle f, g \rangle := \frac{1}{2\pi \mathrm{i}} \int_{C_S} f \cdot g \; .$$

Central extensions

- the second Lie algebra cohomology $\mathrm{H}^2(\mathcal{U},\mathbb{C})$ of $\mathcal{U}$ with values in the trivial module $\mathbb{C}$ classifies equivalence classes of central extensions.
- perfect Lie algebras admit universal central extensions
- A Lie algebra $\mathcal{U}$ is called perfect if $[\mathcal{U},\mathcal{U}] = \mathcal{U}$.
- $\widehat{\mathcal{U}} = \mathbb{C} \oplus U$ $\hat{x} := (0, x)$, $t := (1, 0)$

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \Phi(x, y) \cdot t, \quad [t, \widehat{U}] = 0, \quad x, y \in U.$$

- Φ Lie algebra 2-cocycle if it is antisymmetric and

$$0 = d_2\Phi(x, y, z) := \Phi([x, y], z) + \Phi([y, z], x) + \Phi([z, x], y).$$

- A 2-cocycles Φ is a coboundary if there exists a $\phi : \mathcal{U} \to \mathbb{C}$ such that

$$\Phi(x, y) = d_1\phi(x, y) = \phi([x, y]).$$

GEOMETRIC COCYCLES

- A cocycle $\gamma : \mathcal{U} \times \mathcal{U} \to \mathbb{C}$ is called a geometric cocycle if

$$\gamma = \gamma_C := \frac{1}{2\pi \mathrm{i}} \int_C \widehat{\gamma},$$

  with C a curve on Σ_g and $\widehat{\gamma} : \mathcal{U} \times \mathcal{U} \to \mathcal{F}^1$ defined in a *universal geometric* way.
- Given $\widehat{\gamma}$ only the class of C in $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$ plays a role,
-

$$\dim \mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C}) = \begin{cases} 2g, & \#A = 0, 1, \\ 2g + (N-1), & \#A = N \geq 2\,. \end{cases}$$

- genus zero and $N \geq 1$: $\dim \mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C}) = (N-1)$
- basis e.g. given by circles C_i around the points P_i, where we leave out one of them.
  For example $[C_i]$, $i = 1, \ldots, N-1$.
- better choice: e.g. for the standard splitting take $[C_S] = -[C_\infty]$ and $[C_i]$, $i = 1, \ldots, N-2$

Local and bounded cocycles

- γ a cocycle for the almost-graded Lie algebra $\mathcal{U}$ is called a local cocycle if $\exists T_1, T_2$ such that $\gamma(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies T_2 \leq n + m \leq T_1$
- γ is called bounded (from above) if $\exists T_1$ such that $\gamma(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies n + m \leq T_1$
- for the classes it means that it contains one representing cocycle of this type.

- Importance: Only local cocycles allow to extend the almost-grading to the central extension.

- The speaker classified for the above algebras local and bounded cocycle classes. They are given by geometric cocycles of certain type (see below).

Example of results

Take $\mathcal{L}$ the vector field algebra on Σ_g with a fixed but arbitrary splitting, i.e. an almost-grading, and $K = \#I$ then

- Up to equivalence and rescaling there is only one nontrivial cocycle class which is local.
- Up to equivalence and rescaling the space of bounded cocycle classes is $K-$dimensional.
- The cocycles are given by integrating a universal 1-form over a separating cycle C_S, respectively over the circles C_i, $i = 1, \ldots, K$.

MAIN RESULT – PHILOSOPHY - (GENUS ZERO !!)

- we show that in genus zero all cocycles classes are geometric cocycles classes with respect to certain explicitely given one-forms
- this is done by showing that all cocycles are bounded cocycles with respect to the almost-grading induced by the standard splitting,
- now the classification result of bounded cocycle classes of the author is used which gives a complete classification and explicit expressions given by integrals over curves
- note that in genus zero the geometric cocycles can be obtained via integration over circles around the points in I, or alternatively around ∞
- and they can be calculate via residues
- In case that the Lie algebra is perfect the universal central extension can directly be given.

FUNCTION ALGEBRA – HEISENBERG ALGEBRA

- γ is $\mathcal{L}$-invariant if $\gamma(e\,.\,f, g) + \gamma(f, e\,.\,g) = 0$, for all $f, g \in \mathcal{A}$, for all $e \in \mathcal{L}$,
- multiplicative if $\gamma(fg, h) + \gamma(gh, f) + \gamma(hf, g) = 0$, for all $f, g, h \in \mathcal{A}$
- Theorem: If one of the above properties is fulfilled then it is a geometric cocycle with $\widehat{\gamma}(f, g) = fdg$.
- basis

$$\gamma_i^{\mathcal{A}}(f, g) = \frac{1}{2\pi\mathrm{i}} \int_{C_i} fdg = \mathrm{res}_{a_i}(fdg), \quad i = 1, \ldots, N-1.$$

- γ is bounded from above with respect to the almost-grading given by the standard splitting.

- Every $\mathcal{L}$-invariant cocycle is multiplicative and vice versa.
- Two point situation: $\gamma(A_n, A_m) = \alpha \cdot (-n) \cdot \delta_m^{-n}$
- Heisenberg algebra is such a central extension (the local one or the "full" one).
- for the full one the center is $(N-1)$-dimensional

Results: $g = 0$

- Every cocycle class is geometric and given by

$$\gamma_{C,R}^{\mathcal{L}}(e,f) = \frac{1}{2\pi \mathrm{i}} \int_C (\frac{1}{2}(ef''' - e'''f) - R(ef' - e'f)dz.$$

- R is a projective connection, with our coordinates we can take $R = 0$.
- after cohomological changes they are bounded
- $\mathrm{H}^2(\mathcal{L}, \mathbb{C})$ is $(N-1)$-dimensional
- can be calculate by residues at the points
- these cocycles generate a universal central extension.
- By different techniques Skryabin has shown the existence of a universal central extension for arbitrary genus.

- Main result also here: all cocycle classes are geometric
- $\mathcal{L}$-invariant coycles for $\mathcal{A}$ and arbitrary cocycles for $\mathcal{L}$ define two cocycle types for $\mathcal{D}^1$.
- There is a another type: mixing cocycles

$$\gamma_{C,T}^{(m)}(e,g) := \frac{1}{2\pi\mathrm{i}} \int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A},$$

- T is an affine connection. Can be taken to be zero on the affine part.
- also $\mathcal{D}^1$ is perfect and the universal central extension has $3 \cdot (N-1)$ dimensional center

Others

Current algebra:

- $\mathfrak{g}$ a finite dimensional simple Lie algebra, β Cartan–Killing form

$$\gamma^{\overline{\mathfrak{g}}}_{\beta,C}(x \otimes f, y \otimes g) = \beta(x,y) \cdot \gamma^{\mathcal{A}}_{C}(f,g) = \beta(x,y) \cdot \frac{1}{2\pi \mathrm{i}} \int_C f dg$$

- all cocycles are cohomologous to such cocycles,
- $\widehat{\mathfrak{g}}$ is perfect, universal central extension has again $(N-1)$-dimensional center
- the multiplicativity of $\int_C f dg$ is crucial
- I have corresponding results for $\mathfrak{g}$ reductive.

Also results for Lie superalgebras: Each central extension of $\mathcal{L}$ gives a unique central extension of the superalgebra.

SHORT CUT

Every cocycle class is geometric and given by (for $\mathcal{A}$ we need either $\mathcal{L}$-invariance or multiplicativity)

$$\gamma_C^{\mathcal{A}}(f,g) = \frac{1}{2\pi\mathrm{i}}\int_C f dg$$

$$\gamma_{C,R}^{\mathcal{L}} = \frac{1}{2\pi\mathrm{i}}\int_C (\frac{1}{2}(ef''' - e'''f) - R(ef' - e'f)dz.$$

$$\gamma_{C,T}^{(m)}(e,g) := \frac{1}{2\pi\mathrm{i}}\int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A},$$

$$\gamma_{\beta,C}^{\overline{\mathfrak{g}}}(x\otimes f, y\otimes g) = \beta(x,y)\cdot\gamma_C^{\mathcal{A}}(f,g) = \beta(x,y)\cdot\frac{1}{2\pi\mathrm{i}}\int_C f dg$$

Next use that C_i, $i = 1,\dots,N-1$ is a basis of $\mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C})$ and that the integration over C_i can be done by calculating residues.

THREE-POINT ALGEBRAS

- $A = I \cup O, \qquad I := \{0, 1\}, \quad \text{and} \quad O := \{\infty\}$
- basis elements ("symmetrized" and "anti-symmetrized")

$$A_n(z) = z^n(z-1)^n, \quad B_n(z) = z^n(z-1)^n(2z-1),$$

- structure equations:

$$\begin{aligned} A_n \cdot A_m &= A_{n+m}, \\ A_n \cdot B_m &= B_{n+m}, \\ B_n \cdot B_m &= A_{n+m} + 4A_{n+m+1}. \end{aligned}$$

- space of (multiplicative) cocycles is two-dimensional, e.g. we take the residues around ∞ and around 0

- $$\begin{aligned}\gamma_\infty^{\mathcal{A}}(A_n, A_m) &= 2n\,\delta_m^{-n},\\ \gamma_\infty^{\mathcal{A}}(A_n, B_m) &= 0,\\ \gamma_\infty^{\mathcal{A}}(B_n, B_m) &= 2n\delta_m^{-n} + 4(2n+1)\,\delta_m^{-n-1}\ .\end{aligned}$$

- $$\begin{aligned}\gamma_0^{\mathcal{A}}(A_n, A_m) &= -n\,\delta_m^{-n},\\ \gamma_0^{\mathcal{A}}(A_n, B_m) &= n\,\delta_m^{-n} + 2n\,\delta_m^{-n-1}\\ &\qquad + \sum_{k=2}^{\infty} n\,(-1)^{k-1} 2^k \frac{(2k-3)!!}{k!} \delta_m^{-n-k}\ ,\\ \gamma_0^{\mathcal{A}}(B_n, B_m) &= -n\delta_m^{-n} - 2(2n+1)\,\delta_m^{-n-1}\ .\end{aligned}$$

- vector field algebra
- basis: $e_n := A_{n+1} \frac{d}{dz}, \qquad f_n := B_{n+1} \frac{d}{dz}, \quad n \in \mathbb{Z}$
- structure equation

$$
\begin{aligned}
[e_n, e_m] &= (m-n)\, f_{m+n}, \\
[e_n, f_m] &= (m-n)\, e_{m+n} + (4(m-n)+2)\, e_{n+m+1}, \\
[f_n, f_m] &= (m-n)\, f_{m+n} + 4(m-n)\, f_{n+m+1}.
\end{aligned}
$$

- the universal central extension is two -dimensional, as above obtained by calculating residues at ∞ and 0.
- $$
\begin{aligned}
\gamma_0^{\mathcal{L}}(e,f) &= 1/2\, \mathrm{res}_0(e \cdot f''' - f \cdot e''')dz \\
\gamma_\infty^{\mathcal{L}}(e,f) &= 1/2\, \mathrm{res}_\infty(e \cdot f''' - f \cdot e''')
\end{aligned}
$$

$$\gamma_\infty^{\mathcal{L}}(e_n, e_m) = 2(n^3 - n)\,\delta_m^{-n} + 4n(n+1)(2n+1)\delta_m^{-n-1}$$

$$\gamma_\infty^{\mathcal{L}}(e_n, f_m) = 0,$$

$$\gamma_\infty^{\mathcal{L}}(f_n, f_m) = 2(n^3 - n)\,\delta_m^{-n} + 8n(n+1)(2n+1)\delta_m^{-n-1} + 8(n+1)(2n+1)(2n+3)\delta_m^{-n-2}$$

$$\gamma_0^{\mathcal{L}}(e_n, e_m) = -(n^3 - n)\,\delta_n^{-m} - 2n(n+1)(2n+1)\delta_m^{-n-1}$$

$$\gamma_0^{\mathcal{L}}(e_n, f_m) = (n^3 - n)\,\delta_m^{-n} + 6n^2(n+1)\delta_m^{-n-1} + 6n(n+1)^2\delta_m^{-n-2} + \sum_{k\geq 3} n(n+1)(n+k-1)(-1)^k 2^k \cdot 3 \cdot \frac{(2k-5)!!}{k!}\delta_m^{-n-k}$$

$$\gamma_0^{\mathcal{L}}(f_n, f_m) = -(n^3 - n)\,\delta_m^{-n} - 4n(n+1)(2n+1)\delta_m^{-n-1} - 4(n+1)(2n+1)(2n+3)\delta_m^{-n-2}\,.$$

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