Singular kinetic equations

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arXiv:2108.05042

2021.11.19

Motivation-(Mean field limit/DDSDE)

- Consider the following second order interacting particle systems:

$$\begin{cases} \mathrm{d}X_t^i = V_t^i \mathrm{d}t, \\ \mathrm{d}V_t^i = b(Z_t^i)\mathrm{d}t + \frac{1}{N}\sum_{j\neq i} K(X_t^i - X_t^j)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t^i, \end{cases}$$

where $i = 1, 2, .., N$,
$Z^i = (X^i, V^i) \in \mathbb{R}^{2d}$: position and velocity of particle number i
B_t^i: independent Brownian motions
b: the random enviroment depending on Z^i.
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- Letting $N \to \infty$, we obtain the following Distribution Dependent SDE(DDSDE, also called McKean-Vlasov equation):

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t \\ Z_0 \sim u_0 \mathrm{d}x\mathrm{d}v, \end{cases} \tag{1}$$

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- When b, K are smooth, well-posedness of solutions and propagation of chaos hold

Problem

- Formally, by Itô's formula, the law of solution to DDSDE $=$ the limit u of the empirical measure $u_N := \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^i, V_t^i)}$ solves the following kinetic equation

$$\partial_t u = \Delta_v u - v \cdot \nabla_x u - \mathrm{div}_v((b + K * \langle u \rangle)u), \quad u(0) = u_0, \tag{2}$$

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Global well-posedness of DDSDE (1)? Nonlinear martingale problem.

Kinetic equation

- Consider the following linear kinetic equation:

$$\mathscr{L} u := (\partial_t \pm v \cdot \nabla_x - \Delta_v) u = f \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^{2d}.$$

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- Scaling transform: for $\lambda > 0$ and $a, b, c > 0$, let

$$u_\lambda(t, x, v) := \lambda^a u(\lambda^b t, \lambda^c x, \lambda v), \quad f_\lambda(t, x, v) := f(\lambda^b t, \lambda^c x, \lambda v).$$

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$$\|f\|_{\mathbf{C}_a^\alpha} := \|f\|_{L^\infty} + \sup_{z \neq 0} \frac{\|f(\cdot + z) - f\|_{L^\infty}}{|z|_a^\alpha}, \quad 0 < \alpha < 1$$

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$$P_t f(z) := \Gamma_t p_t * \Gamma_t f(z) = \Gamma_t(p_t * f)(z) \quad \text{and} \quad \mathscr{I} f := \int_0^t P_{t-s} f \mathrm{d}s$$

is a solution to the above equation, where $\Gamma_t f(z) := f(\Gamma_t z), \Gamma_t z := (x + tv, v)$, p_t the density of $\left(\sqrt{2}\int_0^t W_s \mathrm{d}s, \sqrt{2}W_t\right)$.

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- Consider the following nonlinear kinetic equation

$$\partial_t u = \Delta_v u + v \cdot \nabla_x u + b \cdot \nabla_v u + K * \langle u \rangle \cdot \nabla_v u + f, \quad u(0) = u_0, \tag{4}$$

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$$b \in L_T^\infty \mathbf{C}_a^{-\alpha}(\rho_\kappa), \quad f \in L_T^\infty \mathbf{C}_a^{-\alpha}(\rho_\kappa),$$

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- Difficulty: the best regularity of the solution is in $L^\infty_T \mathbf{C}^{2-\alpha}_a$.
  (Ill-defined problem) $b \cdot \nabla_v u$ does not make sense since

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Gubinelli, Imkeller and Perkowski 15 : paracontrolled distribution method
- Aim: develop paracontrolled calculus to get global well-posedness of (4)

Linear equation

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- Consider the following linear kinetic PDE:

$$\mathscr{L}u := (\partial_t - \Delta_v - v \cdot \nabla_x)u = b \cdot \nabla_v u + f, \quad u(0) = u_0. \tag{5}$$

- Suppose that for some $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and ρ_κ, $(b, f) \in L_T^\infty \mathbf{C}_a^{-\alpha}(\rho_\kappa)$.
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- Kinetic Hölder space: $\alpha \in (0,2)$, $T > 0$.

$$\mathbb{S}^\alpha_{T,a}(\rho) := \left\{ f : \|f\|_{\mathbb{S}^\alpha_{T,a}(\rho)} := \|f\|_{L^\infty_T \mathbf{C}^\alpha_a(\rho)} + \|f\|_{\mathbf{C}^{\alpha/2}_{T;\Gamma} L^\infty(\rho)} < \infty \right\},$$

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- Schauder estimates: $\|\mathscr{I} f\|_{\mathbb{S}^{2-\beta}_{T,a}(\rho)} \lesssim \|f\|_{L^\infty_T \mathbf{C}^{-\beta}_a(\rho)}$, for $\mathscr{I} = (\mathscr{L})^{-1}$, $\beta \in (0, 2)$.

Paracontrolled solution to linear PDE

- Paraproducts: if $f \in C_a^\alpha, g \in C_a^\beta$ for $\alpha > 0, \beta < 0$

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$$u = \nabla_v u \prec \mathscr{I} b + \underbrace{u^\sharp}_{\text{regular term}} + \mathscr{I} f, \quad \text{paracontrolled ansatz}$$

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- Aim: Commutator estimate for $[\mathscr{I}, \nabla_v u \prec] b$

Commutator estimate for kinetic operator

Recall $P_t = \Gamma_t p_t * \Gamma_t$ be the kinetic semigroup.

Lemma 2.1

For any $\alpha \in (0,1)$, $\beta \in \mathbb{R}$, $t \in (0,T]$, $\delta \geqslant 0$, $j \geqslant -1$,

$$\|\Delta_j[P_t(f \prec g) - (\Gamma_t f \prec P_t g)]\|_{L^\infty(\rho_1\rho_2)} \lesssim t^{-\frac{\delta}{2}} 2^{-(\alpha+\beta+\delta)j} \|f\|_{\mathbf{C}_a^\alpha(\rho_1)} \|g\|_{\mathbf{C}_a^\beta(\rho_2)}.$$

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$\Rightarrow$

Lemma 2.2

Commutator estimate

$$\|[\mathscr{I}_\lambda, f \prec]g\|_{L_T^\infty \mathbf{C}_a^{\alpha+\beta+2}(\rho_1\rho_2)} \lesssim \|f\|_{\mathbb{S}_{T,a}^\alpha(\rho_1)} \|g\|_{L_T^\infty \mathbf{C}_a^\beta(\rho_2)}. \quad (6)$$

$\Rightarrow u \in C_T \mathbf{C}_a^{2-\alpha}(\rho_\delta)$, $u^\sharp \in C_T \mathbf{C}_a^{3-2\alpha}(\rho_\delta)$

Renormalization

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Assumption: μ is symmetric in second variable and for some $\beta<\alpha$,

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Well-posedness of linear PDE

$$(\partial_t - \Delta_v - v\cdot\nabla_x)u = b\cdot\nabla_v u + f, \quad u(0) = u_0.$$

Theorem 1

Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ *and* $\vartheta := \frac{9}{2-3\alpha}$ *and* $\delta := (2\vartheta+2)\kappa \leqslant 1$. *For any* $T > 0$, (b, f) *as above,* $\exists!$ *paracontrolled solution* $(u, u^\sharp)$ *to PDE* (5) *such that* $\|u\|_{C_T\mathbf{C}_T^{2-\alpha}(\rho_\delta)} + \|u^\sharp\|_{C_T\mathbf{C}_T^{3-2\alpha}(\rho_{2\delta})} \lesssim C(b, f)$.

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- Existence: based on the paracontrolled caculus developed before
  difficulty: Loss of weight from $b \cdot \nabla_v u$
  Solution from [Zhang, Zhu, Z. 20]:
  Step 1: Schauder estimate for b, f in unweighted Besov space
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- Uniqueness: Localization

Nonlinear equation

Nonlinear mean field equation

- Assume that $\mathrm{div}_v b = 0$. Consider the following

$$\mathscr{L} u = b \cdot \nabla_v u + K * \langle u \rangle \cdot \nabla_v u, \quad u(0) = u_0. \tag{7}$$

Here $\langle u \rangle(t,x) := \int_{\mathbb{R}^d} u(t,x,v)\mathrm{d}v$. Assume that

- $$K \in \cup_{\beta > \alpha - 1} \mathbf{C}_x^{\beta/3}, \quad b \circ \nabla_v \mathscr{I}(b) \in \mathbf{C}_a^{1-2\alpha}(\rho_\kappa)$$

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Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and κ be small enough so that $\delta := 2(\frac{9}{2-3\alpha} + 1)\kappa < 1$. $\rho_0 = (1 + |x|^{1/3} + |v|)^{\kappa_0}$ with $\kappa_0 > 0$.

- *for any probability density $u_0 \in L^1(\rho_0) \cap \mathbf{C}_a^\gamma$, $\gamma > 1 + \alpha$, $\exists$ at least a probability density paracontrolled solution $u \in L_T^\infty(\mathbf{C}^{2-\alpha}(\rho_\delta))$ to (7).*
- *If in addition that K is bounded and $H(u_0) := \int u_0 \ln u_0 < \infty$, the solution is unique.*

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$$0 = Ew^t(t, Z_t^z) = w^t(0, z) + E\int_0^t (B \cdot \nabla_v \rho_0)(s, Z_s^z)\mathrm{d}s.$$

Here w^t is the unique solution of the following backward PDE:

$$\partial_s w^t + (\Delta_v + v \cdot \nabla_x + B \cdot \nabla_v) w^t = B \cdot \nabla_v \rho_0, \quad w^t(t) = 0.$$

Idea of proof: Uniqueness

1. Uniqueness: Let $w = u_1 - u_2$

$$\partial_t w = \Delta_v w + v \cdot \nabla_x w + b \cdot \nabla_v w + K * \langle w \rangle \cdot \nabla_v u_1.$$

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$$\mathscr{L} u^n = b^n \cdot \nabla_v u^n + K^n * \langle u_1 \rangle \cdot \nabla_v u^n.$$

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$$\mathrm{d}X_t = V_t \mathrm{d}t, \quad \mathrm{d}V_t = b(X_t, V_t)\mathrm{d}t + (K * \mu_{X_t})(X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t, \tag{8}$$

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  see also [Delarue, Diel 16, Cannizzaro, Chouk 18, Kremp, Perkowski 20]

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Definition 4.1

(Martingale problem) Let $\delta>0$. A probability measure $\mathbb{P}\in\mathcal{P}(\mathcal{C}_T)$ is called a martingale solution to SDE (8), *if for all $f\in C_b$, $\varphi\in\mathbf{C}_a^\gamma$ with some $\gamma>1+\alpha$ and $\mu_t:=\mathbb{P}\circ X_t^{-1}$,*

$$M_t := u_f^\mu(t,Z_t) - u_f^\mu(0,Z_0) - \int_0^t f(s,Z_s)\mathrm{d}s$$

is a martingale under $\mathbb{P}$. Here u_f^μ is a solution to (9).

Main results

Theorem 3

Suppose that $b \circ \nabla_v \mathscr{I}(b) \in \mathbf{C}_a^{1-2\alpha}(\rho_\kappa)$ *and* $K \in \cup_{\beta>\alpha-1} \mathbf{C}_a^{\beta}$*. Then there exists at least one martingale solution* $\mathbb{P}$ *to SDE* (8)*. Moreover, if* K *is bounded measurable, then the solution is unique.*

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Idea of proof

- Existence: convolution approximation

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- Existence: convolution approximation
- Uniqueness: First for $K = 0$ and Girsanov's tansformation

Thank you !