Singular kinetic equations

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1 Background and Motivations

2 Linear equation

3 Nonlinear equation

4 Singular DDSDE
Motivation-(Mean field limit/DDSDE)

Consider the following second order interacting particle systems:

\[
\begin{align*}
\frac{dX_t^i}{dt} &= V_t^i dt, \\
\frac{dV_t^i}{dt} &= b(Z_t^i) dt + \frac{1}{N} \sum_{j \neq i} K(X_t^i - X_t^j) dt + \sqrt{2} dB_t^i,
\end{align*}
\]

where \(i = 1, 2, ..., N\),

\(Z_t^i = (X_t^i, V_t^i) \in \mathbb{R}^{2d}\): position and velocity of particle number \(i\)

\(B_t^i\): independent Brownian motions

\(b\): the random environment depending on \(Z_t^i\).

\(K\): interaction kernel.
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\end{aligned}
\]

where \( i = 1, 2, \ldots, N \),
\( Z^i = (X^i, V^i) \in \mathbb{R}^{2d} \): position and velocity of particle number \( i \)
\( B^i_t \): independent Brownian motions
\( b \): the random environment depending on \( Z^i \).
\( K \): interaction kernel.

Letting \( N \to \infty \), we obtain the following Distribution Dependent SDE (DDSDE, also called McKean-Vlasov equation):

\[
\begin{aligned}
    dX_t &= V_t \, dt \\
    dV_t &= b(Z_t) \, dt + \int_{\mathbb{R}^d} K(X_t - y) \mu_t(dy) \, dt + \sqrt{2} dB_t \\
    Z_0 &\sim u_0 \, dx \, dv,
\end{aligned}
\]

where \( Z_t = (X_t, V_t) \), \( \mu_t \) is the distribution of \( X_t \) and \( B_t \) is a standard BM.
Motivation-(Mean field limit/DDSDE)

Consider the following second order interacting particle systems:

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where \( Z_t = (X_t, V_t) \), \( \mu_t \) is the distribution of \( X_t \) and \( B_t \) is a standard BM.

When \( b, K \) are smooth, well-posedness of solutions and propagation of chaos hold.
Formally, by Itô’s formula, the law of solution to DDSDE = the limit $u$ of the empirical measure $u_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_i^t, v_i^t)}$ solves the following kinetic equation

$$\partial_t u = \Delta_v u - v \cdot \nabla_x u - \text{div}_v ((b + K \ast \langle u \rangle) u), \quad u(0) = u_0, \quad (2)$$

with $\langle u \rangle = \int u \, dv$. 

Aim: For $b, K$ singular, e.g. spatial white noise, global well-posedness of kinetic equation (3) or (2)? Global well-posedness of DDSDE (1)? Nonlinear martingale problem.
Formally, by Itô's formula, the law of solution to DDSDE = the limit $u$ of the empirical measure $u_N := \frac{1}{N} \sum_{i=1}^{N} \delta(x_i^t, v_i^t)$ solves the following kinetic equation

$$\partial_t u = \Delta_v u - v \cdot \nabla_x u - \text{div}_v \left( (b + K \ast \langle u \rangle) u \right), \quad u(0) = u_0, \quad (2)$$

with $\langle u \rangle = \int u \, dv$. If $\text{div}_v b = 0$ then (2) becomes

$$\partial_t u = \Delta_v u - v \cdot \nabla_x u - (b + K \ast \langle u \rangle) \cdot \nabla_v u, \quad u(0) = u_0. \quad (3)$$
Formally, by Itô’s formula, the law of solution to DDSDE $= \text{the limit } u$ of the empirical measure $u_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i^t, v_i^t)}$ solves the following kinetic equation

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$$\partial_t u = \Delta_v u - v \cdot \nabla_x u - (b + K \ast \langle u \rangle) \cdot \nabla_v u, \quad u(0) = u_0.$$  \hspace{1cm} (3)

**Aim**: For $b, K$ singular, (e.g. $b$: spatial white noise)
Global well-posedness of kinetic equation (3) or (2)?
Formally, by Itô’s formula, the law of solution to DDSDE $\equiv$ the limit $u$ of the empirical measure $u_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i^t, v_i^t)}$ solves the following kinetic equation

$$\partial_t u = \Delta_v u - v \cdot \nabla_x u - \text{div}_v((b + K * \langle u \rangle) u), \quad u(0) = u_0,$$

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Global well-posedness of kinetic equation (3) or (2)?
Global well-posedness of DDSDE (1)? Nonlinear martingale problem.
Consider the following linear kinetic equation:

\[ L u := (\partial_t \pm v \cdot \nabla_x - \Delta_v)u = f \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^{2d}. \]
Consider the following linear kinetic equation:

\[ \mathcal{L} u := (\partial_t \pm v \cdot \nabla_x - \Delta_v) u = f \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^{2d}. \]

**Scaling transform:** for \( \lambda > 0 \) and \( a, b, c > 0 \), let

\[ u_\lambda(t, x, v) := \lambda^a u(\lambda^b t, \lambda^c x, \lambda v), \quad f_\lambda(t, x, v) := f(\lambda^b t, \lambda^c x, \lambda v). \]

Then \( \mathcal{L} u_\lambda = f_\lambda \iff a = -2, b = 2, c = 3. \)
Consider the following linear kinetic equation:
\[ \mathcal{L}u := (\partial_t \pm \nu \cdot \nabla_x - \Delta_x)u = f \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^{2d}. \]

**Scaling transform:** for \( \lambda > 0 \) and \( a, b, c > 0 \), let
\[ u_\lambda(t, x, \nu) := \lambda^a u(\lambda^b t, \lambda^c x, \lambda \nu), \]
\[ f_\lambda(t, x, \nu) := f(\lambda^b t, \lambda^c x, \lambda \nu). \]
Then \( \mathcal{L} u_\lambda = f_\lambda \iff a = -2, b = 2, c = 3. \)

**Schauder estimate:** gain 2-regularity in \( \nu \) direction.

Due to transport term \( \nu \cdot \nabla_x \) we gain 2-regularity in \( x \) direction (scaling of \( x \) and \( \nu \) is 3: 1)
Kinetic equation

- Consider the following linear kinetic equation:
  \[ \mathcal{L}u := (\partial_t \pm v \cdot \nabla_x - \Delta v)u = f \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^{2d}. \]

- **Scaling transform**: for \( \lambda > 0 \) and \( a, b, c > 0 \), let
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  Then \( \mathcal{L}u_\lambda = f_\lambda \iff a = -2, b = 2, c = 3. \)

- **Schauder estimate**: gain 2-regularity in \( v \) direction. Due to transport term \( v \cdot \nabla_x \) we gain \( \frac{2}{3} \) regularity in \( x \) direction (scaling of \( x \) and \( v \) is 3:1).
Consider the following linear kinetic equation:
\[ Lu := (\partial_t \pm \mathbf{v} \cdot \nabla_x - \Delta_\mathbf{v})u = f \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^{2d}. \]

**Scaling transform:** for \( \lambda > 0 \) and \( a, b, c > 0 \), let
\[ u_\lambda(t, x, \mathbf{v}) := \lambda^a u(\lambda^b t, \lambda^c x, \lambda \mathbf{v}), \quad f_\lambda(t, x, \mathbf{v}) := f(\lambda^b t, \lambda^c x, \lambda \mathbf{v}). \]

Then \( Lu_\lambda = f_\lambda \iff a = -2, b = 2, c = 3. \)

**Schauder estimate:** gain 2-regularity in \( \mathbf{v} \) direction. Due to transport term \( \mathbf{v} \cdot \nabla_x \) we gain \( \frac{2}{3} \) regularity in \( x \) direction (scaling of \( x \) and \( \mathbf{v} \) is \( 3 : 1 \)) \( \Rightarrow \) study kinetic equation in anisotropic Besov space

\[
\| f \|_{C^\alpha_a} := \| f \|_{L^\infty} + \sup_{z \neq 0} \frac{\| f(\cdot + z) - f \|_{L^\infty}}{|z|_a^{\alpha}}, \quad 0 < \alpha < 1
\]

with \( |z|_a := |x|^{1/3} + |\mathbf{v}|, \quad z = (x, \mathbf{v}). \)
Consider the following linear kinetic equation:

\[ \mathcal{L} u := (\partial_t \pm \nabla_x - \Delta_v) u = f \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^{2d}. \]

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\[ \| f \|_{c_\alpha^a} := \| f \|_{L^\infty} + \sup_{z \neq 0} \frac{\| f(\cdot + z) - f \|_{L^\infty}}{|z|_a^\alpha}, \quad 0 < \alpha < 1 \]

with \( |z|_a := |x|^{1/3} + |v|, \quad z = (x, v). \)

**Kinetic semigroup**

\[ P_t f(z) := \Gamma_t \rho_t * \Gamma_t f(z) = \Gamma_t(\rho_t * f)(z) \quad \text{and} \quad \mathcal{I} f := \int_0^t P_{t-s} f ds \]

is a solution to the above equation, where \( \Gamma_t f(z) := f(\Gamma_t z), \Gamma_t z := (x + t v, v), \rho_t \)

the density of \( \left( \sqrt{2} \int_0^t W_s ds, \sqrt{2} W_t \right). \)
Difficulty

Consider the following nonlinear kinetic equation

$$\partial_t u = \Delta_v u + \nu \cdot \nabla_x u + b \cdot \nabla_v u + K \ast \langle u \rangle \cdot \nabla_v u + f, \quad u(0) = u_0,$$

(4)

with $\langle u \rangle = \int u \, d\nu$.

Difficulty: the best regularity of the solution is in $L^\infty_TC^{2-\alpha_a}(\rho_\kappa)$. (Ill-defined problem)

$b \cdot \nabla_v u$ does not make sense since $C^{\alpha_a} \times C^{\beta_a} \ni (f, g) \rightarrow fg \in C^{\alpha_a+\beta_a}$ only if $\alpha_a + \beta_a > 0$.

Similar difficulty as in singular SPDEs: Hairer 14 the theory of regularity structures Gubinelli, Imkeller and Perkowski 15: paracontrolled distribution method

Aim: develop paracontrolled calculus to get global well-posedness of (4)
Difficulty

Consider the following nonlinear kinetic equation

\[ \frac{\partial}{\partial t} u = \Delta_v u + \mathbf{v} \cdot \nabla_x u + b \cdot \nabla_v u + K \ast \langle u \rangle \cdot \nabla_v u + f, \quad u(0) = u_0, \]  

(4)

with \( \langle u \rangle = \int u \, dv \).

For some \( \alpha \in (\frac{1}{2}, \frac{2}{3}) \), \( \kappa \in (0, 1) \),

\[ b \in L_T^\infty C_a^{-\alpha}(\rho_\kappa), \quad f \in L_T^\infty C_a^{-\alpha}(\rho_\kappa), \]

where \( \rho_\kappa(x, v) := ((1 + |x|^2)^{1/3} + (1 + |v|^2)^{1/2})^{-\kappa/2} \), \( C_a^{-\alpha}(\rho_\kappa) = \{ f : f \rho_\kappa \in C_a^{-\alpha} \} \).
Consider the following nonlinear kinetic equation

\[ \partial_t u = \Delta_v u + v \cdot \nabla_x u + b \cdot \nabla_v u + K \ast \langle u \rangle \cdot \nabla_v u + f, \quad u(0) = u_0, \]  
\[ (4) \]

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**Difficulty**: the best regularity of the solution is in \( L_T^\infty C_a^{2-\alpha} \).

(Ill-defined problem) \( b \cdot \nabla_v u \) does not make sense since

\[ C_a^\alpha \times C_a^\beta \ni (f, g) \to fg \in C_a^{\alpha \wedge \beta} \text{ only if } \alpha + \beta > 0. \]
Difficulty

Consider the following nonlinear kinetic equation

\[
\partial_t u = \Delta_v u + \mathbf{v} \cdot \nabla_x u + b \cdot \nabla_v u + K \ast \langle u \rangle \cdot \nabla_v u + f, \quad u(0) = u_0,
\]

(4)

with \( \langle u \rangle = \int u \, dv \).

For some \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right) \), \( \kappa \in (0, 1) \),

\[b \in L_T^\infty C_{a}^{-\alpha}(\rho_\kappa), \quad f \in L_T^\infty C_{a}^{-\alpha}(\rho_\kappa),\]

where \( \rho_\kappa(x, v) := \left( (1 + |x|^2)^{1/3} + (1 + |v|^2)^{-\kappa/2} \right), \quad C_{a}^{-\alpha}(\rho_\kappa) = \{f : f \, \rho_\kappa \in C_{a}^{-\alpha} \} \).

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Similar difficulty as in singular SPDEs:

Hairer 14 the theory of regularity structures
Gubinelli, Imkeller and Perkowski 15 : paracontrolled distribution method
Difficulty

- Consider the following nonlinear kinetic equation
  \[ \partial_t u = \Delta_x v u + v \cdot \nabla_x u + b \cdot \nabla_v u + K \ast \langle u \rangle \cdot \nabla_v u + f, \quad u(0) = u_0, \]  
  \[ \text{(4)} \]
  with \( \langle u \rangle = \int u \, dv \).

- For some \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right) \), \( \kappa \in (0, 1) \),
  \[ b \in L_T^{\infty} \mathbb{C}_a^{-\alpha}(\rho_\kappa), \quad f \in L_T^{\infty} \mathbb{C}_a^{-\alpha}(\rho_\kappa), \]
  where \( \rho_\kappa(x, v) := ((1 + |x|^2)^{1/3} + (1 + |v|^2)^{-\kappa/2}, \mathbb{C}_a^{-\alpha}(\rho_\kappa) = \{ f : f \rho_\kappa \in \mathbb{C}_a^{-\alpha} \} \).

- **Difficulty**: the best regularity of the solution is in \( L_T^{\infty} \mathbb{C}_a^{2-\alpha} \).
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- Similar difficulty as in singular SPDEs:
  Hairer 14 the theory of regularity structures
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- **Aim**: develop paracontrolled calculus to get global well-posedness of (4)
Linear equation
Consider the following linear kinetic PDE:

$$\mathcal{L} u := (\partial_t - \Delta_v - v \cdot \nabla_x) u = b \cdot \nabla_v u + f, \quad u(0) = u_0.$$  \hspace{1cm} (5)

Suppose that for some $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$ and $\rho_\kappa$, $(b, f) \in L_T^\infty \mathcal{C}_{\alpha}^{-\alpha}(\rho_\kappa)$.

**Aim:** develop paracontrolled distribution method in the kinetic setting to obtain Schauder estimate for (5).
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**Aim:** develop paracontrolled distribution method in the kinetic setting to obtain Schauder estimate for (5).

**Kinetic Hölder space:** \( \alpha \in (0, 2), \ T > 0. \)

\[ \mathbb{S}_{T, a}^{\alpha}(\rho) := \left\{ f : \|f\|_{\mathbb{S}_T^\alpha, a}(\rho) := \|f\|_{L_T^\infty \mathbf{C}_{\alpha}^\alpha(\rho)} + \|f\|_{C_{T, \Gamma}^{\alpha/2} L^\infty(\rho)} < \infty \right\}, \]

where for \( \beta \in (0, 1), \ \Gamma_t f(z) := f(\Gamma_t z), \ \Gamma_t z := (x + tv, v). \)

\[ \|f\|_{\mathbf{C}_{T, \Gamma}^{\beta} L^\infty(\rho)} := \sup_{0 \leq t \leq T} \|f(t)\|_{L^\infty(\rho)} + \sup_{0 < |t-s| \leq 1} \frac{\|f(t) - \Gamma_{t-s} f(s)\|_{L^\infty(\rho)}}{|t-s|^\beta}. \]
Consider the following linear kinetic PDE:

\[ \mathcal{L}u := (\partial_t - \Delta_v - v \cdot \nabla_x)u = b \cdot \nabla_v u + f, \quad u(0) = u_0. \]  

(5)

Suppose that for some \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right) \) and \( \rho_\kappa, (b, f) \in L_T^\infty C_\alpha^- (\rho_\kappa). \)

Aim: develop paracontrolled distribution method in the kinetic setting to obtain Schauder estimate for (5).

Kinetic Hölder space: \( \alpha \in (0, 2), \ T > 0. \)

\[ S^\alpha_{T,a}(\rho) := \left\{ f : \| f \|_{S^\alpha_{T,a}(\rho)} := \| f \|_{L_T^\infty C_\alpha(\rho)} + \| f \|_{C_{T,\Gamma}^{\alpha/2} L^\infty(\rho)} < \infty \right\}, \]

where for \( \beta \in (0, 1), \ \Gamma_t f(z) := f(\Gamma_t z), \ \Gamma_t z := (x + tv, v). \)

\[ \| f \|_{C_{T,\Gamma}^\beta L^\infty(\rho)} := \sup_{0 \leq t \leq T} \| f(t) \|_{L^\infty(\rho)} + \sup_{0 < |t-s| \leq 1} \frac{\| f(t) - \Gamma_{t-s} f(s) \|_{L^\infty(\rho)}}{|t-s|^{\beta}}. \]

1. Recall \( P_t f = \Gamma_t (p_t * f), \ P_t f - \Gamma_t f = \Gamma_t (p_t * f - f) \)
2. \( \Gamma_t f - f = f(x + tv, v) - f(x, v) \)
Consider the following linear kinetic PDE:

\[ \mathcal{L} u := (\partial_t - \Delta v - v \cdot \nabla_x)u = b \cdot \nabla v u + f, \quad u(0) = u_0. \]  

(5)

Suppose that for some \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right) \) and \( \rho, \kappa \), \( (b, f) \in L^\infty T \mathcal{C}^{-\alpha}_a(\rho, \kappa) \).

**Aim:** develop paracontrolled distribution method in the kinetic setting to obtain Schauder estimate for (5).

**Kinetic Hölder space:** \( \alpha \in (0, 2), \ T > 0. \)

\[ S^\alpha_{T, a}(\rho) := \left\{ f : \| f \|_{S^\alpha_{T, a}(\rho)} := \| f \|_{L^\infty T \mathcal{C}^\alpha_a(\rho)} + \| f \|_{C^\alpha_{T, \Gamma} L^\infty(\rho)} < \infty \right\}, \]

where for \( \beta \in (0, 1), \ \Gamma_t f(z) := f(\Gamma_t z), \ \Gamma_t z := (x + tv, v) \).

\[ \| f \|_{C^\beta_{T, \Gamma} L^\infty(\rho)} := \sup_{0 \leq t \leq T} \| f(t) \|_{L^\infty(\rho)} + \sup_{0 < |t - s| \leq 1} \frac{\| f(t) - \Gamma_{t-s} f(s) \|_{L^\infty(\rho)}}{|t - s|^\beta}. \]

1. Recall \( P_t f = \Gamma_t (p_t * f), \ P_t f - \Gamma_t f = \Gamma_t (p_t * f - f) \)
2. \( \Gamma_t f - f = f(x + tv, v) - f(x, v) \)

**Schauder estimates:** \( \| \mathcal{I} f \|_{S^{2-\beta}_{T, a}(\rho)} \lesssim \| f \|_{L^\infty T \mathcal{C}^{-\beta}_a(\rho)}, \text{ for } \mathcal{I} = (\mathcal{L})^{-1}, \ \beta \in (0, 2). \)
Paracontrolled solution to linear PDE

- Paraproducts: if \( f \in C^\alpha_a, g \in C^\beta_a \) for \( \alpha > 0, \beta < 0 \)

\[
fg = f \prec g + f \odot g + f \succ g,
\]

bad term \hspace{1cm} well defined only if \( \alpha + \beta > 0 \)
Paracontrolled solution to linear PDE

- Paraproducts: if $f \in C^\alpha_a$, $g \in C^\beta_a$ for $\alpha > 0$, $\beta < 0$

  \[
  fg = f \prec g + f \circ g + f \succ g,
  \]
  \begin{align*}
  &\text{bad term} \quad \text{well defined only if } \alpha + \beta > 0
  \end{align*}

- \[
  \mathcal{L} u = b \cdot \nabla v u + f = \nabla v u \prec b + \nabla u \succ b + b \circ \nabla v u + f
  \]
  \begin{align*}
  &\text{bad term} \quad \text{not well defined}
  \end{align*}
Paracontrolled solution to linear PDE

- Paraproducts: if \( f \in C^\alpha_a, g \in C^\beta_a \) for \( \alpha > 0, \beta < 0 \)

\[
fg = f \prec g + f \circ g + f \succ g,
\]

bad term well defined only if \( \alpha + \beta > 0 \)

- \( \mathcal{L} u = b \cdot \nabla v u + f = \nabla v u \prec b + \nabla u \succ b + b \circ \nabla v u + f \)

bad term not well defined

- Paracontrolled solution: \( \mathcal{I} = (\mathcal{L})^{-1} \)

\[
u = \nabla v u \prec \mathcal{I} b + u^\# + \mathcal{I} f, \quad \text{paracontrolled ansatz}
\]

regular term

\[
u^\# = \mathcal{I} (\nabla v u \succ b + b \circ \nabla v u) - [\mathcal{I}, \nabla v u \prec] b.
\]
Paracontrolled solution to linear PDE

- Paraproductions: if $f \in C^\alpha_a$, $g \in C^\beta_a$ for $\alpha > 0$, $\beta < 0$

\[ fg = f \prec g + f \circ g + f \succ g, \]

bad term well defined only if $\alpha + \beta > 0$

\[ L u = b \cdot \nabla u + f = \nabla_\nu u \prec b + \nabla u \succ b + b \circ \nabla_\nu u + f \]

bad term not well defined

- Paracontrolled solution: $\mathcal{I} = (L)^{-1}$

\[ u = \nabla_\nu u \prec \mathcal{I} b + u^\# + \mathcal{I} f, \]

paracontrolled ansatz

\[ u^\# = \mathcal{I} (\nabla_\nu u \succ b + b \circ \nabla_\nu u) - [\mathcal{I}, \nabla_\nu u \prec] b. \]

- Aim: Commutator estimate for $[\mathcal{I}, \nabla_\nu u \prec] b$
Commutator estimate for kinetic operator

Recall $P_t = \Gamma_t p_t \ast \Gamma_t$ be the kinetic semigroup.

**Lemma 2.1**

For any $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$, $t \in (0, T]$, $\delta \geq 0$, $j \geq -1$,

$$\|\Delta_j [P_t(f \prec g) - (\Gamma_t f \prec P_t g)]\|_{L^\infty(\rho_1 \rho_2)} \lesssim t^{-\frac{j}{2}} 2^{-(\alpha + \beta + \delta)j} \|f\|_{C_\alpha} \|g\|_{C_\beta}.$$

Here $\Delta_j$ is the $j$-th littlewood block.
Commutator estimate for kinetic operator

Recall $P_t = \Gamma_t \rho_t * \Gamma_t$ be the kinetic semigroup.

**Lemma 2.1**

*For any* $\alpha \in (0, 1), \beta \in \mathbb{R}, t \in (0, T], \delta \geq 0, j \geq -1,$

$$\|\Delta_j [P_t(f \prec g) - (\Gamma_t f \prec P_t g)]\|_{L^\infty(\rho_1 \rho_2)} \lesssim t^{-\frac{\delta}{2}} 2^{-(\alpha+\beta+\delta)j} \|f\|_{c_\alpha^\alpha(\rho_1)} \|g\|_{c_\beta^\beta(\rho_2)}.$$  

*Here* $\Delta_j$ *is the* $j$-*th littlewood block.*

$\Rightarrow$

**Lemma 2.2**

*Commutator estimate*

$$\|[[\mathcal{I}_\lambda, f \prec]g]\|_{L^\infty_T c_\alpha^{\alpha+\beta+2}(\rho_1 \rho_2)} \lesssim \|f\|_{S_\alpha_T(\rho_1)} \|g\|_{L^\infty_T c_\beta^\beta(\rho_2)}.$$ (6)

$\Rightarrow u \in C_T c_\alpha^{2-\alpha}(\rho_\delta), u^\# \in C_T c_\alpha^{3-2\alpha}(\rho_\delta)$
Renormalization

If $b \circ \nabla_v \mathcal{I} b$, $b \circ \nabla_v \mathcal{I} f \in L^\infty_T \mathcal{C}^{1-2\alpha}_a (\rho_\kappa)$
Renormalization

If $b \circ \nabla_v \mathcal{I} b, b \circ \nabla_v \mathcal{I} f \in L_T^\infty \mathcal{C}_a^{1-2\alpha}(\rho_\kappa)$ $\Rightarrow b \circ \nabla u \in L_T^\infty \mathcal{C}_a^{1-2\alpha}(\rho_\kappa)$ by commutator estimate and paracontrolled ansatz
Renormalization

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- Let \( b \) be a Gaussian field with the following covariance:

\[
\mathbb{E}(b(g_1)b(g_2)) = \int_{\mathbb{R}^d} \hat{g}_1(\zeta) \hat{g}_2(-\zeta) \mu(d\zeta).
\]

Assumption: \( \mu \) is symmetric in second variable and for some \( \beta < \alpha \),

\[
\sup_{\zeta' \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{(1 + |\zeta' + \zeta|_a)^{2\beta}} < \infty.
\]

Probabilistic calculation \( \Rightarrow b \in L_T^\infty C_a^{-\alpha}(\rho_\kappa), b \circ \nabla_v \mathcal{I} b \in L_T^\infty C_a^{1-2\alpha}(\rho_\kappa) \)
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- If $b \circ \nabla_v \mathcal{I} b, b \circ \nabla_v \mathcal{I} f \in L_T^\infty \mathbf{C}_a^{1-2\alpha} (\rho_\kappa) \Rightarrow b \circ \nabla u \in L_T^\infty \mathbf{C}_a^{1-2\alpha} (\rho_\kappa)$ by commutator estimate and paracontrolled ansatz.

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Assumption: $\mu$ is symmetric in second variable and for some $\beta < \alpha$,

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\sup_{\zeta' \in \mathbb{R}^2d} \int_{\mathbb{R}^2d} \frac{\mu(d\zeta)}{(1 + |\zeta' + \zeta|_a)^{2\beta}} < \infty.
$$

Probabilistic calculation $\Rightarrow b \in L_T^\infty \mathbf{C}_a^{-\alpha} (\rho_\kappa), b \circ \nabla_v \mathcal{I} \chi b \in L_T^\infty \mathbf{C}_a^{1-2\alpha} (\rho_\kappa)$.

- Example: For $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right)$ and $\gamma_1, \gamma_2 \in [0, d)$ with $3\gamma_1 + \gamma_2 > 4d - 2\beta$, let

$$
\mu(d\xi, d\eta) = |\xi|^{-\gamma_1} |\eta|^{-\gamma_2} d\xi d\eta.
$$

(e.g. noise white in $v$ and colored in $x$)
Renormalization

- If $b \circ \nabla_v \mathcal{I} b, b \circ \nabla_v \mathcal{I} f \in L_T^\infty \mathcal{C}_a^{\alpha - 2\alpha}(\rho_\kappa)$, $b \circ \nabla u \in L_T^\infty \mathcal{C}_a^{\alpha - 2\alpha}(\rho_\kappa)$ by commutator estimate and paracontrolled ansatz.
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Probabilistic calculation $\Rightarrow b \in L_T^\infty \mathcal{C}_a^{-\alpha}(\rho_\kappa), b \circ \nabla_v \mathcal{I} \lambda b \in L_T^\infty \mathcal{C}_a^{1-2\alpha}(\rho_\kappa)$.

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- Interesting point: 0th Wiener chaos is not constant but converges after minus a formally diverging term, which is zero by symmetry.
Renormalization

- If $b \circ \nabla_v \mathcal{I} b, b \circ \nabla_v \mathcal{I} f \in L_T^\infty C_{\alpha}^{1-2\alpha}(\rho_\kappa) \Rightarrow b \circ \nabla u \in L_T^\infty C_{\alpha}^{1-2\alpha}(\rho_\kappa)$ by commutator estimate and paracontrolled ansatz.

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(e.g. noise white in $v$ and colored in $x$)

- Interesting point: 0th Wiener chaos is not constant but converges after minus a formally diverging term, which is zero by symmetry $\Rightarrow$ No renormalization.
Well-posedness of linear PDE

\[(\partial_t - \Delta v - v \cdot \nabla_x) u = b \cdot \nabla v u + f, \quad u(0) = u_0.\]

**Theorem 1**

Let \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right) \) and \( \vartheta := \frac{9}{2-3\alpha} \) and \( \delta := (2\vartheta + 2)\kappa \leq 1. \) For any \( T > 0, \) \((b, f)\) as above, \( \exists! \) paracontrolled solution \((u, u^\#)\) to PDE \((5)\) such that \( \|u\|_{C_T c_T^{2-\alpha}(\rho_\delta)} + \|u^\#\|_{C_T c_T^{3-2\alpha}(\rho_{2\delta})} \lesssim C(b, f). \)
Well-posedness of linear PDE

\[(\partial_t - \Delta_v - v \cdot \nabla_x)u = b \cdot \nabla_v u + f, \quad u(0) = u_0.\]

**Theorem 1**

Let \(\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)\) and \(\vartheta := \frac{9}{2-3\alpha}\) and \(\delta := (2\vartheta + 2)\kappa \leq 1.\) For any \(T > 0, (b, f)\) as above, \(\exists!\) paracontrolled solution \((u, u^\sharp)\) to PDE (5) such that \(\|u\|_{C_T C_T^2 - \alpha(\rho_\delta)} + \|u^\sharp\|_{C_T C_T^3 - 2\alpha(\rho_2\delta)} \lesssim C(b, f).\)

**Idea of proof**

- **Existence**: based on the paracontrolled caculus developed before
  - **difficulty**: Loss of weight from \(b \cdot \nabla_v u\)
  - **Solution** from [Zhang, Zhu, Z. 20]:
    - Step 1: Schauder estimate for \(b, f\) in unweighted Besov space
      \(\|u\|_{L_T^\infty C_T^2 - \alpha} \) depending polynomially on the coefficient \((b, f)\) not exponentially as Gronwall
Well-posedness of linear PDE

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Theorem 1

Let \( \alpha \in \left(\frac{1}{2}, \frac{2}{3}\right) \) and \( \vartheta := \frac{9}{2 - 3\alpha} \) and \( \delta := (2\vartheta + 2)\kappa \leq 1. \) For any \( T > 0, \) \( (b, f) \) as above, \( \exists! \) paracontrolled solution \((u, u^\#)\) to PDE (5) such that
\[
\|u\|_{C_T \mathbb{C}^{2-\alpha} \rho_\delta} + \|u^\#\|_{C_T \mathbb{C}^{3-2\alpha} \rho_{2\delta}} \lesssim C(b, f).
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      \]
    - Step 2: Schauder estimate for \( b, f \) in weighted Besov space
      - **Trick:** Localization+New characterization of Besov space
Well-posedness of linear PDE

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**Theorem 1**

Let \(\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)\) and \(\vartheta := \frac{9}{2 - 3\alpha}\) and \(\delta := (2\vartheta + 2)\kappa \leq 1\). For any \(T > 0\), \((b, f)\) as above, \(\exists!\) paracontrolled solution \((u, u^\#:)\) to PDE (5) such that

\[\|u\|_{C_T c_T^{2-\alpha}(\rho_\delta)} + \|u^\#:\|_{C_T c_T^{3-2\alpha}(\rho_{2\delta})} \lesssim C(b, f).\]

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  - **Uniqueness**: Localization
Nonlinear equation
Nonlinear mean field equation

- Assume that $\text{div}_v b = 0$. Consider the following

$$
\mathcal{L} u = b \cdot \nabla_v u + K * \langle u \rangle \cdot \nabla_v u, \quad u(0) = u_0. \tag{7}
$$

Here $\langle u \rangle(t, x) := \int_{\mathbb{R}^d} u(t, x, v) \, dv$. Assume that

- $K \in \bigcup_{\beta > \alpha - 1} C_x^{\beta/3}$, $b \circ \nabla_v \mathcal{J}(b) \in C^{1-2\alpha}_a(\rho_\kappa)$
Nonlinear equation

Assume that $\text{div}_v b = 0$. Consider the following

$$L u = b \cdot \nabla_v u + K * \langle u \rangle \cdot \nabla_v u, \quad u(0) = u_0. \quad (7)$$

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$$K \in \bigcup_{\beta > \alpha - 1} C^\beta_x, \quad b \circ \nabla_v \mathcal{I}(b) \in C^{1-2\alpha}_a (\rho^\kappa)$$

Theorem 2

Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and $\kappa$ be small enough so that $\delta := 2(\frac{9}{2-3\alpha} + 1)\kappa < 1$. $\rho_0 = (1 + |x|^{1/3} + |v|)^{\kappa_0}$ with $\kappa_0 > 0$.

- for any probability density $u_0 \in L^1(\rho_0) \cap C^\gamma_a, \gamma > 1 + \alpha$, there exists at least a probability density paracontrolled solution $u \in L^\infty_T(C^{2-\alpha}(\rho_\delta))$ to (7).
- If in addition that $K$ is bounded and $H(u_0) := \int u_0 \ln u_0 < \infty$, the solution is unique.
Idea of proof: Existence

A priori estimate from linear equation

\[ \|u\|_{S_T^{2-\alpha}(\rho)} \lesssim 1. \]
Idea of proof: Existence

A priori estimate from linear equation

$$\|u\|_{S_{T}^{2-\alpha}(\rho)} \lesssim 1.$$  

This is not enough to obtain the convergence of the nonlocal term $K * \langle u \rangle$.  

Solution:

$$\|u(t)\|_{L^{1}(\rho_{0})} \leq C \|u_{0}\|_{L^{1}(\rho_{0})},$$

Moment estimate of associated SDE: By Itô's formula, we have

$$E\rho_{0}(Z_{t}) = \rho_{0}(z) + E\int_{0}^{t}(\Delta v \rho_{0} + v \cdot \nabla x \rho_{0})(Z_{s})\,ds + E\int_{0}^{t}(B \cdot \nabla v \rho_{0})(s, Z_{s})\,ds,$$

with $B = b + K * \langle u \rangle$.

By Itô's formula again, we have

$$0 = Ew_{t}(t, Z_{z}) = w_{t}(0, z) + E\int_{0}^{t}(B \cdot \nabla v \rho_{0})(s, Z_{z_{s}})\,ds.$$
Idea of proof: Existence

1. A priori estimate from linear equation

\[ \|u\|_{S^{2-\alpha}_T} \lesssim 1. \]

This is not enough to obtain the convergence of the nonlocal term \( K \ast \langle u \rangle \).

2. Solution: \( \|u(t)\|_{L^1(\rho_0)} \leq C \|u_0\|_{L^1(\rho_0)}, \|u(t)\|_{L^1(\rho_0)} = E\rho_0(Z_t) \)

Moment estimate of associated SDE: By Itô’s formula, we have

\[
E\rho_0(Z_t) = \rho_0(z) + E \int_0^t (\Delta v \rho_0 + v \cdot \nabla_x \rho_0)(Z_s)ds + E \int_0^t (B \cdot \nabla v \rho_0)(s, Z_s)ds,
\]

with \( B = b + K \ast \langle u \rangle \).
Idea of proof: Existence

1. A priori estimate from linear equation

$$\|u\|_{S^2_T-\alpha(\rho)} \lesssim 1.$$  

This is not enough to obtain the convergence of the nonlocal term $K * \langle u \rangle$.

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with $B = b + K * \langle u \rangle$. By Itô’s formula again, we have

$$0 = Ew^t(t, Z^z_t) = w^t(0, z) + E\int_0^t (B \cdot \nabla v \rho_0)(s, Z^z_s) ds.$$  

Here $w^t$ is the unique solution of the following backward PDE:

$$\partial_s w^t + (\Delta v + v \cdot \nabla_x + B \cdot \nabla v)w^t = B \cdot \nabla v \rho_0, \quad w^t(t) = 0.$$
Idea of proof: Uniqueness

Uniqueness: Let \( w = u_1 - u_2 \)

\[
\partial_t w = \Delta_v w + \mathbf{v} \cdot \nabla_x w + \mathbf{b} \cdot \nabla_v w + K \ast \langle w \rangle \cdot \nabla_v u_1.
\]

\( L^1 \) estimate and \( \|\nabla_v u_1\|_{L^2_t L^1_x}^2 \)
Idea of proof: Uniqueness

1. Uniqueness: Let $w = u_1 - u_2$

$$
\partial_t w = \Delta_v w + v \cdot \nabla_x w + b \cdot \nabla_v w + K \ast \langle w \rangle \cdot \nabla_v u_1.
$$

$L^1$ estimate and $\|\nabla_v u_1\|_{L^2_t L^1}$

2. Entropy estimate: Formally let $\beta(u) = u \ln u$

$$
\partial_t \beta(u) = \Delta_v \beta(u) - v \cdot \nabla_x \beta(u) - b \cdot \nabla_v \beta(u) - \beta''(u) |\nabla_v u|^2.
$$

$\Rightarrow H(u(t)) + \|\nabla_v u\|^2_{L^2_t L^1} \leq H(u_0)$
Idea of proof: Uniqueness

1. **Uniqueness:** Let \( w = u_1 - u_2 \)

\[
\partial_t w = \Delta_v w + v \cdot \nabla_x w + b \cdot \nabla_v w + K \ast \langle w \rangle \cdot \nabla_v u_1.
\]

\( L^1 \) estimate and \( \| \nabla_v u_1 \|_{L^2_t L^1}^2 \)

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\]

\( \Rightarrow H(u(t)) + \| \nabla_v u \|_{L^2_t L^1}^2 \leq H(u_0) \)

3. \( \beta'(u) \nabla_x u \) is not well defined since \( \beta'(u) \in C_a^{2-\alpha}, \nabla_x u \in C_a^{-1-\alpha} \).
Idea of proof: Uniqueness

1. Uniqueness: Let \( w = u_1 - u_2 \)

\[
\partial_t w = \Delta_v w + \nu \cdot \nabla_x w + b \cdot \nabla_v w + K \ast \langle w \rangle \cdot \nabla_v u_1.
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\( \Rightarrow \) Linear approximation

\[
\mathcal{L} u^n = b^n \cdot \nabla_v u^n + K^n \ast \langle u_1 \rangle \cdot \nabla_v u^n.
\]
Singular DDSDE
Consider the following kinetic DDSDE with singular drift: $Z = (X, V)$

$$
\begin{align*}
    \mathrm{d}X_t &= V_t \mathrm{d}t, \\
    \mathrm{d}V_t &= b(X_t, V_t) \mathrm{d}t + (K \ast \mu_t)(X_t) \mathrm{d}t + \sqrt{2} \mathrm{d}B_t,
\end{align*}
$$

$B_t$: a $d$-dimensional Brownian motion, $b$ is singular

$\mu_t$: law of $X_t$, $K \ast \mu(x) := \int_{\mathbb{R}^d} K(x-y) \mu(\mathrm{d}y)$.
Consider the following kinetic DDSDE with singular drift: \( Z = (X, V) \)

\[
dX_t = V_t dt, \quad dV_t = b(X_t, V_t) dt + (K * \mu_t)(X_t) dt + \sqrt{2} dB_t,
\]

(8)

\( B_t \): a \( d \)-dimensional Brownian motion, \( b \) is singular

\( \mu_X \): law of \( X_t \), \( K * \mu(x) := \int_{\mathbb{R}^d} K(x - y) \mu(dy) \).

Problem: How to understand (8)? What is the meaning of \( \mathcal{L}^{b, \mu} f(X_s, V_s) \)? Here

\[
\mathcal{L}^{b, \mu} = (\partial_t + \Delta_v + v \cdot \nabla_x) + b \cdot \nabla_v + K * \mu_t \cdot \nabla_v.
\]
Consider the following kinetic DDSDE with singular drift: 

\[ Z = (X, V) \]

\[ dX_t = V_t dt, \quad dV_t = b(X_t, V_t)dt + (K * \mu_{X_t})(X_t)dt + \sqrt{2} dB_t, \quad (8) \]

- \( B_t \): a \( d \)-dimensional Brownian motion, \( b \) is singular
- \( \mu_{X_t} \): law of \( X_t \), \( K * \mu(x) := \int_{\mathbb{R}^d} K(x - y) \mu(dy) \).

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\[ L^{b, \mu} = (\partial_t + \Delta_v + v \cdot \nabla_x) + b \cdot \nabla_v + K * \mu_t \cdot \nabla_v. \]

Solution: Consider the following linear equation for given \( \mu : [0, T] \rightarrow \mathcal{P}(\mathbb{R}^{2d}) \)

\[ L^{b, \mu} u = f, \quad u(T) = \varphi. \quad (9) \]

see also [Delarue, Diel 16, Cannizzaro, Chouk 18, Kremp, Perkowski 20]
Singular DDSDE

- Consider the following kinetic DDSDE with singular drift: \( Z = (X, V) \)

\[
\begin{align*}
\mathrm{d}X_t &= V_t \mathrm{d}t, \\
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B_t: & \text{ a } d\text{-dimensional Brownian motion, } b \text{ is singular} \\
\mu_t: & \text{ law of } X_t, \quad K * \mu(x) := \int_{\mathbb{R}^d} K(x - y) \mu(\mathrm{d}y).
\end{align*}
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Problem: How to understand (8)? What is the meaning of \( \mathcal{L}^{b, \mu} f(X_s, V_s) \)? Here

\[
\mathcal{L}^{b, \mu} = (\partial_t + \Delta_v + v \cdot \nabla_x) + b \cdot \nabla_v + K * \mu_t \cdot \nabla_v.
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\mathcal{L}^{b, \mu} u = f, \quad u(T) = \varphi.
\]

see also [Delarue, Diel 16, Cannizzaro, Chouk 18, Kremp, Perkowski 20]

Definition 4.1

(Martingale problem) Let \( \delta > 0 \). A probability measure \( \mathbb{P} \in \mathcal{P}(\mathcal{C}_T) \) is called a martingale solution to SDE (8), if for all \( f \in \mathcal{C}_b \), \( \varphi \in \mathcal{C}_a^{\gamma} \) with some \( \gamma > 1 + \alpha \) and \( \mu_t := \mathbb{P} \circ X_t^{-1} \),

\[
M_t := u^\mu_f(t, Z_t) - u^\mu_f(0, Z_0) - \int_0^t f(s, Z_s) \mathrm{d}s
\]

is a martingale under \( \mathbb{P} \). Here \( u^\mu_f \) is a solution to (9).
Main results

Theorem 3

Suppose that $b \circ \nabla_v J(b) \in C^{1-2\alpha}_a(\rho_\kappa)$ and $K \in \bigcup_{\beta \geq \alpha - 1} C^\beta_a$. Then there exists at least one martingale solution $\mathbb{P}$ to SDE (8). Moreover, if $K$ is bounded measurable, then the solution is unique.
Main results

Theorem 3

Suppose that \( b \circ \nabla_v \mathcal{I} (b) \in C^{1-2\alpha}_a(\rho_\kappa) \) and \( K \in \bigcup_{\beta > \alpha-1} C^\beta_a \). Then there exists at least one martingale solution \( \mathbb{P} \) to SDE (8). Moreover, if \( K \) is bounded measurable, then the solution is unique.

Idea of proof

- Existence: convolution approximation
Main results

Theorem 3

Suppose that $b \circ \nabla_v F(b) \in C^{1-2\alpha}_a(\rho_\kappa)$ and $K \in \bigcup_{\beta > \alpha - 1} C^\beta_a$. Then there exists at least one martingale solution $\mathbb{P}$ to SDE (8). Moreover, if $K$ is bounded measurable, then the solution is unique.

Idea of proof

- Existence: convolution approximation
- Uniqueness: First for $K = 0$ and Girsanov’s transformation
Thank you!