27/10/2025 Lecture 1 Tonegawa

Mean Curvature flow in GMT: Brakke flow

Plateau problem: given $\Gamma^1 \subset \mathbb{R}^3$ closed curved find a surface M with $\partial M = \Gamma$ with the least area

Direct method of Calc Var

- minimizing sequence
- $\exists$ a subseq that converges to a limit

→ 1930 Douglas-Radó

1960 Federer-Fleming normal currents (oriented surface)

<u>Not constructive</u>

(minimizing seq exists because of def. of infimum ma non costruisco)

- Brakke (Almgren)

Plateau problem by gradient method: mean curvature flow ← more constructive (la sequenza è in <u>time</u>)

(1978) Princeton lecture notes

Notation $I = [0, \infty)$ or $[0, T)$, $\Gamma_t^k \subset \mathbb{R}^n$ $t \in I$ (time) is called mean curvature flow (MCF) if the normal velocity V_{Γ_t} is equal to the mean curvature vector $\mathbb{H}_{\Gamma_t}$. $V_{\Gamma_t} = \mathbb{H}_{\Gamma_t}$ for $t \in I$ on Γ_t

example: $k = n-1$ ($(n-1)$-dim ~~##~~ surface)

$$R(t) = \sqrt{R_0^2 - 2(n-1)t} \qquad t \in \left[0, \frac{R_0^2}{2(N-1)}\right)$$

$$\Gamma_t = \{x \in \mathbb{R}^n ; |x| = R(t)\}$$ ← sphere che diventano PUNTO

$$\underbrace{\frac{dR(t)}{dt}}_{\text{normal velocity}} = -\frac{n-1}{R(t)} \rightsquigarrow V_{\Gamma(t)} = -\frac{n-1}{R(t)} \frac{x}{|x|}$$

cioè con normale uscente

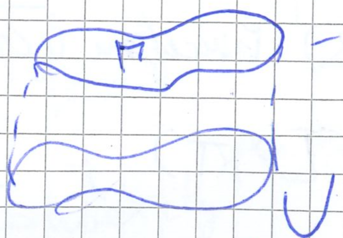
principal curvature $= -\dfrac{1}{R(t)}$

<u>scalar</u> mean curvature $= -\dfrac{n-1}{R(t)}$ (= sum of principal curvatures)

mean curvature vector $\underline{H} = -\dfrac{n-1}{R(t)} \dfrac{x}{|x|}$ ← wrt to ~~exterior~~ NORMAL!!

$$\Rightarrow \underline{V}_{\Gamma(t)} = \underline{H}$$

Example 2: $k = n-1$, suppose $\Gamma_t = \{(x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1})) : (x_1, \dots, x_{n-1}) \in U \subset \mathbb{R}^n\}$ (U: bdd domain in $\mathbb{R}^{n-1}$)



→ Γ come graph

→ unit normal $= \underline{\nu} = \dfrac{1}{\sqrt{1+|\nabla f|^2}}(-f_{x_1}, -f_{x_2}, \dots, -f_{x_{n-1}}, 1)$

$$f_{x_i} = \frac{\partial f}{\partial x_i}$$

$$\nabla f = (f_{x_1}, \dots, f_{x_{n-1}}), \quad V_{\Gamma(t)} = \frac{\partial_t f}{\sqrt{1+|\nabla f|^2}} \underline{\nu}$$

plus normal

$$h_{\Gamma_t} = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \underline{\nu}$$

$\Rightarrow$ $|\nabla f| \approx 0$ $\quad f_t = \Delta f$ $(= \mathrm{div}(\nabla f))$ linearization of MCF è Heat equation!

MCF is geometric heat equation

↳ parabolic version of minimal surfaces

Assume Γ_t is fixed on ∂U

In this case: $\frac{d}{dt} \mathcal{H}^{n-1}(\Gamma_t)$ $\quad$ $(n-1)$-dim Hausdorff measure

$$= \frac{d}{dt} \int_U \sqrt{1+|\nabla f|^2}\, dx_1 \dots dx_{n-1}$$

$$= \int_U \frac{\nabla f \cdot (\nabla f)_t}{\sqrt{1+|\nabla f|^2}}\, dx_1 \dots dx_{n-1}$$

$$- \int \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \partial_t f + \int_{\partial U} \frac{\partial_t f}{\sqrt{1+|\nabla f|^2}}\, \underline{\nu} \cdot \nabla f \, d\mathcal{H}^{n-2}$$

$\underline{\nu}$ outer normal to ∂U

$$= -\int \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \partial_t f \, dx_1 \dots dx_{n-1}$$

$f_t = 0$ on ∂U

superficie FISSATA al bordo

$$= -\int \underbrace{\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \frac{\partial_t f}{\sqrt{1+|\nabla f|^2}}}_{H_{\Gamma_t} \cdot V_{\Gamma_t}} \underbrace{\sqrt{1+|\nabla f|^2}\, dx_1 \dots dx_{n-1}}_{d\mathcal{H}^{n-2}}$$

$$= -\int_{\Gamma_t} H_{\Gamma_t} \cdot V_{\Gamma_t}\, d\mathcal{H}^{n-1} \underset{\text{se MCF}}{=} -\int |H_{\Gamma_t}|^2 d\mathcal{H}^{n-1} \le 0$$

questo è in general

$\to$ MCF decrease surface area unless $H_\Gamma = 0$

when $t \to \infty$ $\leadsto$ I expect convergence to MINIMAL SURFACE

Γ is called MINIMAL if $h_\Gamma = 0$

$\Gamma_0 \rightsquigarrow$ regularization but singularity can happen in finite time

contracts

Brakke flow

$\rightsquigarrow$ multi-phase mean curvature flow

grain boundary motion

$\Rightarrow$ Weak solution of MCF

- Brakke flow
- level set flow (only two-phase case)
- BV solution

relazioni tra le soluzioni not fully understood

idea of Brakke flow $\Gamma_t \rightsquigarrow \mathcal{H}^k \llcorner \Gamma_t$

look at surface measure induced from Γ_t

$$\mathcal{H}^k \llcorner \Gamma_t (A) = \mathcal{H}^k (A \cap \Gamma_t)$$

+ utile vedere flow con misura

$\rightarrow$ Weak notion of normal velocity vector

$$V_{\Gamma_t} \rightsquigarrow \forall \phi(x,t) \in C_c^\infty(\mathbb{R}^n_{(x} \times \mathbb{R}_{t)})$$

$$\frac{d}{dt}\left(\int_{\Gamma_t} \phi(x,t)\, d\mathcal{H}^k\right) =$$

$\nabla\phi = (\partial_{x_1}\phi, \dots, \partial_{x_n})$ transport thm con ∂ con derivata Totale

change of surface element

$$= \int_{\Gamma_t} (\nabla\phi - \underline{H}_{\Gamma_t}\phi) \cdot \underline{V}_{\Gamma_t} + \phi_t \, d\mathcal{H}^k(x) \qquad (1)$$

the rate of change perché cambia Γ_t → directional derivative

if $\phi = 1$ $\quad \frac{d}{dt}\mathcal{H}^k(\Gamma_t) = -\int H_{\Gamma_t} \cdot V_{\Gamma_t} \, d\mathcal{H}^k$

Prop 1. For a <u>smooth</u> family $\{\Gamma_t\}$, the normal vector field $\tilde{v}$ defined on Γ_t is the normal velocity vector V_{Γ_t} iff

$$\frac{d}{dt}\int_{\Gamma_t} \phi(x,t) \, d\mathcal{H}^k \leq \int_{\Gamma_t} (\nabla\phi - H_{\Gamma_t}\phi) \cdot \tilde{v} + \phi_t \, d\mathcal{H}^k \qquad (2)$$

$\forall \phi \in \mathcal{C}^1_c(\mathbb{R}^n)$ with $\phi \geq 0$

something normal velocity vector

Proof → If $\tilde{v} = V_{\Gamma_t}$, then (2) is true with equality due to (1)

Conversely, suppose (2) is true for $\tilde{v}$.

Since V_{Γ_t} satisfies (1), it satisfies (2) as well.

<u>Take the difference</u> → $w = \tilde{v} - V_{\Gamma_t}$

$$0 \leq \int_{\Gamma_t} (\nabla\phi - H_{\Gamma_t}\phi) \cdot w \, d\mathcal{H}^k \text{ for}$$

any $\phi \geq 0$ → claim $\underline{w} \equiv 0$.

$\forall x_0 \in \Gamma_t$. We may assume $T_{x_0}\Gamma_t$ = tangent space of Γ_t at x_0

$= \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$, $x_0 = 0$ after some rotation

Let $\phi \in C^1_c(\mathbb{R}^n)$, $\phi \geq 0$. Let $\lambda > 0$

<u>Define:</u> $\phi_\lambda(x) = \frac{1}{\lambda^{k-1}} \phi\left(\frac{x}{\lambda}\right) \rightsquigarrow \nabla_x \phi_\lambda(x) = \frac{1}{\lambda^k} \nabla\phi\left(\frac{x}{\lambda}\right)$

$$0 \leq \int_{M_t} \frac{1}{\lambda^k} \left(\nabla\phi\left(\tfrac{x}{\lambda}\right) - H_{\Gamma_t}(x) \frac{1}{\lambda^{k-1}} \phi\left(\tfrac{x}{\lambda}\right)\right) \cdot w \, d\mathcal{H}^k(x)$$

Let $\tilde{x} = \frac{x}{\lambda}$ $(\lambda \searrow 0)$ faccio blow up

$d\mathcal{H}^k(x) = \lambda^k d\mathcal{H}^k(\tilde{x})$

$$0 \leq \int_{\lambda^{-1}\Gamma_t} \left(\nabla\phi(\tilde{x}) - \underbrace{H_{\Gamma_t}(\lambda\tilde{x})}_{\rightsquigarrow H_{\Gamma_t}(0)} \lambda \phi(\tilde{x})\right) \cdot w(\lambda\tilde{x}) \, d\mathcal{H}^k(\tilde{x})$$

diventa tangente space

as $\lambda \to 0$

$$0 \leq \int_{T_0\Gamma_t} \overbrace{\nabla\phi(\tilde{x})}^{(\partial_{x_1}\ldots\partial_{x_n}\phi)} \cdot \underbrace{w(0)}_{\text{è CONST.}} \, d\mathcal{H}^k(\tilde{x})$$

$T_0\Gamma_t = \mathbb{R}^k \times \{0\}$

vedo solo le prime k componenti

$$= \int_{\mathbb{R}^k \times \{0\}} (\phi_{k+1}(\tilde{x}), \ldots, \phi_n(\tilde{x})) \, d\mathcal{H}^k \cdot (w_{k+1}(0) \ldots w_n(0))$$

$\Rightarrow w_{k+1}(0) = \ldots = w_n(0) = 0$

choosing ϕ

But w normal vector $\to (w_1(0) \equiv \ldots \equiv w_k(0) = 0)$

$\Rightarrow \tilde{V} = V_{\Gamma_t}$

NB Same argument if in (2) we set $\geq$

Prop 2 A smooth family $\{\Gamma_t\}$ is a MCF iff $\forall \phi \in C^1_c$, $\phi \geq 0$, $\forall t_1 < t_2$, $t_1, t_2 \in I$,

$$\int_{\Gamma_{t_2}} \phi(x, t_2)\, d\mathcal{H}^k - \int_{\Gamma_{t_1}} \phi(x, t_1)\, d\mathcal{H}^k(x) \qquad (3)$$

$$\leq \int_{t_1}^{t_2} \int_{\Gamma_t} (\nabla\phi - \phi H_{\Gamma_t}) \cdot H_{\Gamma_t} + \phi_t \, d\mathcal{H}^k(x)$$

NB questa formulazione vale ANCHE se NON sono smooth

questa se ho MCF vale (2) ⊕ con $V_{\Gamma_t} = H_{\Gamma_t}$ ho ottenuto questa integrando (2).

Viceversa: Se vale (3) $\longrightarrow$ vale (2) con $\tilde{V} = H_{\Gamma_t}$ e quindi $\underset{H_{\Gamma_t}}{\tilde{V}} = V_{\Gamma_t}$ ■

NB

H_{Γ_t} è normal vector FIELD per cui da (3) ho (2).

→ Basic notions of GMT

Def 1 A set $\Gamma \subset \mathbb{R}^n$ is called a countably k-rectifiable if $\exists$ Lipschitz map $F_j : \mathbb{R}^k \to \mathbb{R}^n$ ($j = 1, 2, 3, \dots$) st $\mathcal{H}^k(\Gamma \setminus \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^k)) = 0$

copro con ∞ k dimensional subsets

Remark → Lipschitz can be equivalently replaced by C^1

Notation: Γ is rec_k if it is countably k-rectifiable and $\mathcal{H}^k(K\cap\Gamma)<\infty$, locally finite

Γ $\mathcal{H}^k$ measurable

$\forall K \subset\subset \mathbb{R}^n$

Prop 3: If $\Gamma \in \mathrm{rec}_k$, for $\mathcal{H}^k$ a.e. $x\in\Gamma$, $\exists!$ k-dim subspace denoted by $T_x\Gamma$ called approximate tangent space

$\Longleftrightarrow$ let $\Gamma_{x,\lambda} = \left\{ \frac{y-x}{\lambda} : y\in\Gamma \right\} \to$ " $\frac{\Gamma-\{x\}}{\lambda}$ " for $\lambda\to 0$

blow up — move x to origin and expand

$\mathcal{H}^k \llcorner \Gamma_{x,\lambda} \xrightarrow[\text{as measures}]{*} \mathcal{H}^k \llcorner T_x\Gamma$ as Radon Measures

(concept of piano ... col blow up)

Def 2. A Radon measure μ on $\mathbb{R}^n$ is called <u>k-integral</u> if $\exists\,\Gamma\in\mathrm{rec}_k$ $\exists\,\theta:\Gamma\to\mathbb{N}$ ($\mathcal{H}^k$-measurable) st $\mu=\theta\,\mathcal{H}^k\llcorner\Gamma$

$\to$ i.e. $\forall\,\phi\in\mathcal{C}_c(\mathbb{R}^n)$ $\displaystyle\int_{\mathbb{R}^n}\phi(x)\,d\mu = \int_\Gamma \phi(x)\,\underset{\sim}{\theta(x)}\,d\mathcal{H}^k$

θ multiplicity function — natural numbers

If $\theta\equiv 1$ then $\mu=\mathcal{H}^k\llcorner\Gamma$

↓ è come quanti "fogli" hanno appiamo

→ Mean curvature vector (for C^2 Γ ē usual notation)

→ $\forall g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_\Gamma \mathrm{div}_{T_x\Gamma}\, g(x)\, d\mathcal{H}^k = -\int_\Gamma \underline{H}_\Gamma \cdot \underline{g}\, d\mathcal{H}^k \qquad (4)$$

↳ $\mathrm{div}_{T_x\Gamma}\, g = \sum_{i,j=1}^{n} (T_x\Gamma)_{ij} \frac{\partial g_i}{\partial x_j} = \mathrm{tr}(\overset{\text{MATRIX}}{T_x\Gamma} : \nabla g)$

$(T_x\Gamma)_{ij} = (i,j)$ component of the matrix $T_x\Gamma : (\mathbb{R}^n \to T_x\Gamma)$

→ check this for graph $\Gamma = \mathrm{graph}\, f$ → ē ovvio, nota

$\mathrm{div}_{T_x\Gamma}\, g = (I - \underbrace{\underline{n} \otimes \underline{n}}_{\text{tag, trace normale}}) : \nabla g$

$\left(I - \frac{(\nabla f, 1)}{\sqrt{1+|\nabla f|^2}} \otimes \frac{(\nabla f, 1)}{\sqrt{1+|\nabla f|^2}}\right) : \nabla g$

$f_{,i}\ f_{,j}\ g_{i,j}$

$f_{,ij}\ f_{,j}\ g_i$

by parts → $-\int \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \frac{(\nabla f, 1)}{\sqrt{1+|\nabla f|^2}} \cdot g$

$T_x\Gamma = I - \underline{\nu} \otimes \underline{\nu}$

▷ $\int \left(I - \frac{(\nabla f, 1)}{\sqrt{}} \otimes \frac{(\nabla f, 1)}{\sqrt{}}\right) : \nabla g \sqrt{1+|\nabla f|^2}\, dx$

NB $g(x_1, \dots, x_{n-1}, f(x \dots))$ → should integrate by parts con la composta