Discussion on "Singular kinetic equations"
by Rongchan Zhu

Workshop "Higher structures emerging from renormalisation"
ESI, Vienna, November 18, 2021

Singular SPDEs

• Parabolic SPDEs on $\mathbb{T}^d$

$$\partial_t \phi = \Delta\phi + F(\phi, \nabla\phi, \xi)$$

ξ: space-time white noise

Important examples:

* Φ^4_d model: $F = -\phi^3 + \xi$ (stochastic quantization)

* KPZ eq: $F = -(\partial_x \phi)^2 + \xi$ (interface growth)

* PAM: $F = \phi\, \xi(x)$ (diffusion in random medium)

↑ spatial white noise

• Extensions featured in this talk:

– $\mathbb{R}^d$ instead of $\mathbb{T}^d$
using new type of weighted Besov spaces
Prior work: e.g. Hairer & Labbé, Mourrat & Weber, Moinat & Weber (using "coming down from infinity")

– less regularising differential operator using kinetic semigroup

– nonlocal non-linearity F (McKean-Vlasov equ.)

Why one needs to renormalise

Mild solution: $\phi = \underbrace{(\partial_t - \Delta)^{-1}}_{\text{heat kernel}} F(\phi, \nabla\phi, \xi)$

Two important facts:

1) $\xi \in \mathcal{C}_{\mathfrak{s}}^{-\frac{d+2}{2}-\varepsilon} \quad \forall \varepsilon > 0$

Besov-Hölder space with parabolic scaling

2) $\phi \in \mathcal{C}_{\mathfrak{s}}^{\alpha} \Rightarrow (\partial_t - \Delta)^{-1}\phi \in \mathcal{C}_{\mathfrak{s}}^{\alpha+2}$ (Schauder estimate)

Consequence: $(\partial_t - \Delta)^{-1}\xi \in \mathcal{C}_{\mathfrak{s}}^{\frac{2-d}{2}-\varepsilon} \quad \forall \varepsilon > 0$

Problem: $\frac{2-d}{2} - \varepsilon < 0 \quad \forall d \geq 2$

$\Rightarrow$ stoch. convolution $(\partial_t - \Delta)^{-1}\xi$ is a distribution

$\Rightarrow$ its powers not well-defined

Renormalisation:

Stoch. convolution solves stoch. heat equation

$$\partial_t \phi = \Delta\phi + \xi$$

Fourier basis: $\{e_k\}_{k\in\mathbb{Z}^d}$, $-\Delta e_k = \lambda_k e_k$, $\lambda_k \sim \|k\|^2$

$$\phi(t,x) = \sum_k \phi_k(t)\, e_k(x)$$

$\Rightarrow \dot{\phi}_k = -\lambda_k \phi_k + \xi_k$ ← i.i.d. Wiener processes

$\Rightarrow \phi_k$ is an OU process with asymptotic variance λ_k^{-1}

$\Rightarrow \phi_k$ approaches Gaussian Free Field (GFF)

$$\phi_{GFF}(x) = \sum_k \frac{Z_k}{\sqrt{\lambda_k}} e_k(x) \qquad Z_k \text{ iid} \sim \mathcal{N}(0,1)$$

(say we forget $k=0$ mode)

$$\phi_{GFF}^2 = \sum_k (\phi * \phi)_k e_k$$

$$\mathbb{E}[\phi_{GFF}^2] \to (\phi*\phi)_0 = \sum_k \phi_k^2 = \sum_k \frac{Z_k^2}{\lambda_k}$$

$$\to \mathbb{E}[\phi_{GFF}^2] \sim \sum_k \frac{1}{\lambda_k} \sim \sum_k \frac{1}{\|k\|^2} \begin{cases} < \infty & \text{if } d = 1 \\ +\infty & \text{if } d \geq 2 \end{cases}$$

Renormalisation: subtract divergent expectations (e.g. after spectral Galerkin approx, then removing UV cut-off)

Remark: almost sure improvement of Sobolev embedding

→ "Probabilistic calculations", very important e.g. for Schrödinger equation

Fractional Sobolev space:

$$\phi = \sum_k \phi_k e_k \Rightarrow \|\phi\|_{H^s}^2 := \sum_k \langle k \rangle^{2s} \phi_k^2$$

"Japanese bracket"
$\langle k \rangle = \sqrt{1 + \|k\|^2}$

Deterministic: Say $d = 1$

$H^{1/2} \hookrightarrow L^p \ \forall p < \infty$, false for $H^s, s < \frac{1}{2}$

Stochastic: $\phi = \sum_k \frac{Z_k}{\sqrt{\lambda_k}} e^{ikx}$ $\quad Z_k$ iid $\sim \mathcal{N}(0,1)$

By rotation inv, $\phi(x) \sim \mathcal{N}_{\mathbb{C}}(0, \sum_k \frac{1}{\lambda_k}) = \mathcal{N}_{\mathbb{C}}(0, \mathbb{E}[\|\phi\|_{L^2}^2])$

$\Rightarrow$ if $\phi \in L^2$, then $\phi \in L^p \ \forall p < \infty$ a.s.

One gains 1/2 of regularity in stochastic case

General theories allowing to solve fixed-point equations for renormalised SPDEs

Regularity Structures Hairer 2014	⟺	Paracontrolled calculus Gubinelli-Imkeller-Perkowski 2015
Hölder-type (local) conditions General results Strongly algebraic	Dictionary: Bailleul-Hoshino 2021	Spectral conditions Not as general More analytical

Remark on mean-field particle systems: Dean's equ.

Consider for simplicity case with no interaction

$X_i(t)$, $i=1,\dots,N$ indep. Brownian paths

Empirical distribution: $\hat{\rho}_N(t,x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - X_i(t))$

φ test Function

Itô's formula: $d\varphi(X_i(t)) = \frac{1}{2}\varphi''(X_i)\,dt + \varphi'(X_i)\,dX_i$

$$d\langle \hat{\rho}_N, \varphi\rangle = \frac{1}{2}\langle \hat{\rho}_N, \varphi''\rangle dt + \frac{1}{N}\sum_i \varphi_i'(X_i)\,dX_i$$

$$= \frac{1}{2}\langle \partial_{xx}\hat{\rho}_N, \varphi\rangle dt + \frac{1}{N}\sum_i \underbrace{\langle \delta(x-X_i), \varphi'\rangle}_{= -\langle \partial_x \delta(x-X_i), \varphi\rangle} dX_i$$

$\Rightarrow$ Formally,

$$d\hat{\rho}_N = \frac{1}{2}\partial_{xx}\hat{\rho}_N + d\eta(x,t)$$

where $d\eta(x,t) = -\frac{1}{N}\sum_i \partial_x(\delta(x-X_i))\,dX_i$

Covariance:

$$\mathbb{E}[d\eta(x,t)\,d\eta(y,s)] = \frac{1}{N^2}\sum_{i,j} \partial_x(\delta(x-X_i))\,\partial_y(\delta(y-X_j))\,\underbrace{\mathbb{E}[dX_i(t)\,dX_j(s)]}_{=\delta_{ij}\,\delta(t-s)}$$

$$= \frac{1}{N^2}\partial_x\partial_y \sum_i \delta(x-X_i)\,\delta(y-X_i)\,\delta(t-s)$$

$$= \frac{1}{N}\delta(t-s)\,\partial_x\partial_y\left[\hat{\rho}_N(x)\,\delta(x-y)\right]$$

Let $\tilde{\eta}(x,t) = \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\hat{\rho}_N}\,\underbrace{\xi(x,t)}_{\text{space-time white noise}}\right)$

Then $\mathbb{E}[d\tilde{\eta}(x,t)\,d\tilde{\eta}(y,s)] = \mathbb{E}[d\eta(x,t)\,d\eta(y,s)]$

Hence one has formally

$$\boxed{\partial_t \hat{\rho}_N - \frac{1}{2}\partial_{xx}\hat{\rho}_N = \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\hat{\rho}_N}\,\xi(t,x)\right)}$$

However, $\partial_x(\sqrt{\rho}\,\xi)$ does not make sense, and does not seem to be accessible to regularity structures / paracontrolled calculus...

c.f. Konarovskyi, Lehmann, von Renesse, arXiv: 1812.11068 / 1806.05018