

### 3 Sheaves and Cohomology

How can we work effectively with sheaves on a toric variety? We exploit methods from computational commutative algebra.

*Sheaves & Modules.* Let  $X$  be a normal toric variety with no torus factors and set  $n := |\Sigma(1)|$ . The Cox ring  $S := \mathbb{C}[x_0, x_1, \dots, x_{n-1}]$  has the grading determined by  $\deg(x_i) := [D_i] \in \text{Cl}(X)$  for all  $0 \leq i < n$ , and the irrelevant ideal is  $B := \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle = \bigcap_{\gamma} \langle x_{\gamma_0}, x_{\gamma_1}, \dots, x_{\gamma_k} \rangle$ .

An  $S$ -module  $F$  is  $B$ -torsion if, for all  $f \in F$ , there exists a positive integer  $m$  such that  $B^m f = 0$ .

The support of a  $B$ -torsion module is contained in the irrelevant set.

**Cox (1995).** *The category of coherent  $\mathcal{O}_X$ -modules is equivalent to the quotient category of finitely-generated  $\text{Cl}(X)$ -graded  $S$ -modules modulo the full subcategory of  $B$ -torsion modules.*

*Ideas.* For any  $\text{Cl}(X)$ -graded  $S$ -module  $F$ , there is a quasicoherent sheaf  $\tilde{F}$  on  $X$  such that, for all  $\sigma \in \Sigma$ ,  $H^0(U_\sigma, \tilde{F}) = (F_{x^{\hat{\sigma}}})_{[0]}$ . The sheaf  $\tilde{F}$  is coherent when  $F$  is finitely generated. For any  $\alpha \in \text{Cl}(X)$ , the shifted  $S$ -module  $S(\alpha)$  produces  $\mathcal{O}_X(\alpha) := \tilde{S}(\alpha)$ . Given a Weil divisor  $D := a_0 D_0 + a_1 D_1 + \dots + a_{n-1} D_{n-1}$  such that  $\alpha = [D]$ , we also have  $\mathcal{O}_X(\alpha) = \mathcal{O}_X(D)$ .

The shifted  $S$ -module  $F(\alpha)$  satisfies  $F(\alpha)_\beta = F_{\alpha+\beta}$  for all  $\beta \in \text{Cl}(X)$ .

For the opposite direction, consider a coherent sheaf  $\mathcal{F}$  on  $X$ . For any  $\alpha \in \text{Cl}(X)$ , set  $\mathcal{F}(\alpha) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha)$  and

$$\Gamma_*(\mathcal{F}) := \bigoplus_{\alpha \in \text{Cl}(X)} H^0(X, \mathcal{F}(\alpha)).$$

One verifies that  $\Gamma_*(\tilde{\mathcal{F}}) \cong \mathcal{F}$ . However, one must choose a suitable finitely generated submodule of  $\Gamma_*(\mathcal{F})$ . If  $f \in H^0(X, \mathcal{F})$  restricts to 0 in  $H^0(U_\sigma, \mathcal{F})$ , then there exists a positive  $m \in \mathbb{Z}$  such that  $x^{\hat{\sigma}} f = 0$ , so any two choices “differ” by a  $B$ -torsion module.  $\square$

**Warning:** In general,  $\Gamma_*(\tilde{F}) \neq F$ .

**Example.** When  $X$  is smooth, there exists an exact sequence

$$0 \longrightarrow \Omega_X \longrightarrow \bigoplus_{j=0}^{n-1} \mathcal{O}_X(-D_j) \longrightarrow \text{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \longrightarrow 0.$$

Given the grading matrix  $\mathbf{W} : \mathbb{Z}^n := \text{Div}_T(X) \rightarrow \text{Pic}(X)$ , there is a first choice of finitely generated  $S$ -module for  $\Omega_X$ : the kernel of

$$\bigoplus_{j=0}^{n-1} S(-\deg x_j) \xrightarrow{\mathbf{W} \begin{bmatrix} x_0 & & & \\ & x_1 & & \\ & & \dots & \\ & & & x_{n-1} \end{bmatrix}} S^{n-d}.$$

The theory of toric vector bundles identifies distinguished  $S$ -modules representing  $T_N$ -equivariant locally-free sheaves on  $X$ .

*Cohomology.* As a benefit of this equivalence, homological techniques over  $S$  lead to effective methods for calculating sheaf cohomology.

**Eisenbud–Mustață–Stillman (2000).** *For any positive  $p \in \mathbb{Z}$ , the map*

$$\text{Ext}_S^{p+1}(S/B^{[m]}, F) \longrightarrow \bigoplus_{\alpha \in \text{Cl}(X)} H^p(X, \tilde{F}(\alpha)),$$

*is an isomorphism for all sufficiently positive  $\alpha$ . Explicit lower bounds on the parameter  $m$  depend on the minimal graded free resolution of  $F$ .*

*Idea.* Replace  $F$  by its minimal free resolution and force the spectral sequence to degenerate at the first page.

The case  $F = S$  gives the cohomology of line bundles. The minimal free resolution of  $S/B$  is a cellular resolution determined by the dual fan (or polytope when  $X$  is projective). For instance, when  $X$  is simplicial, we have

$$\sum_{\alpha \in \text{Cl}(X)} h^i(X, \mathcal{O}_X(\alpha)) = \sum_{\sigma \subseteq \{0,1,\dots,n-1\}} \frac{\dim \tilde{H}^{i-1}(\Delta_\sigma) \prod_{j \in \sigma} t^{-w_j}}{\prod_{j \notin \sigma} (1 - t^{w_j}) \prod_{j \in \sigma} (1 - t^{-w_j})},$$

where  $\Delta_\sigma$  is the induced subcomplex of the simplicial  $(d-1)$ -sphere defined by the fan  $\Sigma$ . □

As a feature, this approach simultaneously computes the cohomology of all sufficiently large twists.

*Vanishing Cohomology.* We highlight a few of the basic results.

**Theorem (Demazure).** *Let  $X$  be a complete toric variety. For any nef  $\mathbb{Q}$ -Cartier divisor  $D$ , we have  $H^p(X, \mathcal{O}_X(D)) = 0$  for all  $p > 0$ .*

**Theorem (Toric Serre Duality).** *Let  $X$  be a complete toric variety. For any  $\mathbb{Q}$ -Cartier divisor  $D$ , we have*

$$H^p(X, \mathcal{O}_X(D))^\vee \cong H^{n-p}(X, \mathcal{O}_X(K_X - D)).$$

**Theorem (Bott–Steenbrink–Danilov).** *Let  $X$  be a smooth projective toric variety. For any ample divisor  $D$ , we have*

$$H^p(X, \Omega_X^q(D)) = 0 \quad \text{for all } p > 0 \text{ and all } q \geq 0.$$

### Computational Examples in *Macaulay2*.

```
Macaulay2, version 1.26.06
Type "help" to see useful commands
i1 : needsPackage "NormalToricVarieties";
i2 : kk = ZZ/101;
i3 : PP4 = toricProjectiveSpace(4, CoefficientRing => kk);
i4 : S = ring PP4;
i5 : F = sheaf(PP4, S^{1} ++ S^{2} ++ S^{3})
o5 = 00      1      1      1
      PP4      PP4      PP4
o5 : coherent sheaf on PP4, free of rank 3
i6 : G = 00_PP4(7)
o6 = 00      1
      PP4      (7)
o6 : coherent sheaf on PP4, free of rank 1
i7 : G ** F
o7 = 00      1      1      1
      PP4      PP4      PP4
o7 : coherent sheaf on PP4, free of rank 3
i8 : G ++ F
o8 = 00      1      1      1      1
      PP4      PP4      PP4      PP4
o8 : coherent sheaf on PP4, free of rank 4
```

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i9 : module (G++F)
o9 = S4
o9 : S-module, free, degrees {-7, -1, -2, -3}
i10 : Omega = cotangentSheaf PP4
o10 = cokernel {2} | x_2 0 0 x_3 0 0 0 0 0 x_4 |
                {2} | x_0 x_3 0 0 0 0 x_4 0 0 0 |
                {2} | -x_1 0 x_3 0 0 0 0 0 x_4 0 |
                {2} | 0 x_1 x_0 0 x_4 0 0 0 0 0 |
                {2} | 0 -x_2 0 x_0 0 x_4 0 0 0 0 |
                {2} | 0 0 -x_2 -x_1 0 0 0 x_4 0 0 |
                {2} | 0 0 0 0 x_2 x_1 0 x_0 0 0 |
                {2} | 0 0 0 0 -x_3 0 x_1 0 x_0 0 |
                {2} | 0 0 0 0 0 -x_3 -x_2 0 0 x_0 |
                {2} | 0 0 0 0 0 0 -x_3 0 -x_3 -x_2 -x_1 |
o10 : coherent sheaf on PP4, quotient of  $\mathcal{O}_{PP4}^{10}(-2)$ 
i11 : K = ker ((transpose matrix degrees S) * (diagonalMatrix gens S))
o11 = image {1} | -x_1 0 -x_2 0 0 -x_3 0 0 0 -x_4 |
              {1} | x_0 -x_2 0 0 -x_3 0 0 0 -x_4 0 |
              {1} | 0 x_1 x_0 -x_3 0 0 0 -x_4 0 0 |
              {1} | 0 0 0 x_2 x_1 x_0 -x_4 0 0 0 |
              {1} | 0 0 0 0 0 0 x_3 x_2 x_1 x_0 |
o11 : S-module, submodule of S5
i12 : prune K == module Omega
o12 = true
i13 : HH0(PP4,  $\mathcal{O}_{PP4}(1)$ )
o13 = kk5
o13 : kk-module, free
i14 : HH4(PP4,  $\mathcal{O}_{PP4}(-5)$ )
o14 = kk1
o14 : kk-module, free
i15 : matrix table (reverse toList (0..4), toList (-10..9),
                   (i,j) -> rank HHi(PP4,  $\mathcal{O}_{PP4}(j-i)$ ))
o15 = | 715 495 330 210 126 70 35 15 5 1 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 1 5 15 35 70 126 210 330 495 715 |
o15 : Matrix ZZ5 <-- ZZ20
i16 : F = S1/random(5,S);
o16 : S-module, quotient of S1
i17 : for i from 0 to 4 list HHi(PP4, F~)
o17 = {kk1, 0, 0, kk1, 0}
o17 : List
i18 : X = hirzebruchSurface 2;
i19 : picardGroup X
o19 = ZZ2
o19 : ZZ-module, free

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i20 : OO(X_1)
o20 = OO1X (-2, 1)
o20 : coherent sheaf on X, free of rank 1
i21 : matrix table (reverse toList (-7..7), toList (-7..7),
    (i,j) -> rank HH0(X, OO_X(j,i)))
o21 = | 20 25 30 36 42 49 56 64 72 80 88 96 104 112 120 |
      | 12 16 20 25 30 36 42 49 56 63 70 77 84 91 98 |
      | 6 9 12 16 20 25 30 36 42 48 54 60 66 72 78 |
      | 2 4 6 9 12 16 20 25 30 35 40 45 50 55 60 |
      | 0 1 2 4 6 9 12 16 20 24 28 32 36 40 44 |
      | 0 0 0 1 2 4 6 9 12 15 18 21 24 27 30 |
      | 0 0 0 0 1 2 4 6 8 10 12 14 16 18 |
      | 0 0 0 0 0 0 1 2 3 4 5 6 7 8 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
o21 : Matrix ZZ15 <-- ZZ15
i22 : matrix table (reverse toList (-7..7), toList (-7..7),
    (i,j) -> rank HH1(X, OO_X(j,i)))
o22 = | 12 9 6 4 2 1 0 0 0 0 0 0 0 0 0 |
      | 12 9 6 4 2 1 0 0 0 0 0 0 0 0 0 |
      | 12 9 6 4 2 1 0 0 0 0 0 0 0 0 0 |
      | 12 9 6 4 2 1 0 0 0 0 0 0 0 0 0 |
      | 12 9 6 4 2 1 0 0 0 0 0 0 0 0 0 |
      | 12 9 6 4 2 1 0 0 0 0 0 0 0 0 0 |
      | 10 8 6 4 2 1 0 0 0 0 0 0 0 0 0 |
      | 6 5 4 3 2 1 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 |
      | 0 0 0 0 0 0 0 0 0 0 1 2 4 6 8 10 |
      | 0 0 0 0 0 0 0 0 0 0 1 2 4 6 9 12 |
      | 0 0 0 0 0 0 0 0 0 0 1 2 4 6 9 12 |
      | 0 0 0 0 0 0 0 0 0 0 1 2 4 6 9 12 |
      | 0 0 0 0 0 0 0 0 0 0 1 2 4 6 9 12 |
o22 : Matrix ZZ15 <-- ZZ15
i23 : matrix table (reverse toList (-7..7), toList (-7..7),
    (i,j) -> rank HH2(X, OO_X(j,i)))
o23 = | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
      | 8 7 6 5 4 3 2 1 0 0 0 0 0 0 0 |
      | 18 16 14 12 10 8 6 4 2 1 0 0 0 0 0 |
      | 30 27 24 21 18 15 12 9 6 4 2 1 0 0 0 |
      | 44 40 36 32 28 24 20 16 12 9 6 4 2 1 0 |
      | 60 55 50 45 40 35 30 25 20 16 12 9 6 4 2 |
      | 78 72 66 60 54 48 42 36 30 25 20 16 12 9 6 |
o23 : Matrix ZZ15 <-- ZZ15

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i24 : Omega = cotangentSheaf X
o24 = cokernel {2, 0} | 2x_1x_3 |
               {-1, 2} | x_0   |
               {-1, 2} | -x_2  |

o24 : coherent sheaf on X, quotient of  $00_X^1(-2, 0) ++ 00_X^2(1, -2)$ 

i25 : HH^1(X, Omega(2,1))
o25 = 0
o25 : QQ-module

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