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Workshop on

"Rigidity and Flexibility of Geometric Structures"

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organized by

Georg Nawratil (TU Vienna), Anthony Nixon (Lancaster U), Josef Schicho (JKU Linz), Brigitte Servatius (Worcester Polytechnic Inst.)

Abstracts

Ciprian Borcea (Rider)

Ellipsoids and auxetics

Abstract:

Auxetic behavior refers to structural deformations where stretching in some direction involves lateral widening. This type of behavior has been observed in phase transitions of certain crystals and is of interest for rational design of metamaterials. Periodic framework structures allow a strictly geometric approach for understanding auxetic deformations. We highlight the essential role played by ellipsoids. This is joint work with Ileana Streinu.

Hans-Christian Graf von Bothmer (Hamburg)

Hexapods with a small linear span

Abstract:

We classify the possible Euler-curves of mobile n-pods under suitable genericity conditions. In the language of bond-theory we look at those n-pods whose only bonds are inversion bonds. The main new tool is an embedding of the group of direct isometries of $\mathbb{R}^3$ into weighted projective space $\mathbb{P}(1^4, 2^7)$. This is joint work with Matteo Gallet and Josef Schicho.

James Cruickshank (Galway)

Periodic packings of curves

Abstract:

We consider the problem of embedding certain classes of curves in the plane so that the relative interiors are pairwise disjoint. In particular we want to understand the contact graphs of such packings. For example, it is known that plane Laman graphs are precisely those that are representable by packings of line segments in the plane. I will discuss some variations of this result involving different types of curves and also involving some periodicity.

Sean Dewar (Lancaster)

Rigidity in the normed plane

Abstract:

It is a known result of Laman that a finite graph will have an infinitesimally rigid placement in the Euclidean plane if and only if it contains a (2,3)-tight subgraph. Recent research into rigidity in normed spaces has proven that a graph will have an infinitesimally rigid placement in the l_p and polyhedral normed planes if and only if it contains a (2,2)-tight subgraph; the change from 3 to 2 corresponding with the dimension of the isometry group for the space. In my talk I will discuss rigidity in normed spaces and give a classification for rigidity in all two dimensional normed spaces; the proof of this result, which will be covered briefly, utilizes infinite frameworks and involves results from convex geometry.

Matteo Gallet (JKU Linz)

Tropical geometry for minimally rigid graphs

Abstract:

I will report about a joint work with Capco, Grasegger, Koutschan, Lubbes, and Schicho concerning the computation of the number of (complex) realizations of a minimally rigid graph for a general choice of edge lengths. Using techniques from tropical geometry, we provide a recursive formula for this number. In the talk, after introducing the main result, I will mainly focus on the translation of the initial question into a problem suitable for a tropical analysis.

Zeyuan He (Cambridge)

Rigid-foldable quadrilateral creased papers

Abstract:

A quadrilateral creased paper is the union of an orientable 2-manifold and a quadrilateral mesh embedded on this 2-manifold, which is not necessarily developable. The “Rigid-foldability” we discuss here corresponds to flexibility in rigidity theory, where each quadrilateral is considered as a rigid panel. Based on a nearly-complete classification of rigid-foldable Kokotsakis quadrilaterals from Ivan Izmestiev, here we will present some large rigid-foldable quadrilateral creased papers with the following additional requirements: 1) For at least one rigid folding motion no folding angle remains constant. 2) Not solving complex equations. 3) The sector angles can be

solved quadrilateral by quadrilateral 4) The quadrilateral creased paper can be extended in both longitudinal and transverse directions infinitely. All these quadrilateral creased papers have one degree of freedom in each branch of their rigid folding motion.

Grigory Ivanov (EPFL)

Orthodiagonal Kokotsakis polyhedra

Abstract:

The full classification of all possible configurations of flexible Kokotsakis polyhedron has been given by Ivan Izmestiev recently. However, it is still unknown whether there exist polyhedra of several types proposed by Ivan or not. In this talk, we will briefly discuss geometric ideas behind an associated spherical linkage and explain how to construct a new type of flexible Kokotsakis polyhedron which we call orthodiagonal antiinvolutive type. The talk is based on a joint work in progress with Ivan Izmestiev, Georg Nawratil, and Hellmuth Stachel.

Ivan Izmestiev (Fribourg)

Flexible Kokotsakis polyhedra and elliptic functions

Abstract:

A Kokotsakis polyhedron is a plate and hinge structure made of one quadrilateral in the middle, four quadrilaterals on the sides and four triangles at the corners. It is rigid in general, but flexible examples were known since 1930's. Flexibility of a Kokotsakis polyhedron can be reformulated in terms of the flexibility of an associated spherical linkage as recognized by Stachel and Nawratil. The basic building blocks of this spherical linkage are spherical quadrilaterals. A recently obtained classification of flexible Kokotsakis polyhedra uses the parametrization of the configuration space of a quadrilateral linkage by elliptic functions. In this talk we describe the main ideas of the classification.

Bill Jackson (Queen Mary)

Rigidity of graphs and frameworks

Abstract:

The first reference to the rigidity of frameworks in the mathematical literature occurs in a problem posed by Euler in 1776. Consider a polyhedron P in 3-space. We view P as a 'panel-and-hinge framework' in which the faces are 2-dimensional panels and the edges are 1-dimensional hinges. The panels are free to move continuously in 3-space, subject to the constraints that the shapes of the panels and the adjacencies between them are preserved, and that the relative motion between pairs of adjacent panels is a rotation about their common hinge. The polyhedron P is rigid if every such motion results in a polyhedron which is congruent to P. Euler's conjecture was that every polyhedron is rigid.

The conjecture was verified for the case when P is convex by Cauchy in 1813. Gluck showed in 1975 that it is true when P is 'generic' i.e. there are no algebraic dependencies between the coordinates of the vertices of P. Connelly finally disproved the conjecture in 1982 by constructing a polyhedron which is not rigid.

I will describe results and open problems concerning the rigidity of various other types of frameworks. I will be mostly concerned with the generic case for which the problem of characterizing rigidity usually reduces to pure graph theory.

Oleg Karpenkov (Liverpool)

Configuration spaces of tensegrities on graphs and CW-complexes

Abstract:

In this talk we consider a natural stratification of the configuration space of tensegrities. We discuss several results about the structure of the configuration space of two-dimensional tensegrities with a small number of points. We conclude the talk with a nice generalization of tensegrities by R.M. Erdahl, K.A. Rybnikov and S.S. Ryshkov to the case of CW-complexes.

Jan Legersky (JKU Linz)

Graphs with flexible labelings allowing injective realizations

Abstract:

Given a graph, we ask whether it is possible to find a flexible labeling, namely, edge lengths such that there are infinitely many compatible realizations, modulo rigid motions. Even if a graph is generically rigid, the non-generic edge lengths may still be flexible.

In case realizations are not required to be injective, the graphs with a flexible labeling are characterized by the existence of a special edge coloring that is called NAC-coloring. In this talk we focus on graphs and labelings allowing infinitely many injective realizations. We provide a necessary condition on the movability of a graph that is also based on the NAC-colorings of the graph. We show that the condition is also sufficient for graphs with at most 8 vertices, which is not true in general.

This is joint work with Georg Grasegger and Josef Schicho.

Leo Liberti (CNRS & LIX, Ecole Polytechnique)

Distance Geometry in data science

Abstract:

Many problems in data science are addressed by mapping entities of various kind to vectors in a Euclidean space of some dimension. Most of these methods (e.g. Multidimensional Scaling, Principal Component Analysis, K-means clustering, random projections) are based on the proximity of pairs of vectors. In order for the results of these methods to make sense when mapped back, the proximity of entities in the original problem must be well approximated in the Euclidean space setting. If proximity were known for each pair of original entities, this mapping would be a good example of isometric embedding. Usually, however, this is not the case, as data are partial, wrong and noisy. I shall survey some of the methods above from the point of view of Distance Geometry.

Niels Lubbes (RICAM)

On an invariant for linkages

Abstract:

We start by proposing a coordinate free definition for linkages in Euclidean 3-space. Morally, such a linkage is graph whose vertices are copies of Euclidean 3-spaces (=links) and the edges (=joints) are labeled with a subtorsor of isometries between the vertices. A configuration is a choice of elements (=displacements) in the subtorsors that makes the graph commute along all paths. For a given configuration, we can embed each subtorsor as a subgroup of SE3. A projective closure of SE3 is the Study quadric and is useful due to its underlying algebraic structure defined by dual quaternions. Like in bond theory (see [Li-Schicho-Schröcker,2018]) we obtain special configurations (=bonds) if, after projective closure, a limit of displacements lies in the boundary of the Study quadric. We will introduce restrictions on configurations with such "boundary displacements". As an application we classify certain linkages that have both rotational and prismatic joints.

Andrea Micheletti (Rome)

Some questions on molecular structures and rigid-panel structures

Abstract:

This talk is divided into two parts. First, a molecular structure is considered as a set of points in space whose positions are restricted by a set of distance constraints, defined on node pairs, and a set of angle constraints, defined on node triplets. Moreover, constraints on dihedral angles and/or solid angles, defined on node quadruplets, can also be included. After discussing the equivalence of such structures with bar frameworks, the problem of giving a suitable definition of stress operator for molecular structures is presented. Next, a rigid-panel structure is considered as a set of rigid panels connected to each other either by door hinges or sliding hinges, a sliding hinge differs from a door hinge in that it allows the relative translation along the hinge axis. The problem of finding an isostatic assignment of door hinges and sliding hinges is presented for the case of a Miura-ori-like assembly.

Andreas Mueller (JKU Linz)

Higher-order rigidity revisited from screw theory

Abstract:

Shaky structures are finitely immobile but possess differential mobility. The order of this mobility is called the order of shakiness. That is, a shaky structure may exhibit higher-order differential motions. A similar phenomenon is observed for mechanisms with finite mobility, where the differential DOF exceeds the finite DOF in generic, i.e. regular, configurations. These so-called underconstrained mechanisms can also be characterized by the degree of shakiness. A shaky system exhibits higher-order differential motions that do not correspond to finite motions. A mechanism, i.e. a mobile system, possesses differential motions of arbitrary order. It seems thus to be intuitively clear that a system exhibiting higher-order differential motions of finite order only is finitely rigid, and it was believed for a long time that this is a sufficient rigidity criterion. Connelly and Servatius presented a mobile linkage that contradicts this assumption. This raised two fundamental questions: 1. What is the proper definition

of higher-order rigidity? and 2. How can one check whether a shaky system is eventually mobile? These issues are discussed in this presentation. The problem is approached with a formulation in terms of joint screw coordinates making use of methods from the theory of transformation groups. This geometric Lie group framework allows for a systematic treatment of general linkages. A central concept that will be used is that of a tangent cone to a (analytic or algebraic) variety. Several definitions of tangent cone are recalled, and it is concluded that the notion of a tangent semi-cone is most adequate to the problem at hand. There is, however, no constructive definition that allows determining the tangent semi-cone. Therefore, the concept of kinematic tangent cone is recalled, which is the set of tangents to smooth curves in the configuration space variety. The latter is commonly used in mechanism theory to investigate possible finite motions in a given configuration. A constructive formulation for the kinematic tangent cone is presented and applied to several examples. It fails describe finite motions for the example proposed by Connelly and Servatius, however, due to the existence of a cusp singularity in the configuration space. As a method to investigate higher-order rigidity, a local approximation of the configuration space variety is proposed. All involved relations are given algebraically in terms of instantaneous screws. The presented formulations allow for a definition of higher-order rigidity, respectively shakiness.

Karla Mundilova (TU Vienna)

“Rigid” curved crease origami

Abstract:

Manual curved crease origami is obtained by folding paper along curves and offers a variety of possible shapes. We give an introduction to mathematical paper folding, i.e. the constraints for the common (crease) curve of two developable surface patches. Moreover, we motivate the concept of rigidcurved crease folding by considering only folds with fixed developed rulings. Finally, we give illustrations that are based on David Huffman’s designs of folds of constant angle between cylinders and cones.

John Owen (Siemens)

Symmetry, redundancy and quadratic solvability in point-line-plane frameworks

Abstract:

I will discuss some point-line-plane frameworks for which point group symmetry can predict finite mechanisms, locally redundant constraints and consequent quadratic solvability. I will also describe a symmetry (extrusion symmetry) which is not generated by an isometry but which leads to the identification of additional finite mechanisms.

Gaiana Panina (St. Petersburg)

Oriented area as a Morse function

Abstract:

It makes sense to equip the configuration space of a linkage by geometrically meaningful functions; I will speak of the oriented area. We will discuss planar polygonal linkages, polygonal linkages in 3D, linkages with sliding endpoints, and partial two-trees.

Joint research with J. Gordon, G. Khimshiashvili, D. Siersma, Ya. Teplitskaya, and A. Zhukova.

Stephen Power (Lancaster)

Counting 3-periodic nets and their isotopy classes

Abstract:

Entangled crystals are intriguing material structures that have been investigated intensively by chemists since the 1990s. On the other hand there have been few investigations of entangled periodic structures with regard to deformations avoiding edge collisions. We formally define embedded 3-periodic nets (with nonintersecting line segment bonds) together with their adjacency depth, isomorphism type and periodic isotopy type. (Most physically observed nets have adjacency depth 1.) We prove that, as expected, there are finitely many types (isomorphic or isotopic) of nets having a fixed quotient graph and fixed depth, and so counting and classification problems abound. We obtain some results in this direction for depth 1 connected nets with small quotient graph and for some multicomponent nets. In the case of isotopy type we introduce the category of linear graph knots (and links) on the flat 3-torus to provide invariants for periodic isotopy classification. This is ongoing joint work with Igor Baburin and Davide Proserpio.

Daniel Scharler (Innsbruck)

Rational motions with trajectories of low degree

Abstract:

In the dual quaternion model of Euclidean kinematics, a rational motion corresponds to a polynomial $p \in \mathbb{DH}[t]$ in one indeterminate t and dual quaternion coefficients such that $p\overline{p}$ is real. In general, the degree of the trajectories is $2n$ if the degree of p is n. Polynomials p where the degree of the trajectories is less than $2n$ do exist. If so, we say that a degree reduction occurs. A necessary condition for degree reduction is the existence of real zeros of the primal part of p. In general (but not always!) each such zero decreases the trajectory degree by one. An example of a motion with an exceptionally high degree reduction is the Darboux motion (see [1]) where $\deg p = 3$, the primal part of p has two real zeros but the degree of the trajectories is only 2. The Darboux motion also exhibits the rather strange property that the trajectory degree of the inverse motion, given by the conjugate polynomial $\overline{p}$, has trajectory degree 4.

An explanation for these phenomena can be found in the geometry of the so called kinematic image space. The kinematic image space is a special quadric (Study quadric) in the seven dimensional real projective space. In [2], the authors observed that there is a quadric contained in the an exceptional generator of the Study quadric which is key in the above mentioned geometry. Exceptional degree reduction can be explained in terms of one family of rulings on this quadric – a geometric entity which is *not* invariant with respect to conjugation. Our considerations also yield a method to systematically construct rational motions with exceptional degree reduction. So far, the Darboux motion and its planar version were the only examples known to us.

References.

[1] Bottema, O. and Roth, B.: *Theoretical kinematics*. NORTH-HOLLAND PUBL. CO., N. Y., 1979, 558, 1979

[2] Rad, T. and Scharler, D. and Schröcker, H.P.: *The kinematic image of RR, PR, and RP dyads*. Robotica, 36(10), 1477-1492.

Jean-Marc Schlenker (Luxembourg)

Polyhedra inscribed in quadrics

Abstract:

Steiner asked in 1832 what the combinatorial types of convex polyhedra with their vertices on a quadric in 3-dimensional projective space are. We will describe two recent progresses on this question. One result (joint with Jeff Danciger and Sara Maloni) describes the combinatorial types of polyhedra inscribed in a one-sheeted hyperboloid, while the other (joint with Hao Chen) deals with polyhedra having their vertices on a sphere in projective space which are not contained in the ball. The first result is based on anti-de Sitter geometry, while the second uses a natural extension of the hyperbolic space by the de Sitter space. In both cases, rigidity questions play a key role.

Bernd Schulze (Lancaster)

Frameworks with coordinated edge motions

Abstract:

In this talk we will discuss the rigidity of bar-joint frameworks in Euclidean d-space in which specified classes of edges are allowed to change length in a coordinated fashion, subject to a linear constraint for each class. This is a tensegrity-like setup that is amenable to combinatorial “Maxwell-Laman-type” analysis.

We describe the rigidity of generic d-dimensional coordinated frameworks in terms of the rigidity properties of their underlying bar-joint frameworks. Moreover, we present Laman-type characterisations for generic coordinated rigidity in the plane, for frameworks with one and two coordination classes.

This is joint work with Hattie Serocold and Louis Theran.

Brigitte Servatius (WPI)

Delta matroids and rigidity matroids for graphs and maps on surfaces

Abstract:

Bouchet(1989) showed that cellular maps on a compact surface give rise to an associated delta-matroid using topology. We give a purely combinatorial description of these delta-matroids and propose a definition for a rigidity matroid for graphs on surfaces derived from their delta-matroid.

This is joint work with Remi Cocou Avohou and Herman Servatius.

Meera Sitharam (Florida)

Realization certificates for solving geometric constraint systems (and a polynomial time algorithm for planar, minimally rigid, bar-joint systems)

Abstract:

Consider the computational problem of finding coordinates of joints given only the labeled graph and bar lengths (formally, the linkage) of a bar-joint framework. The problem involves finding a real solution(s) to an input geometric constraint (polynomial) system (GCS) if one exists. Even deciding the existence of a solution for Henneberg-I linkages is hard for NP. All known deterministic algorithms take double exponential time and there is no known difference in deterministic complexity between finding one solution and finding all solutions (of which there could be exponentially many). The existence problem, however, is in the class NP over the realsïntroduced by Blum, Shub and Smale [BSS]. The latter implies that once a certificateör ïndexfor a solution is provided, the indexed solution can be checked/found in polynomial time. However, the BSS model of computation permits the certificate to be the solution itself which is not particularly illuminating. Yet, this inspires the quest for meaningful certificates for solutions, which, when input, permit the solution to be found/checked in polynomial time in a standard finite precision setting. Meaningful certificates clearly exist for so-called ruler-and-compass constructible (formally Quadratically Radically Solvable (QRS) systems), of which Henneberg-I linkages are a subclass. This is intuitively because these systems can be recursively decomposed into a series of small, polynomial-time solvable, rigid subsystems each of whose solutions has a natural order-type that can then be concatenated into a solution certificate for the entire system. Hence the question of meaningful certificates is related to the existence of a so-called Decomposition-Recombination (DR) plan of small size as well as the complexity of finding it. While we have shown the the (decision version of the) optimal DR problem is NP-hard even for 2D bar-joint linkages, it can be solved in polynomial time in this and other cases when the underlying graph is minimally rigid (they key is that the rigidity matroid is a sparsity matroid). The talk will then focus on a more recent result. For planar, minimally rigid graphs, permitting modification to a single boundary facet, we have shown the existence of a small flex DR-plan (a generalization of DR-plan) that permits meaningful certificates, and a polynomial time algorithm to check and find the solution or realization corresponding to the certificate.

Hellmuth Stachel (TU Vienna)

Between rigidity and flexibility

Abstract:

The rigidity of structures is of importance for various areas of science such as chemistry, physics, biology, engineering, and material science. In this lecture the emphasis is on basic science rather than on applications. We focus on the geometry of various flexible or rigid structures such as polyhedra or frameworks, and we do not care about mechanics of materials or clearances of joints.

We recall that a polyhedron is said to be flexible if its spatial form can be changed analytically with respect to a parameter, due to changes of its dihedral angles while every face remains congruent to itself during the flex. Conversely, it is said to be rigid if it admits only trivial deformations like translations and rotations. Besides, we can define an infinitesimal flexibility of structures, which lies somehow between flexibility and rigidity. Physical models of infinitesimally flexible structures show a slight mobility.

We present several examples of structures where flexibility or infinitesimal flexibility is caused by particular dimensions, while in the generic case a structure of this type is rigid. Sometimes, a global or local symmetry causes the flexibility, but there are also overconstrained frameworks where the underlying algebraic conditions for flexibility have no relation to symmetry at all. In a few cases physical models will be provided to illustrate the presented results. These models are taken from the collection of kinematic models at the Institute of Discrete Mathematics and Geometry, TU Vienna.

Louis Theran (St Andrews)

Unlabeled generalisations of global rigidity

Abstract:

A d-dimensional framework (G, p) is globally rigid if any other d-dimensional framework with the same edge lengths is related to it by a rigid motion. An important result of Gortler, Healy, and Thurston is that global rigidity is a generic property: for every graph G either (G, p) is globally rigid for every generic p or (G, p) is not globally rigid for every generic p. In particular, this implies that if G has n vertices, p is a generic configuration of n points in dimension d, and q is any other configuration of n points in dimension d so that the framework (G, p) has the same edge lengths as (G, q), then q is related to p by a rigid motion.

The unlabeled version of global rigidity, by analogy, asks when, for a generic configuration p of n points in dimension d and a graph G with n vertices, we know that if q is any other configuration of n points in dimension d and H is any other graph with n vertices, if (H, q) has the same edge length measurements as (G, p), then G and H are isomorphic and q is related to p by a rigid motion. It turns out that G being generically globally rigid is necessary and sufficient.

I'll also briefly describe a further generalisation to measurements of unknown paths and loops.

This is joint work with Steven Gortler and Dylan Thurston / Ioannis Gkioulekas, Steven Gortler, and Todd Zickler

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