The chain polynomials of noncrossing partition lattices are real-rooted

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Let L be a finite Partially Ordered Set (poset) and $c_k(L)$ be the number of k-element chains of L.
We consider the chain polynomial of L

$$p_L(x) = f(\Delta(L), x) = \sum_{k \geq 0} c_k(L) x^k$$

which is the f-polynomial of the order complex $\Delta(L)$ of L.

- $\Delta(L)$ is the set of all chains of L (it is a simplicial complex)

For some purposes we may focus on the corresponding h-polynomial of the poset.

$$h_L(x) = h(\Delta(L), x) = (1-x)^n p_L\left(\frac{x}{1-x}\right) =$$

$$= \sum_{k\geq 0} c_k(L) x^k (1-x)^{n-k}$$

where n is the largest size of a chain in L.

$$h_L(x) = h_0 + h_1x + \cdots + h_nx^n$$

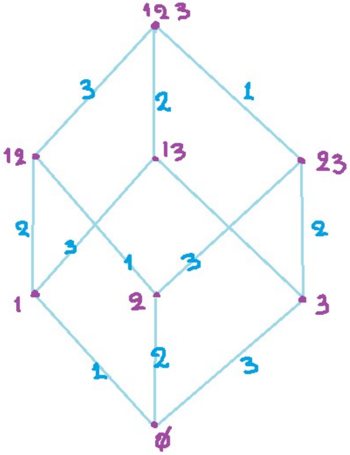
where the coefficients add to the number of n-chains of L.

- If L is Cohen-Macaulay the h-polynomial has nonnegative coefficients
- If L has an R-labeling the coefficients of $h_L(x)$ are given a combinatorial interpretation

- Let $\lambda : C(L) \to (\Lambda, \leq)$ be an edge labeling of the Hasse diagram of L.
- We say that λ is an R-labeling of L if in each closed interval $[x, y]$ of L there exists a unique increasing maximal chain.

If λ is an R-labeling of L we get that $h_L(x) = \sum_c x^{\mathrm{des}(w)}$
where c runs in the maximal chains of L

Example: Let L_n be the lattice of subsets of [n]
(Boolean algebra of rank n)



$$h_{L_n}(x) = A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)}$$

which is the classical n-th Eulerian Polynomial
that counts descents in the Symmetric Group

$$A_n(x) = \begin{cases} 1 & n = 1 \\ 1 + x & n = 2 \\ 1 + 4x + x^2 & n = 3 \\ 1 + 11x + 11x^2 + x^3 & n = 4 \\ 1 + 26x + 66x^2 + 26x^3 + x^4 & n = 5 \end{cases}$$

The Eulerian polynomial $A_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ is

- unimodal

$$a_0 \le a_1 \le \cdots \le a_k \ge a_{k+1} \ge \cdots \ge a_{n-1}$$

- palindromic

$$a_k = a_{n-1-k}$$

- log-concave

$$a_k^2 \ge a_{k-1}a_{k+1}$$

- gamma-positive

$$A_n(x) = \sum_{k \geq 0} \gamma_k x^k (1+x)^{n-1-2k}$$

$\gamma_k \geq 0$ for every k

- real-rooted

every root of $A_n(x)$ is real

These are properties we often encounter in algebraic and geometric combinatorics.
We focus on the property of real-rootedness
which has strong implications for a polynomial.
Let $f(x) = f_0 + f_1 x + \cdots + f_n x^n$ be a polynomial with nonnegative integer coefficients.
If it is real-rooted, then:

- it is unimodal
- it is log-concave
- if it is also palindromic, it is gamma-positive

Question: For which finite posets does the chain polynomial (equivalently the h-polynomial) have only real roots?

- Is it true for all geometric lattices?
- Is it true for the face lattices of all convex polytopes?
- Is it true for all double Cohen-Macaulay posets?

We will prove the following:

Theorem

For every finite Coxeter Group W, the h-polynomial of the noncrossing partition lattice $NC(W)$ is real-rooted.

Let W be a finite Coxeter group and T the set of all reflections.

Definition (absolute order)

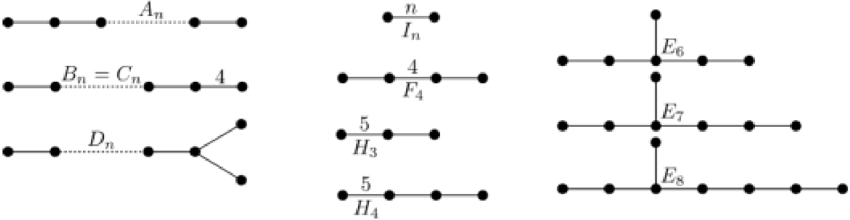
We define a partial order $\leq$ on W, called the absolute order, by letting $a \leq_T b$ if $l_T(b) = l_T(a) + l_T(a^{-1}b)$

the subword property:
$a \leq_T b \Leftrightarrow$ *a occurs as an arbitrary subword of some reduced T-word for b.*

We set $NC(W) = [e, \gamma]$ where e is the identity and γ is a Coxeter element of W.

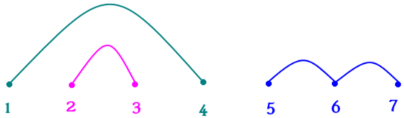
It is called the The Noncrossing Partition Lattice.

We will now describe the h-polynomials of $NC(W)$ for every finite, irreducible Coxeter group W.



$W = A_{n-1}$ (symmetric group of degree n),$\gamma = (12 \cdots n)$

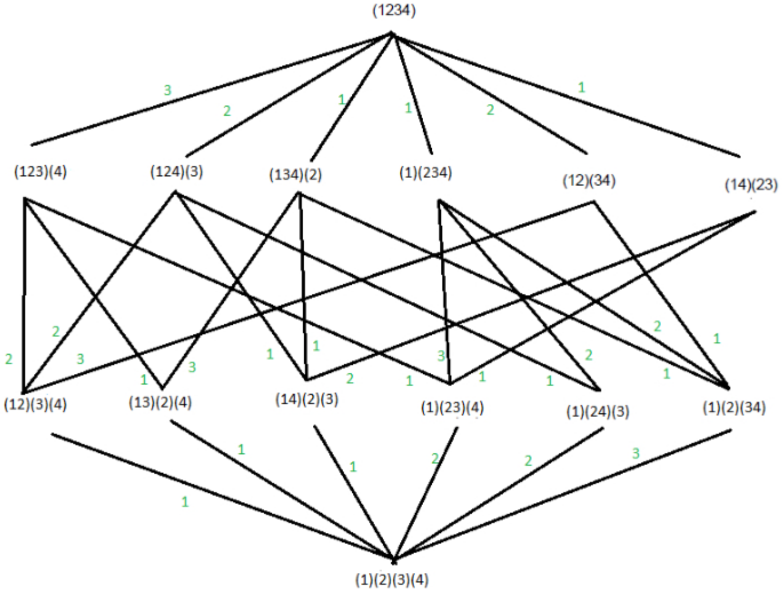
- The elements of $NC(A_{n-1})$ can be viewed as the set partitions of $[n]$ that "do not cross"



$\pi = \{\{1,4\},\{2,3\},\{5,6,7\}\}$
$w = (1,4)(2,3)(5,6,7)$

- The cardinality of $NC(A_{n-1})$ is given by the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

Hasse diagram of $NC(A_3)$



- $NC(A_{n-1})$ has n^{n-2} maximal chains
- $\mathrm{h}_{NC(A_{n-1})}(x)$ counts chains on the set of parking functions of length n-1 (P_{n-1})

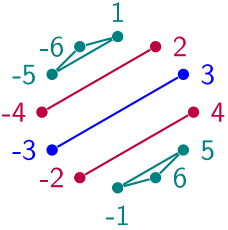
$$\mathrm{h}_{NC(A_{n-1})}(x) = \sum_{w \in P_{n-1}} x^{\mathrm{des}(w)} = \frac{1}{n} \sum_{w \in [n]^{n-1}} x^{\mathrm{des}(w)}$$

P_3

$123, 13|2, 2|13, 23|1, 3|12, 3|2|1$ $1|12, 12|1, 2|1|1,$
$1|13, 13|1, 3|1|1, 12|2, 2|12, 2|2|1, 1|1|1$

$W = B_n$ (the hyperoctahedral group of degree n), $\gamma = [12 \cdots n]$
Set partitions of $\{1, 2, \cdots, n, -1, -2, \cdots, -n\}$ such that:

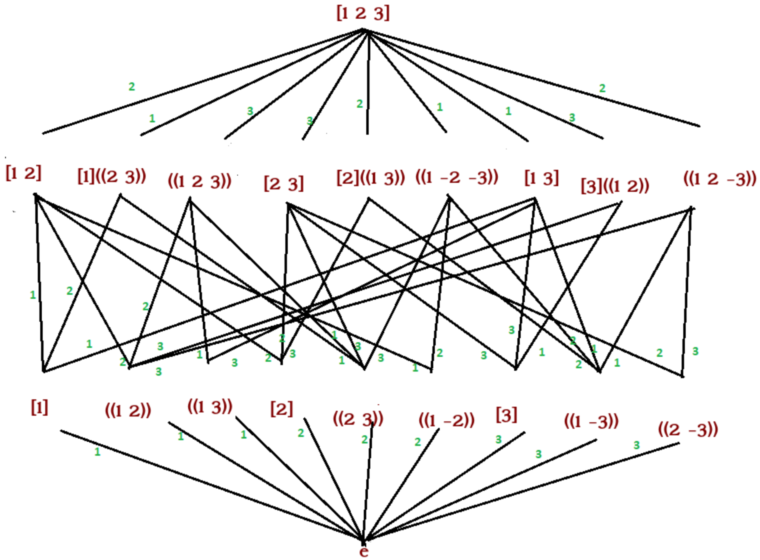
- $B \in \pi \Rightarrow -B \in \pi$
- there is at most one zero part
- they don't cross



$\pi = \{\{1, -6, 5\}, \{-1, 6, 5\}, \{2, -4\}, \{-2, 4\}, \{3, -3\}\}$
$w = ((5, 6, -1))((2, -4))[3]$

$$\#NC(B_n) = \binom{2n}{n}$$

Hasse diagram of $NC(B_3)$



- $NC(B_n)$ has n^n maximal chains

$$\mathrm{h}_{NC(B_n)}(x) = \sum_{w \in [n]^n} x^{\mathrm{des}(w)}$$

$$n = 3$$

$$123, 13|2, 2|13, 23|1, 3|12, 3|2|1$$

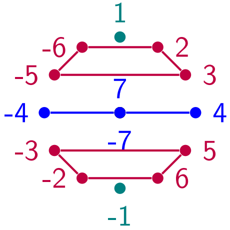
$$1|12, 12|1, 2|1|1, 1|13, 13|1, 3|1|1, 2|23, 23|2, 3|2|2$$

$$1|1|1, 2|2|2, 3|3|3$$

$$12|2, 2|12, 2|2|1, 23|3, 3|23, 3|3|2, 13|3, 3|13, 3|3|1$$

$W = D_n, \gamma = [12 \cdots n-1][n]$
Set partitions of $\{1, 2, \cdots, n, -1, -2, \cdots, -n\}$ such that:

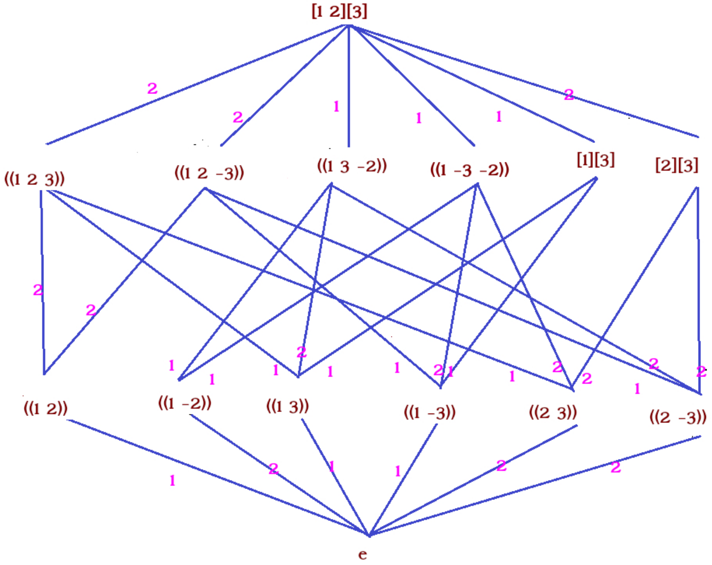
- $B \in \pi \Rightarrow -B \in \pi$
- the zero block, if there, contains more than two elements
- they don't cross (on their interior)



$\pi = \{\{2, 3, -5, -6\}, \{2, 3, 5, 6\}, \{1\}, \{-1\}, \{4, -4, 7, -7\}\}$
$w = ((2, 3, -5, -6))((1))[4][7]$

$$\#NC(D_n) = \binom{2n}{n} - \binom{2n-2}{n-1}$$

Hasse diagram of $NC(D_3)$



- $NC(D_n)$ has $2(n-1)^n$ maximal chains

$$\mathrm{h}_{NC(D_n)}(x) = \sum_{w \in D_n} x^{\mathrm{des}(w)}$$

$$D_n = \{(\pm w_1, w_2, \cdots, w_n) : w_1, w_2, \cdots, w_n \in [n-1]\}$$

and k is considered a descent if $|w_k| > |w_{k+1}|$ or $w_k = w_{k+1} > 0$

$$n = 3$$

1|12	12|1	2|1|1	12|2	2|12	2|2|1	1|1|1	2|2|2
−112	− 12|1	− 2|1|1	− 12|2	− 2|12	− 22|1	− 11|1	− 22|2

We define a total ordering $\leq$ of the set of reflections T, "compatible" with the coxeter element γ.

Theorem (Athanasiadis, Brady, Watt)

Let λ be the natural edge labeling of $NC_w(\gamma) = NC(W)$.

$$\lambda : C(NC(W)) \to (T, \leq)$$

$$\lambda(u, v) = u^{-1}v$$

Then λ is an EL-labeling of $NC(W)$.

$$h_{NC(W)}(x) = \sum_{\tau \in M(W)} x^{\mathrm{des}(\tau)}$$

$M(W)$=set of minimal factorizations of γ in reflections

In D_5:

$$((1,2)) < ((1,3)) < ((1,4)) < ((1,5)) < ((1,-5)) <$$

$$((2,3)) < ((2,4)) < ((1,-2)) < ((2,5)) < ((2,-5)) <$$

$$((3,4)) < ((1,-3)) < ((2,-3)) < ((3,5)) < ((3,-5)) <$$

$$((1,-4)) < ((2,-4)) < ((3,-4)) < ((4,5)) < ((4,-5))$$

We now define, in each case, a map $l : T \to \mathbb{Z}$, such that $t_1 \leq t_2 \Rightarrow l(t_1) \leq l(t_2)$, which induces:

- a 1-1 correspondence $M(A_{n-1}) \to P_{n-1}$, for type A
- a 1-1 correspondence $M(B_n) \to [n]^n$, for type B
- a 2-1 correspondence $M(D_n) \to [n-1]^n$ for type D

In types A and B, the correspondence preserves the descents, so the result follows.

In case D:
for every $w \in [n-1]^n$ there is a pair of chains c_1, c_2 with $l(c_1) = l(c_2) = w$

- for some: $\mathrm{des}(c_1) = \mathrm{des}(c_2) = \mathrm{des}(w) \longrightarrow f_1(x)$
- for the rest: $\mathrm{des}(c_1) = \mathrm{des}(c_2) + 1 = \mathrm{des}(w) \longrightarrow f_2(x)$

$$\sum_{w \in [n-1]^n} x^{\mathrm{des}(w)} = f_1(x) + f_2(x)$$

$$h_{NC(D_N)}(x) = 2f_1(x) + \left(1 + \frac{1}{x}\right) f_2(x)$$

$$f_2(x) = x \cdot h_{NC(B_{n-1})}(x) = x \sum_{w \in [n-1]^{n-1}} x^{\mathrm{des}(w)}$$

$$\mathrm{h}_{NC(W)}(x) = \begin{cases} 1 + (n-1)x & \text{if } W = I_n \\ 1 + 28x + 21x^2 & \text{if } W = H_3 \\ 1 + 275x + 842x^2 + 232x^3 & \text{if } W = H_4 \\ 1 + 100x + 265x^2 + 66x^3 & \text{if } W = F_4 \\ 1 + 826x + 10778x^2 + 21308x^3 + 8141x^4 + 418x^5 & \\ & \text{if } W = E_6 \\ 1 + 4152x + 110958x^2 + 446776x^3 + 412764x^4 & \\ +85800x^5 + 2431x^6 & \text{if } W = E_7 \\ 1 + 25071x + 1295238x^2 + 9523785x^3 + 17304775x^4 & \\ +8733249x^5 + 1069289x^6 + 17342x^7 & \text{if } W = E_8 \end{cases}$$

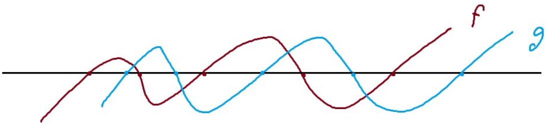
Lemma

For every finite irreducible Coxeter Group W the h-polynomial of the noncrossing partition lattice $NC(W)$ is real-rooted.

A,B,D: we use the method of interlacing polynomials.

Let $f(x), g(x)$ be real-rooted polynomials.

- f(x) interlaces g(x) ($f(x) \preceq g(x)$) :
  their roots are interpolating with g(x) having the largest root



- if $f(x) \preceq g(x)$ then f(x)+g(x) is real-rooted

Cases A and B:

$$h_{r,n}(x) = \sum_{w \in [r]^n} x^{\mathrm{des}(w)} \text{ is real-rooted for } n, r \geq 1$$

for $j \in [r]$ we set $h_{r,n,j}(x) = \sum_{w \in A_{r,n,j}} x^{\mathrm{des}(w)}$

where $A_{r,n,j}$ is the set of words $(w_1, w_2, \cdots, w_n) \in [r]^n$ with $w_n = j$

$$h_{r,n}(x) = \sum_{i=1}^{r} h_{r,n,i}(x) = \frac{1}{x} h_{r,n+1,1}(x)$$

$$h_{r,n+1,j}(x) = \sum_{i=1}^{j-1} h_{r,n,i}(x) + x \sum_{i=j}^{r} h_{r,n,i}(x)$$

by induction on n we prove that:

$(h_{r,n,r}(x), h_{r,n,r-1}(x), \cdots, h_{r,n,1}(x))$

is an interlacing sequence of polynomials

so $h_{n,r}(x) = \sum_{j=1}^{r} h_{n,r,j}(x)$ is real-rooted

Theorem

For every finite Coxeter Group W the h-polynomial of the noncrossing partition lattice $NC(W)$ is real-rooted.

$$NC(W) \cong NC(W_1) \times \cdots \times NC(W_l)$$

where $W_1, \cdots, W_l$ are the irreducible components of W

Lemma

Let P, Q be finite posets. If $h_P(x)$ and $h_Q(x)$ have nonnegative coefficients and only real-roots, then so does $h_{P\times Q}(x)$.

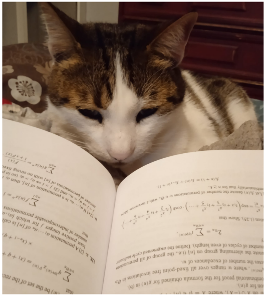
Theorem

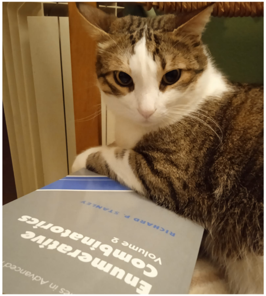
For every finite irreducible Coxeter Group W the h-polynomial of the noncrossing partition lattice $NC(W)$ has a nonnegative, real-rooted symmetric decomposition, with respect to $r_w - 1$, where r_W is the rank of W. In particular, it is unimodal, with a peak at position $\lfloor \frac{r_W}{2} \rfloor$.

$$h_{NC(D_6)} = 1 + 665x + 8330x^2 + 16010x^3 + 5950x^4 + 294x^5 =$$

$$= (x^5 + 372x^4 + 2752x^3 + 2752x^2 + 372x + 1) +$$

$$x \cdot (293x^4 + 5578x^3 + 13258x^2 + 5578x + 293)$$





Thank you for your attention!
Vielen dank für ihre aufmerksamkeit!
Σας ευχαριστώ για την προσοχή σας!