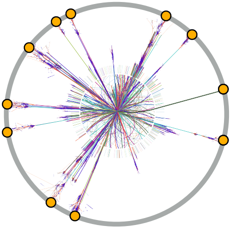
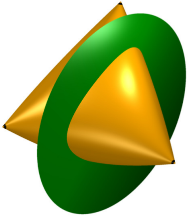
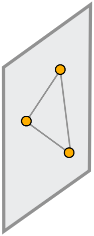
Kinematic Varieties II
Gram Matrices

Bernd Sturmfels
MPI Leipzig







Mini-Course at ESI Vienna, within
Amplitudes and Algebraic Geometry
February 17, 2026

Source and **Puzzle**

This lecture is based on the article

Y. El Maazouz, BSt, S. Sverrisdóttir:
Gram matrices for isotropic vectors
arXiv:2411.08624

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Find all matrices of rank 2:

$$\begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix} \qquad \mathcal{V}_{1,4,2}$$

```
R = QQ[a,b,c,d,e,f];
M = matrix{{0,a,b,c},{a,0,d,e},{b,d,0,f},{c,e,f,0}}
decompose minors(3,M)
```

A Tale of Six Vectors

Six vectors $P_1, W_1, P_2, W_2, P_3, W_3$ in $\mathbb{R}^r$ satisfy

$$P_i \cdot P_i \;=\; P_i \cdot W_i \;=\; W_i \cdot W_i \;=\; 0 \qquad \text{for } i = 1, 2, 3.$$

The dot is a quadratic form, given by a symmetric $r \times r$ matrix Q.

All vectors are isotropic.

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All vectors are isotropic.

In **physics**, one works in $1+3$ dim'l spacetime, where $r = 4$ and

$$Q \ = \ \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have three massless particle. Their momentum vectors P_i and their polarization vectors W_i lie on the **lightcone** in $\mathbb{R}^{1+3} = \mathbb{R}^r$.

Everyone prefers quantities that are invariant under $\mathrm{SO}(r, Q)$.

Gram Matrix

The Gram matrix records pairwise inner products.
It has zero 2×2 blocks along the main diagonal:

$$X = \begin{bmatrix} 0 & 0 & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{13} & x_{23} & 0 & 0 & x_{35} & x_{36} \\ x_{14} & x_{24} & 0 & 0 & x_{45} & x_{46} \\ x_{15} & x_{25} & x_{35} & x_{45} & 0 & 0 \\ x_{16} & x_{26} & x_{36} & x_{46} & 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 \\ W_1 \\ P_2 \\ W_2 \\ P_3 \\ W_3 \end{bmatrix} Q \begin{bmatrix} P_1 \\ W_1 \\ P_2 \\ W_2 \\ P_3 \\ W_3 \end{bmatrix}^T$$

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Q: What is the rank of this matrix?
A: The Gram matrix X has rank $\leq r$.

In **computer algebra**, we fix a polynomial ring in 12 variables x_{ij} and study the ideal generated by the $(r+1) \times (r+1)$ minors of X.

Q: What is its variety in $\mathbb{C}^{12}$?

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A: Ask `Macaulay2` or ask `OSCAR`

Ideals, Varieties and Algorithms

Using Gröbner bases, we explore six determinantal varieties.
They are nested in our space $\mathbb{C}^{12}$ of 6×6 matrices X:

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There are also four embedded primes.

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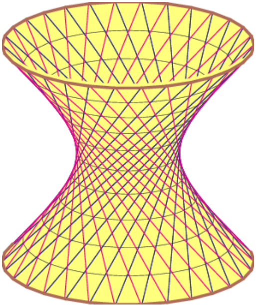
$r = 5$: The determinant of X is irreducible. $\dim = 11$
It generates a principal ideal that is prime.

Rulings on the Quadric Surface



$$\begin{bmatrix} 0 & 0 & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{13} & x_{23} & 0 & 0 & x_{35} & x_{36} \\ x_{14} & x_{24} & 0 & 0 & x_{45} & x_{46} \\ x_{15} & x_{25} & x_{35} & x_{45} & 0 & 0 \\ x_{16} & x_{26} & x_{36} & x_{46} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \end{bmatrix}$$

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Determinantal Varieties

Fix a symmetric $kn \times kn$ matrix $X = (x_{ij})$ with zero $k \times k$ blocks:

$$\begin{array}{rrcl} & x_{ij} &=& x_{ji} \quad \text{for } 1 \leq i \leq j \leq kn \\ \text{and} & x_{jk+l,jk+m} &=& 0 \quad \text{for } j = 0, 1, \ldots, n-1 \text{ and } 1 \leq l \leq m \leq k. \end{array}$$

The polynomial ring $\mathbb{C}[X]$ has variables x_{ij}.

Their number is $M = \binom{kn+1}{2} - n\binom{k+1}{2}$.

Let $\mathcal{V}_{k,n,r}$ be the variety in $\mathbb{C}^M$ given by matrices X of rank $\leq r$.

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Theorem

Let $r \ge 2k+1$. The variety $\mathcal{V}_{k,n,r}$ is irreducible of dimension

$$\dim(\mathcal{V}_{k,n,r}) = n\left(kr - \binom{k+1}{2}\right) - \binom{r}{2}.$$

The minors of X generate the prime ideal, which is Cohen-Macaulay.

Blocks of Size One

Example ($k = 1$)

Here X is a symmetric $n \times n$ matrix with zeros on the diagonal. The variety $\mathcal{V}_{1,n,r}$ is irreducible for $r \geq 3$. For $r = 3$ it equals the *squared Grassmannian* $\mathrm{sGr}(2, n)$. For $r = 0, 1$, it is just the origin.

What about $r = 2$?

Puzzle: Find all matrices of rank 2:

$$\begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix} \qquad \mathcal{V}_{1,4,2}$$

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$\mathcal{V}_{1,4,2}$

```
{ideal(a,b,d), ideal(a,c,e), ideal(b,c,f), ideal(f,e,d),
 ideal(a,f,b*e-c*d),ideal(b,e,a*f-c*d),ideal(c,d,a*f-b*e)}
```

Proposition

The variety $\mathcal{V}_{1,n,2}$ has $2^{n-1} - 1$ irreducible components. Each has codimension $\binom{n-1}{2}$. The total degree is $\binom{2n-3}{n-2}$. The 3×3 minors of X generate the radical ideal.

Main Results

Theorem

The dimension of our variety equals

$$\dim(\mathcal{V}_{k,n,r}) = \begin{cases} n\left(\ell(k-\ell)+\ell r-\binom{\ell+1}{2}\right)-\binom{r}{2} & \text{if } r \le 2k+2, \\ n\left(kr-\binom{k+1}{2}\right)-\binom{r}{2} & \text{if } r \ge 2k-1. \end{cases}$$

Here $\ell = \lfloor r/2 \rfloor$. *In the second case, the ideal is Cohen-Macaulay.*

The two dimension formulas agree for $r \in \{2k-1, 2k, 2k+1, 2k+2\}$.

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Theorem

The variety $\mathcal{V}_{k,n,r}$ is irreducible if and only if r is odd or $r > 2k$.
For r even with $4 \leq r \leq 2k$, the number of irreducible components of $\mathcal{V}_{k,n,r}$ is 2^{n-1}. For $r = 2$, it is $2^{n-1}-1$.

The Joy of Data

We have explicit parametrizations of all irreducible components of $\mathcal{V}_{k,n,r}$.

We computed their ideals in many cases.

Dimension, degree and number of ideal generators:

$r \backslash (k, n)$	$(2, 4)$	$(2, 5)$	$(3, 3)$	$(4, 2)$	$(5, 2)$
1	$0, 41, 284$	$0, 121, 740$	$0, 28, 378$	$0, 17, 136$	$0, 26, 325$
2	$7, 84, 1092$	$9, 600, 4770$	$8, 63, 1950$	$7, 20, 416$	$9, 70, 1700$
3	$9, \mathbf{4224}, 1764$	$12, 183040, 13860$	$9, 3915, 4599$	$7, 200, 626$	$9, 1190, 4550$
4	$14, \mathbf{2772}, 1176$	$19, 306735, 19404$	$15, 930, 4977$	$12, 20, 416$	$16, 175, 6202$
5	$18, 672, \mathbf{336}$	$25, 151008, 13860$	$17, 9504, 2520$	$12, 100, 136$	$16, 1750, 4550$
6	$21, 84, 36$	$30, 28314, 4950$	$21, 1386, 540$	$15, 4, 16$	$21, 50, 1700$
7	$23, 8, 1$	$39, 2640, 825$	$24, 120, 45$	$15, 8, 1$	$21, 250, 325$

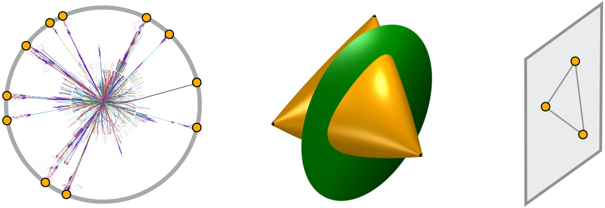
Table: The determinantal varieties $\mathcal{V}_{k,n,r}$ for matrix sizes $8 \leq kn \leq 10$.

Cosmology

Our project started in August 2024, after the posting of

A New Twist on Spinning (A)dS Correlators

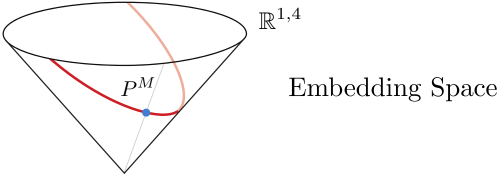
Daniel Baumann[1,2,3,4], Grégoire Mathys[5],
Guilherme L. Pimentel[6] and Facundo Rost[1,3]



We now turn to Physics.

Conformal Field Theory

The embedding space formalism in conformal field theory takes vectors in $\mathbb{R}^d$ onto the **Lorentz cone** *in $\mathbb{R}^r$ where $r = d+2$.*



Conformal symmetries in $\mathbb{R}^d$ are encoded by the action of the Lorentz group $\mathrm{SO}(1, r-1)$ on $\mathbb{R}^r$. We consider n fields, given by momentum vectors P_i and polarization vectors W_i for $i = 1, 2, \ldots, n$.

This leads to $2n \times 2n$ Gram matrices of rank r with blocks of size $k = 2$.

M. Costa, J. Penedones, D. Poland and S. Rychkov: *Spinning conformal correlators*, Journal of High Energy Physics **11** (2011)

Conformally Invariant Coordinates

Conformal field theory considers functions of the Gram matrix X that are invariant under translations $W_i \mapsto W_i + \alpha_i P_i$. We define

$$\begin{aligned} P_{ij} &= x_{2i-1,2j-1}, \\ H_{ij} &= x_{2i-1,2j-1} \cdot x_{2i,2j} - x_{2i,2j-1} \cdot x_{2j,2i-1}, \\ V_{i,jk} &= \big(x_{2i,2j-1} \cdot x_{2k-1,2i-1} - x_{2i,2k-1} \cdot x_{2j-1,2i-1}\big)/x_{2j-1,2k-1}. \end{aligned}$$

These are the well-known *conformally-invariant structures*:

$$\begin{aligned} P_{ij} &= P_i \cdot P_j, \\ H_{ij} &= (W_i \cdot W_j)(P_i \cdot P_j) - (W_i \cdot P_j)(W_j \cdot P_i), \\ V_{i,jk} &= \big[(W_i \cdot P_j)(P_k \cdot P_i) - (W_i \cdot P_k)(P_j \cdot P_i)\big]/(P_j \cdot P_k). \end{aligned}$$

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We wish to express $\mathcal{V}_{2,n,r}$ in terms of these. From now on $k = 2$.

Example: The main component for $n = 3, r = 4$ is

$$I_{3,4} = \left\langle H_{12} + V_{123}V_{213}\,,\, H_{13} - V_{123}V_{312}\,,\, H_{23} + V_{213}V_{312} \right\rangle.$$

The Joy of Prime Ideals

Let $I_{n,r}$ be the ideal in $\mathbb{C}[P, H, V]$ of relations among the $2\binom{n}{2} + n\binom{n-1}{2}$ basic building blocks P_{ij}, H_{ij}, $V_{i,jk}$.

Example ($d = 3, r = 5, n = 3$)

For three fields in cosmology, $I_{3,5}$ is the principal ideal generated by

$$4H_{12}H_{23}H_{13} \; - \; (V_{1,23}H_{23} - V_{2,13}H_{13} + V_{3,12}H_{12} + V_{1,23}V_{2,13}V_{3,12})^2.$$

This equals the determinant of the Gram matrix

$$X \; = \; \begin{bmatrix} 0 & 0 & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{13} & x_{23} & 0 & 0 & x_{35} & x_{36} \\ x_{14} & x_{24} & 0 & 0 & x_{45} & x_{46} \\ x_{15} & x_{25} & x_{35} & x_{45} & 0 & 0 \\ x_{16} & x_{26} & x_{36} & x_{46} & 0 & 0 \end{bmatrix}$$

An Orbit Space

Corollary

Let $n \geq 3$ and $r \geq 4$. The variety of $I_{n,r}$ has dimension

$$\dim(V(I_{n,r})) \;=\; (2r-4)n - \binom{r}{2}.$$

This equals $6n - 10$ for $r = 5$, and it equals $4n - 6$ for $r = 4$.

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Lemma (Four fields in cosmology   dim 14 in $\mathbb{C}^{24}$)

The ideal $I_{4,5}$ contains the 3×3 minors of the 4×3 matrix

$$V \;=\; \begin{bmatrix} V_{1,23} & V_{1,24} & V_{1,34} \\ V_{2,14} & V_{2,13} & -V_{2,34} \\ V_{3,14} & V_{3,24} & -V_{3,12} \\ V_{4,23} & V_{4,13} & V_{4,12} \end{bmatrix}.$$

The product $V\cdot\begin{bmatrix} P_{23}P_{14} & -P_{24}P_{13} & P_{34}P_{12}\end{bmatrix}^T$ is zero on $V(I_{4,5})$, so its four entries are in $I_{4,5}$. But there are many more generators.

Spinor-Helicity Formalism

... takes us further for $r = 4$.

The orbit space $V(I_{n,4})$ has the parametric representation:

$$P_{ij} = [\,i\,j\,]\langle\,i\,j\,\rangle\,, \quad H_{st} = \langle s\,t\rangle^2[\,s\,\bar{s}\,][\,t\,\bar{t}\,]\,, \quad V_{s,ij} = \frac{\langle s\,i\,\rangle\langle\,s\,j\,\rangle[\,s\,\bar{s}\,]}{\langle\,i\,j\,\rangle}.$$

The brackets are the 2×2 minors of a $2 \times 3n$ matrix of parameters.

Proposition

The prime ideal $I_{5,4}$ is minimally generated by 180 quadrics, 20 cubics, 95 quartics and 156 quintics in the 50 variables P, H, V.

The variety $V(I_{5,4})$ has dimension 14 and degree 10145 in $\mathbb{C}^{50}$.

Conformal Correlators

A fundamental **observable** for physics is the *two-point function*

$$\frac{H_{ij}}{P_{ij}^3} \;=\; \frac{(W_i \cdot W_j)(P_i \cdot P_j) - (W_i \cdot P_j)(W_j \cdot P_i)}{(P_i \cdot P_j)^3}.$$

These are the coordinates of a rational map

$$\mathcal{V}_{2,n,r} \dashrightarrow \mathbb{P}^{\binom{n}{2}-1}.$$

The image is the *conformal two-point variety* $\mathcal{C}_{n,r}^{(2)}$.

Conformal Correlators

A fundamental **observable** for physics is the *two-point function*

$$\frac{H_{ij}}{P_{ij}^3} \;=\; \frac{(W_i \cdot W_j)(P_i \cdot P_j) - (W_i \cdot P_j)(W_j \cdot P_i)}{(P_i \cdot P_j)^3}.$$

These are the coordinates of a rational map

$$\mathcal{V}_{2,n,r} \dashrightarrow \mathbb{P}^{\binom{n}{2}-1}.$$

The image is the *conformal two-point variety* $\mathcal{C}_{n,r}^{(2)}$.

Computing $\mathcal{C}_{n,r}^{(2)}$ is very difficult. Experiments suggest:

Conjecture ($r = 5$, Cosmology)

The dimension of the conformal two-point variety equals

$$\dim\big(\mathcal{C}_{n,5}^{(2)}\big) \;=\; \min\Big(5n - 11\,,\binom{n}{2} - 1\Big).$$

One Last Theorem

... about the conformal two-point variety for $r = 4$:

$$\mathcal{C}_{n,4}^{(2)} \subset \mathbb{P}^{\binom{n}{2}-1}.$$

Spinor-helicity gives the parametrization

$$\frac{H_{st}}{P_{st}^3} = \frac{[s\,\bar{s}]\,[t\,\bar{t}]}{[s\,t]^3\langle s\,t\rangle} \qquad \text{for } 1 \leq s < t \leq n.$$

We use **tropical geometry** (phylogenetic trees) to prove:

Theorem

The dimension of the two-point variety $\mathcal{C}_{n,4}^{(2)}$ equals $3n-7$.

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... about the conformal two-point variety for $r = 4$:

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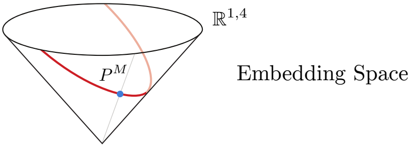
The dimension of the two-point variety $\mathcal{C}^{(2)}_{n,4}$ equals $3n-7$.

Corollary

The variety $\mathcal{C}^{(2)}_{n,4}$ has positive codimension for $n \geq 5$.
For $n = 5$ it is a hypersurface of degree 270 in $\mathbb{P}^9$.

Conclusion



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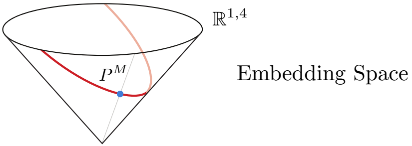

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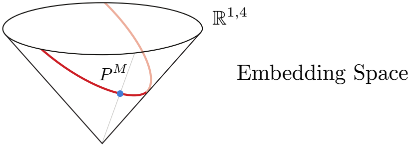

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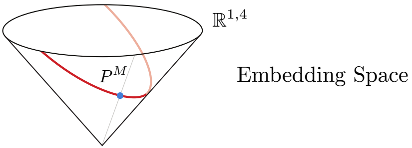

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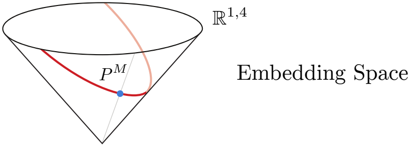
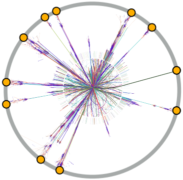
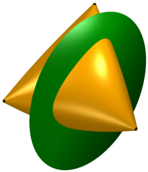
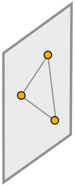


- The project started from discussions with cosmologists.
- We studied symmetric matrices with diagonal zero blocks.
- We understand their rank strata in terms of commutative algebra and algebraic geometry.
- We introduced algebraic varieties for conformal field theory.
- These govern the relations among the two-point functions.







Homework

Consider the ideal generated by the 5×5 minors of the matrix

$$X = \begin{bmatrix} 0 & 0 & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{13} & x_{23} & 0 & 0 & x_{35} & x_{36} \\ x_{14} & x_{24} & 0 & 0 & x_{45} & x_{46} \\ x_{15} & x_{25} & x_{35} & x_{45} & 0 & 0 \\ x_{16} & x_{26} & x_{36} & x_{46} & 0 & 0 \end{bmatrix}.$$

Determine all associated primes of this ideal.