# Tutorial on Gauge invariant structures

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# References

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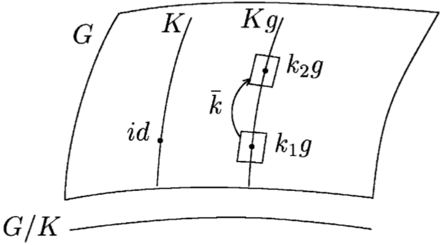



Fig. 2. Defining a right-invariant form on the space of right cosets $G/K$.

Figure: B. Khesin, G. Misiolek, Euler equations on homogeneous spaces and Virasoro orbits, Advances in Mathematics 176 (2003) 116–144.
Caution: inverse convention to Bourbaki, Topologie Générale, TGIII.12

# Example of the Hunter-Saxton equation

The Hunter-Saxton equation

$$-v_{xxt} = 2v_x v_{xx} + v v_{xxx}$$

is the Euler equation describing the geodesic flow on the homogeneous space $\mathrm{Diff}^+(\mathbb{S}^1)/\,\mathrm{Rot}(\mathbb{S}^1)$ of the group of orientation-preserving diffeomorphisms modulo the group of rotations, with respect to the right-invariant homogeneous $\dot{H}^1$-metric [KM03]:

$$\langle\!\langle v(x)\partial_x, w(x)\partial_x \rangle\!\rangle_{\dot{H}^1} = \int_{\mathbb{S}^1} v_x(x) w_x(x) dx. \qquad (1)$$

Remark: The $\dot{H}^1$-metric is degenerated: any constant vector field $c\partial_x$ is in the kernel of the $\dot{H}^1$-metric:

# Example of the Hunter-Saxton equation

Note that the constant vector fields are generated by the action of the rotation group $\mathrm{Rot}(\mathbb{S}^1) \simeq \mathbb{S}^1 \subset \mathrm{Diff}^+(\mathbb{S}^1)$. Consider the quotient space

$$_{\mathrm{Rot}(\mathbb{S}^1)}\backslash^{\mathrm{Diff}^+(\mathbb{S}^1)}$$

on which $\mathrm{Diff}^+(\mathbb{S}^1)$ acts from the right and $\mathrm{Rot}(\mathbb{S}^1) \simeq \mathbb{S}^1$ acts from the left, i.e. $_{\mathrm{Rot}(\mathbb{S}^1)}\backslash^{\mathrm{Diff}^+(\mathbb{S}^1)}$ is the space of cosets of the form $\mathrm{Rot}(\mathbb{S}^1) \circ g$, $g \in \mathrm{Diff}^+(\mathbb{S}^1)$ (we use here the Bourbaki convention).

## Proposition

*Since the $\dot{H}^1$-bilinear form* (1) *is* $\mathrm{Ad}\left(\mathrm{Rot}(\mathbb{S}^1)\right)$*-invariant, with kernel equal to the space generated by the infinitesimal action of the rotation group* $\mathrm{Rot}(\mathbb{S}^1)$*, it defines a Riemannian metric on the quotient space* $_{\mathrm{Rot}(\mathbb{S}^1)}\backslash^{\mathrm{Diff}^+(\mathbb{S}^1)}$.

# Proof

**Ad $\left(\mathrm{Rot}(\mathbb{S}^1)\right)$-invariance of the $\dot{H}^1$ bilinear pairing.** Identify the circle $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$ and consider the Adjoint action by a rotation by angle $\theta$ on the space of vector fields:

$$\mathrm{Ad}(\mathrm{Rot}(\theta))(v(x)\partial_x) : x \mapsto x+\theta \mapsto v(x+\theta)\partial_x \mapsto \big(v(x+\theta)-\theta\big)\partial_x.$$

We have

$$\begin{aligned}\langle\langle (v(x+\theta)-\theta)\partial_x, (w(x+\theta)-\theta)\partial_x\rangle\rangle_{\dot{H}^1} &= \int_{\mathbb{S}^1} v_x(x+\theta)w_x(x+\theta)dx \\ &= \int_{\mathbb{S}^1} v_x(x)w_x(x)dx,\end{aligned}$$

hence Ad $\left(\mathrm{Rot}(\mathbb{S}^1)\right)$ preserves the $\dot{H}^1$-bilinear form (1).

**Degeneracy of the $\dot{H}^1$ bilinear pairing.** Since the kernel of this bilinear pairing is exactly the space of constant vector fields, it defines a scalar product on the tangent space to the quotient at the coset $[\mathrm{Id}]$ which can be translated by the right action of $\mathrm{Diff}^+(\mathbb{S}^1)$ to any other tangent space to ${}_{\mathrm{Rot}(\mathbb{S}^1)}\backslash^{\mathrm{Diff}^+(\mathbb{S}^1)}$. The fact that the resulting Riemannian metric is well-defined is a direct consequence of the $\mathrm{Ad}\left(\mathrm{Rot}(\mathbb{S}^1)\right)$-invariance. $\square$

Remark: **Gauge invariance**

A curve $[\gamma] : [a, b] \to {}_{\mathrm{Rot}(\mathbb{S}^1)}\backslash^{\mathrm{Diff}^+(\mathbb{S}^1)}$ on the quotient space has the same length as any lift $\gamma : [a, b] \to \mathrm{Diff}^+(\mathbb{S}^1)$ of $[\gamma]$. In other words, the length functional

$$\mathrm{Length}(\gamma) = \int_a^b \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle_{\dot{H}^1}^{\frac{1}{2}} dt$$

is invariant by the action of the gauge group consisting of group-valued curves: $R_\theta : [a, b] \to \mathrm{Rot}(\mathbb{S}^1)$, $t \mapsto R_{\theta(t)}$ where

$$R_\theta \cdot \gamma : [a, b] \to R_{\theta(t)} \circ \gamma(t).$$

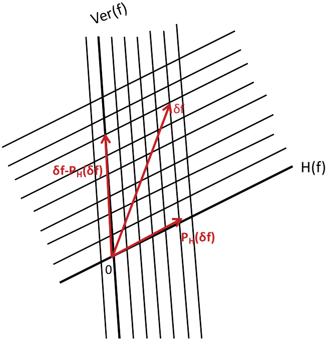
## Definition

Gauge invariant metrics consist of degenerate metrics $\tilde{\mathrm{g}} : T\mathscr{M} \times T\mathscr{M} \to \mathbb{R}$ on a fiber bundle $\pi : \mathscr{M} \to \mathscr{M}/K$, whose kernel at $m \in \mathscr{M}$ coincides with the vertical space $\ker T_m\pi$ and which descend to a Riemannian metric on $\mathscr{M}/K$.

## Construction from a $K$-invariant metric

Suppose that we have a vector bundle $\mathbb{T}$ over $\mathscr{M}$ which is a $K$-invariant subbundle of $T\mathscr{M}$ transverse to the vertical bundle $\mathrm{Ver} := \ker(T\pi)$ (i.e. a connexion). Using any $K$-invariant metric $\mathrm{g}_{\mathscr{M}}$ on $\mathscr{M}$, one can define a $K$-invariant metric $\mathrm{g}_{GI}$ on $\mathscr{M}$ that is degenerate along the fiber of the projection $\pi : \mathscr{M} \to \mathscr{Q} = \mathscr{M}/K$.

(a)




(b)

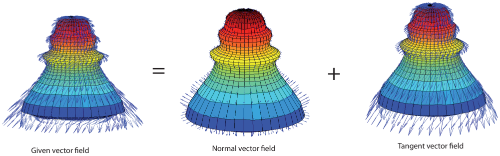



Figure: A.B. Tumpach, H. Drira, M. Daoudi, A. Srivastava, Gauge Invariant Framework for Shape Analysis of Surfaces. IEEE Transactions on Pattern Analysis and Machine Intelligence, January 2016, Volume 38, Number 1.

## Theorem

*Let $g_{\mathscr{M}}$ be a $K$-invariant metric on $\mathscr{M}$ and suppose that there exists a $K$-invariant subbundle $\mathbb{T}$ of $T\mathscr{M}$ such that*

$$T_m\mathscr{M} = \ker(T_m\pi) \oplus \mathbb{T}_m, \forall m \in \mathscr{M}. \qquad (2)$$

- *Then there exists a unique gauge invariant metric $g_{GI}$ on $T\mathscr{M}$ which coincides with $g_{\mathscr{M}}$ on $\mathbb{T}$ and is degenerate exactly along the fibers of $\pi : \mathscr{M} \to \mathscr{M}/K$.*
- *It induces a Riemannian metric $g$ on the quotient space $\mathscr{Q} = \mathscr{M}/K$ such that $T_m\pi : \mathbb{T}_m \to T_{\pi(m)}\mathscr{Q}$ is an isometry. One has*

$$g\left(T_m\pi(X_m), T_m\pi(Y_m)\right) = g_{\mathscr{M}}\left(p_{\mathbb{T}}(X_m), p_{\mathbb{T}}(Y_m)\right)$$

*where $m \in \mathscr{M}$ and $X_m, Y_m \in T_m\mathscr{M}$, and $p_{\mathbb{T}} : T_m\mathscr{M} \to \mathbb{T}_m$ is the projection onto $\mathbb{T}_m$ parallel to $\mathrm{Ver}_M = \ker T_m\pi$.*

## Theorem (Riemannian immersion Theorem)

*Consider a quotient manifold $\mathscr{M}/K$ and suppose that $\pi : \mathscr{M} \to \mathscr{M}/K$ admits a globally defined smooth section $s : \mathscr{M}/K \to \mathscr{M}$. Denote by $\mathscr{S}$ the smooth submanifold $\mathscr{S} = s(\mathscr{M}/K) \subset \mathscr{M}$. Suppose that $\mathscr{M}$ is endowed with a Riemannian metric $\mathrm{g}_{\mathscr{M}}$. Then the Riemannian metric $\mathrm{g}_{\mathscr{M}}$ naturally induces a unique Riemannian metric $\mathrm{g}_{imm}$ on $\mathscr{Q}$ such that the projection $\pi$ restricted to $\mathscr{S}$ is an isometry. One has:*

$$\mathrm{g}_{imm}\left(\bar{X}, \bar{Y}\right) = \mathrm{g}_{\mathscr{M}}\left(T_{\bar{m}}s(\bar{X}), T_{\bar{m}}s(\bar{Y})\right) \tag{3}$$

*where $\bar{X}, \bar{Y} \in T_{\bar{m}}(\mathscr{M}/K)$ and $\bar{m} := \pi(m) \in \mathscr{M}/K$. Equivalently, one has*

$$\mathrm{g}_{\mathscr{M}}\left(X_{s(\pi(m))}, Y_{s(\pi(m))}\right) = \mathrm{g}_{imm}\left(T_{s(\pi(m))}\pi(X_{s(\pi(m))}), T_{s(\pi(m))}\pi(Y_{s(\pi(m))})\right), \tag{4}$$

*where $X_{s(\pi(m))}, Y_{s(\pi(m))} \in T_{s(\pi(m))}\mathscr{S}$, and $m \in \mathscr{M}$.*

# Riemannian submersion Thm in the non-complemented case

**Theorem (Preston-T)**

*Let $\mathscr{M}$ be a manifold endowed with a Riemannian metric $\mathrm{g}_{\mathscr{M}}$, and suppose that a Banach Lie group $K$ acts on $\mathscr{M}$ in such a way that the quotient $\mathscr{Q} = \mathscr{M}/K$ is a smooth manifold. Then*

(i) *the normal bundle $\mathrm{Nor} = T\mathscr{M}/\ker T\pi$ over $\mathscr{M}$ is canonical endowed with a inner product, i.e. a positive definite symmetric bilinear form,*

(ii) *If moreover $\mathrm{g}_{\mathscr{M}}$ is $K$-invariant, then there exists a unique Riemannian metric $\mathrm{g}_{sub}$ on the quotient space $\mathscr{Q} = \mathscr{M}/K$ such that the canonical projection $p : \mathrm{Nor} \to \mathrm{Nor}/K \simeq T(\mathscr{M}/K)$ is an isometry.*

Recall:

**Proposition**

*Let $G$ be a topological group and $K$ a subgroup of $G$. The quotient space $G/K$ is Hausdorff if and only if $K$ is a closed subgroup of $G$.*

**Conjecture (An-Neeb)**

*Consider a closed subgroup $K$ of a Banach–Lie group $G$. The following are equivalent:*

1. *$K$ is a Banach–Lie subgroup, i.e it has the structure of a Banach–Lie group with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$ endowed with the subspace topology*
2. *$G/K$ carries the structure of a Banach manifold for which the quotient map $\pi : G \to G/K$, $g \mapsto gK$ has a surjective differential at each point and $G$ acts smoothly on $G/K$.*

It was shown by An and Neeb that $2 \Rightarrow 1$, but the implication $1 \Rightarrow 2$ is, as far as we know, still an open problem. It is for instance know to be true when $K$ is a split Banach–Lie subgroup, i.e the Lie algebra $\mathfrak{k}$ is closed in $\mathfrak{g}$ and has a closed complement (see for instance Bourbaki), or when $K$ is a closed normal subgroup (see Glockner-Neeb).