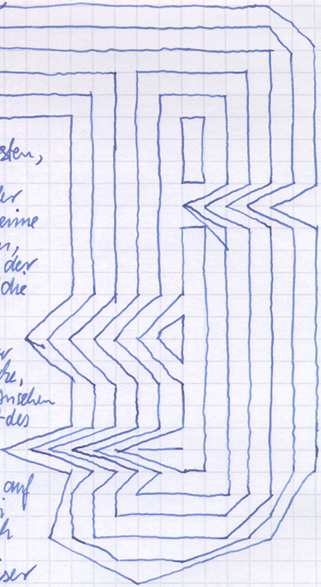
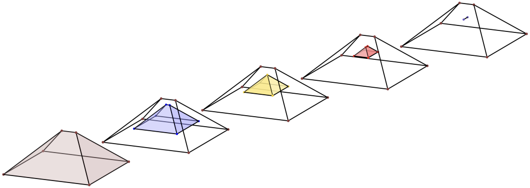


Nathan: Vor grauen Jahren lebt' ein Mann im Osten,
Der einen Ring von unschätzbarem Wert
Aus lieber Hand besaß. Der Stein war ein Opal, der
Hundert schöne Farben spielte, und hatte die geheime
Kraft, vor Gott und Menschen angenehm zu machen,
Wer in dieser Zuversicht ihn trug. Was Wunder, daß der
Mann im Osten ihn darum nie vom Finger ließ; und die
Verfügung traf, auf ewig ihn bei seinem Hause zu
erhalten? Nämlich so. Er ließ den Ring von seinen
Söhnen dem geliebtesten; und setzte fest, daß dieser
wiederum den Ring von seinen Söhnen dem vermache,
der ihm der liebste sei; und stets der liebste, ohn' Ansehen
der Geburt, in Kraft allein des Rings, das Haupt, der Fürst des
Hauses werde. – Versteh mich, Sultan.
Saladin: Ich versteh dich. Weiter!
Nathan: So kam nun dieser Ring von Sohn zu Sohn, auf
einen Vater endlich von drei Söhnen; die alle drei
ihm gleich gehorsam waren, die alle drei er folglich
gleich zu lieben sich nicht entbrechen konnte.
Nur von Zeit zu Zeit schien ihm bald der, bald dieser
bald der dritte.. – sowie jeder sich mit ihm allein

Shrinking Polytopes and Varieties

Sofía  Garzón Mora & Christian  Haase



with help of Benjamin Nill and Andreas Paffenholz

Advanced Studies in Pure Mathematics 10, 1987
Algebraic Geometry, Sendai, 1985
pp. 167–178

On Polarized Manifolds Whose Adjoint Bundles Are Not Semipositive

Takao Fujita

Introduction

Let L be an ample (not necessarily very ample) line bundle on a projective variety M with $\dim M=n$ having only rational normal Gorenstein singularities. Let ω be the dualizing sheaf and let K be the line bundle such that $\mathcal{O}_M(K)=\omega$. We will study the line bundles $K+tL$, where t is a positive integer. By the base point free theorem (cf. [K2; Theorem 2.6]), we have $\mathrm{Bs}|m(K+tL)|=\emptyset$ for $m\gg 0$ if $K+tL$ is *numerically semipositive* (=*nef*, for short), which means $(K+tL)C\geqq 0$ for any curve C in M. We do not know, however, how large m should be. Here we just pose the following:

Theorem 1. *$K+nL$ is nef unless $(M, L)\simeq(\boldsymbol{P}^n, \mathcal{O}(1))$. In particular, $K+(n+1)L$ is always nef.*

Theorem 2. *Suppose that $K+nL$ is nef. Then $K+(n-1)L$ is nef except in the following cases:*

(a) *M is a hyperquadric in $\boldsymbol{P}^{n+1}$ and $L=\mathcal{O}_M(1)$.*
(b) *$(M, L)\simeq(\boldsymbol{P}^2, \mathcal{O}(2))$.*
(c) *(M, L) is a scroll over a smooth curve.*

Corollary 4.2. *Let P be a smooth 3-dimensional polytope with no interior lattice points. Then P is of one of the following types.*

(1) $P = \Delta_3, 2\Delta_3, 3\Delta_3$.

(2) *There is a projection $P \twoheadrightarrow \Delta_2$, where any preimage of each vertex is an interval. Equivalently, there are $a, b, c \in \mathbb{Z}$ such that*

$$P = \operatorname{conv}\begin{bmatrix} 0 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(3) *There is a projection $P \twoheadrightarrow 2\Delta_2$, where any preimage of each vertex is an interval. Equivalently, there are $a, b, c \in \mathbb{Z}$ such that*

$$P = \operatorname{conv}\begin{bmatrix} 0 & 0 & 0 & a & b & c \\ 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{bmatrix}.$$

(4) *There is a projection $P \twoheadrightarrow \Delta_1 \times \Delta_1$. Equivalently, there are $a, b, c \in \mathbb{Z}$ such that*

$$P = \operatorname{conv}\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c & a+b-c \end{bmatrix}.$$

(5) $P = P_0 * P_1$, *where P_0 and P_1 are smooth polygons with the same normal fan.*

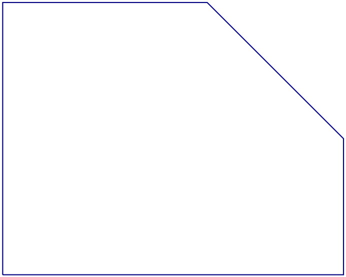
(6) *There is a smooth 3-dimensional polytope P' with no interior lattice points and a unimodular simplex $S \not\subseteq P$ such that*

$$P' = P \cup S$$

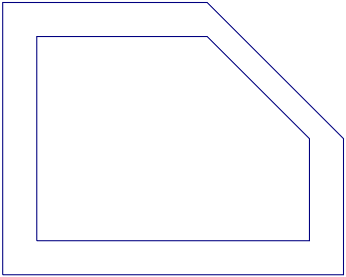
and $P \cap S$ is a common facet of P and S.

(Fine) Polyhedral Adjoints

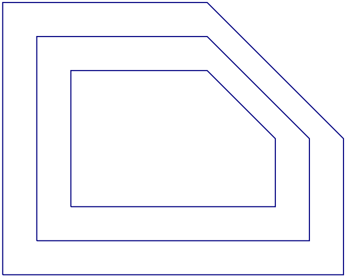
Polyhedral Adjoints



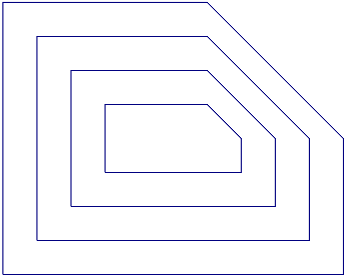
Polyhedral Adjoints



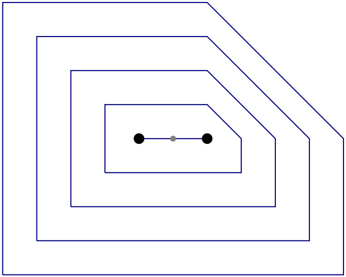
Polyhedral Adjoints



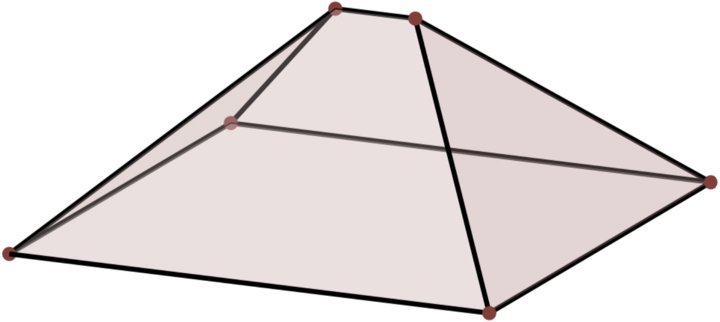
Polyhedral Adjoints



Polyhedral Adjoints

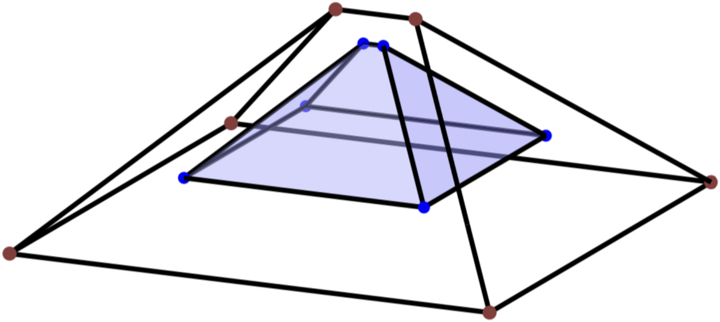


Polyhedral Adjoints



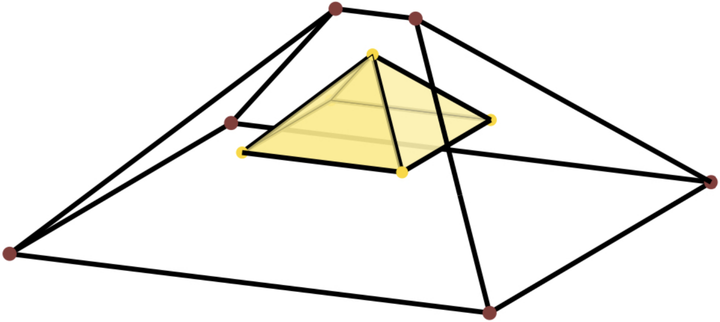
$P^{(0)} = P$

Polyhedral Adjoints



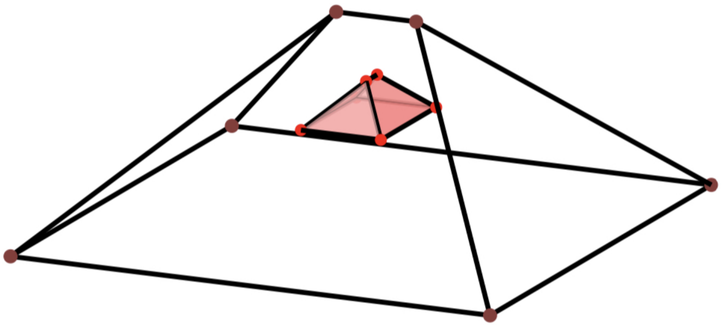
$P^{(s)}$ for $s = 3/2$

Polyhedral Adjoints



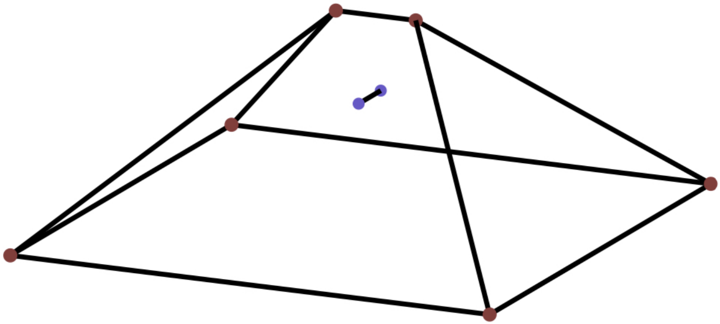
$P^{(s)}$ for $s = 2$

Polyhedral Adjoints



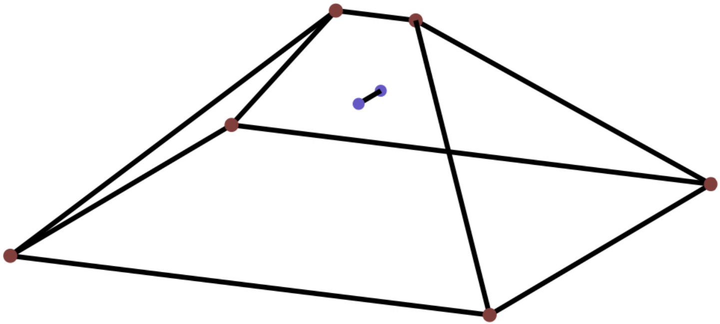
$P^{(s)}$ for $s = 5/2$

Polyhedral Adjoints



$P^{(s)}$ for $s = 3$

Polyhedral Adjoints

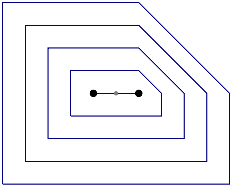


$P^{(s)}$ for $s = 3$

$$\eta = \eta(P) = \sup\{s > 0 \ : \ P^{(s)} \neq \emptyset\}$$

Polyhedral Adjoints

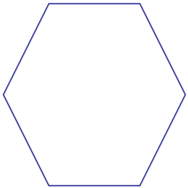
$$P^{(s)} = \big\{ x \in \mathbb{R}^d \ : \langle a, x \rangle \le b - s \text{ whenever } a \in \mathbb{Z}^d, b \in \mathbb{R} \text{ s.t.} \\ \langle a, \bullet \rangle \le b \text{ defines a facet of } P \big\}$$



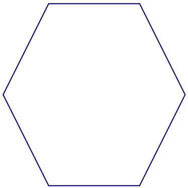
$$\eta = \eta(P) = \sup\{ s > 0 \ : \ P^{(s)} \ne \emptyset \}$$

Fine Polyhedral Adjoints

$$P^{F(s)} = \big\{ x \in \mathbb{R}^d \ : \langle a, x \rangle \leq b - s \text{ whenever } a \in \mathbb{Z}^d \setminus \{0\}, b \in \mathbb{R} \text{ s.t. } \langle a, \bullet \rangle \leq b \text{ is valid for } P \big\}$$



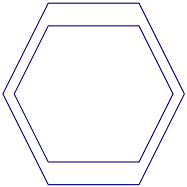
Original Adjoints



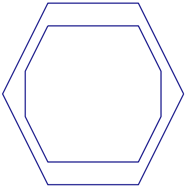
Fine Adjoints

Fine Polyhedral Adjoints

$$P^{F(s)} = \big\{x \in \mathbb{R}^d \ : \langle a, x\rangle \le b - s \text{ whenever } a \in \mathbb{Z}^d \setminus \{0\}, b \in \mathbb{R} \text{ s.t.} \\ \langle a, \bullet\rangle \le b \text{ is valid for } P\big\}$$



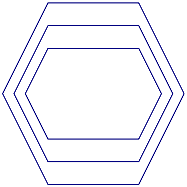
Original Adjoints



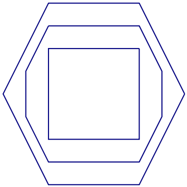
Fine Adjoints

Fine Polyhedral Adjoints

$$P^{F(s)} = \Big\{x \in \mathbb{R}^d \ : \langle a, x\rangle \leq b - s \text{ whenever } a \in \mathbb{Z}^d \setminus \{0\}, b \in \mathbb{R} \text{ s.t.}$$
$$\langle a, \bullet\rangle \leq b \text{ is valid for } P\Big\}$$



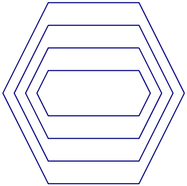
Original Adjoints



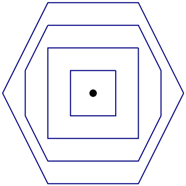
Fine Adjoints

Fine Polyhedral Adjoints

$$P^{F(s)} = \big\{ x \in \mathbb{R}^d \ : \langle a, x \rangle \leq b - s \text{ whenever } a \in \mathbb{Z}^d \setminus \{0\}, b \in \mathbb{R} \text{ s.t. } \langle a, \bullet \rangle \leq b \text{ is valid for } P \big\}$$



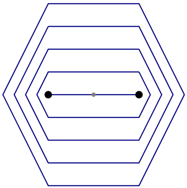
Original Adjoints



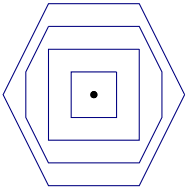
Fine Adjoints

Fine Polyhedral Adjoints

$$P^{F(s)} = \big\{ x \in \mathbb{R}^d \ : \langle a, x \rangle \leq b - s \text{ whenever } a \in \mathbb{Z}^d \setminus \{0\}, b \in \mathbb{R} \text{ s.t. } \langle a, \bullet \rangle \leq b \text{ is valid for } P \big\}$$



Original Adjoints



Fine Adjoints

Fine Polyhedral Adjoints

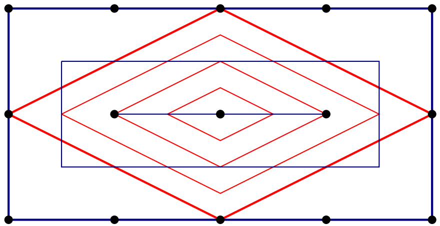
$$P^{F(s)} = \big\{ x \in \mathbb{R}^d \ : \langle a, x \rangle \le b - s \text{ whenever } a \in \mathbb{Z}^d \setminus \{0\}, b \in \mathbb{R} \text{ s.t. } \langle a, \bullet \rangle \le b \text{ is valid for } P \big\}$$

Original Adjoints

Fine Adjoints

$$\eta^F = \eta^F(P) = \sup\{ s > 0 \ : \ P^{F(s)} \ne \emptyset \}$$

(NON) MONOTONICITY



$$P \subseteq Q \quad \not\Longrightarrow \quad P^{(s)} \subseteq Q^{(s)}$$

$$P \subseteq Q \quad \Longrightarrow \quad P^{F(s)} \subseteq Q^{F(s)} \quad \Longrightarrow \quad \eta^F(P) \leq \eta^F(Q)$$

Finiteness

Conjecture (Fujita 1992)

The set $\{\eta(X)\ :\ X$ *a smooth polarized d-fold* $\}$ *is discrete.*
The set $\{\eta(P)\ :\ P$ *a smooth lattice d-polytope* $\}$ *is discrete.*

Finiteness

Theorem (Paffenholz 2015)

The set $\{\eta(P)\ :\ P \text{ a smooth lattice } d\text{-polytope }\}$ *is discrete.*

Theorem (Di Cerbo 2018)

The set $\{\eta(X)\ :\ X \text{ a smooth polarized } d\text{-fold }\}$ *is discrete.*

Finiteness

Theorem (Paffenholz 2015)

The set $\{\eta(P) \;:\; P$ a smooth lattice d-polytope $\}$ is discrete.

Theorem (Di Cerbo 2018)

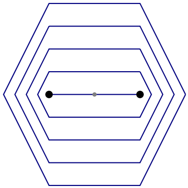
The set $\{\eta(X) \;:\; X$ a smooth polarized d-fold $\}$ is discrete.

Theorem (Garzón Mora,  2023)

The set $\{\eta^F(P) \;:\; P$ a lattice d-polytope $\}$ is discrete.

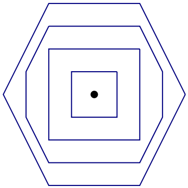
(Fine) Core & Projections

(Fine) Core



Original Adjoints

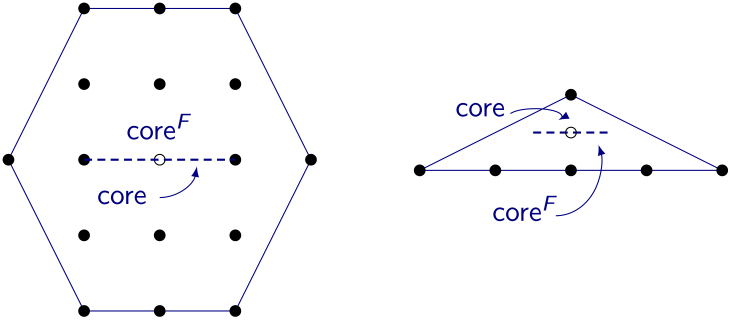
$\mathrm{core}(P) = P^{(\eta)}$



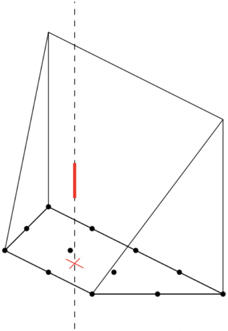
Fine Adjoints

$\mathrm{core}^F(P) = P^{F(\eta^F)}$

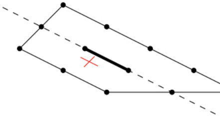
Core vs. Fine core



Natural Projection



$P = \mathrm{conv}\begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & h & h \end{bmatrix}$



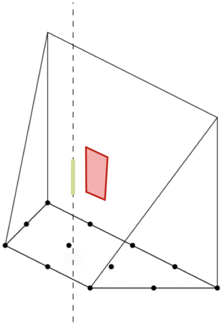
$Q = \mathrm{conv}\left[\begin{smallmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 4 & 2 \end{smallmatrix}\right]$

Theorem (Di Rocco, , Nill, Paffenholz 2014)

The image $Q := \pi_P(P)$ of the natural projection of P satisfies

$$\eta(Q) \leq \eta(P).$$

Natural Projection (Fine)



$$P = \operatorname{conv}\begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & h & h \end{bmatrix}$$

Theorem (Garzón Mora, 2023)

The image $Q := \pi_P(P)$ of the natural projection of P satisfies

$$\eta^F(Q) = \eta^F(P).$$

Moreover, $\mathrm{core}^F(Q)$ *is the point* $\pi_P(\mathrm{core}^F(P))$.

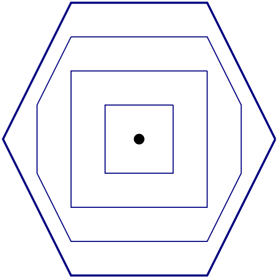
Fine Core Normals

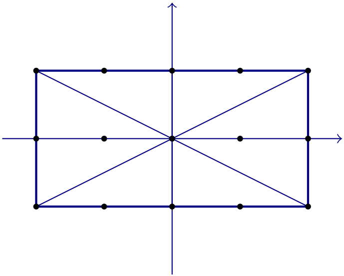
Fine Core Normals

Call $a \in \mathbb{Z}^d \setminus \{0\}$ a Fine core normal of P if

$$\langle a, x \rangle = b - \eta^F$$

along $x \in \mathrm{core}^F(P)$ for some $\langle a, \bullet \rangle \leq b$ is valid for P.

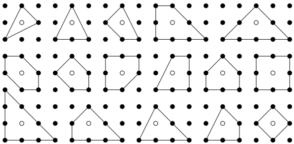




Finiteness

Proposition (Batyrev 2023, Garzón Mora, 2023)

All Fine core normals belong to the convex hull of the facet normals.
The convex hull of the Fine core normals has no relative interior lattice point but 0.



Corollary

The set $\{\eta^F(P) \ : \ P \text{ a lattice } d\text{-polytope } \}$ *is discrete.*

Exploring the Fine Spectrum

Segments

Lemma (Garzón Mora,)

Let P be a dimension 1 lattice polytope. Then

$$\eta^F(P) = \frac{k}{2}$$

for some $k \in \mathbb{Z}_{>0}$.

Baletti's Data Base

"$\mathbb{Q}$-codegree $= 1/\eta$"

```
Classical Q-codegree ; Fine Q-codegree ; Vertices of Polytope
3/8;1/2;[[0,0],[1,0],[1,4],[3,7],[2,7],[-1,-5]]
1;1;[[0,0],[1,0],[1,7],[2,7],[0,-1],[2,2]]
8/17;2/3;[[0,0],[6,0],[1,2],[2,-1],[0,-1]]
1/2;1/2;[[0,0],[1,0],[0,1],[3,3],[2,5],[1,4],[4,5]]
7/15;2/3;[[0,0],[1,0],[1,5],[2,1],[3,5],[3,3]]
4/17;2/3;[[0,0],[1,0],[4,7],[-7,-13]] 2;2;[[0,0],[6,0],[0,1],[-14,1]]
1;1;[[0,0],[3,0],[0,1],[6,1],[2,2],[7,2]]
1/2;1/2;[[0,0],[1,0],[1,3],[3,3],[3,6],[4,6],[0,-1],[2,5]]
5/8;1;[[0,0],[1,0],[1,8],[2,10],[2,6]]
3/5;3/5;[[0,0],[1,0],[0,2],[3,2],[4,4],[1,4]]
10/21;2/3;[[0,0],[1,0],[6,7],[3,5],[1,3]]
4/9;2/3;[[0,0],[6,0],[0,1],[13,-2],[14,-2]]
9/20;1/2;[[0,0],[1,0],[5,8],[5,6],[3,2]]
11/34;1/2;[[0,0],[4,0],[0,1],[3,-4]]
1/2;2/3;[[0,0],[1,0],[0,2],[-5,1],[-3,3]]
9/14;1;[[0,0],[1,0],[1,7],[2,4],[2,10]]
2/3;2/3;[[0,0],[1,0],[2,4],[3,3],[4,7],[6,9]]
7/18;1/2;[[0,0],[4,0],[0,1],[2,-4]] 13/38;1/2;[[0,0],[4,0],[0,1],[1,-4]]
```

Baletti's Data Base

"$\mathbb{Q}$-codegree $= 1/\eta$"

Dimension	Max. Volume	Fine $\mathbb{Q}$-codegree spectrum
1	∞	$2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{1}{4}, \ldots$
2	50	$3, 2, \frac{3}{2}, 1, \frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \frac{2}{5}, \frac{3}{8}, \frac{1}{3}, \frac{3}{10}, \frac{2}{7}$
3	23	$4, 3, \frac{5}{2}, 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{7}{6}, 1, \frac{5}{6}, \frac{4}{5}, \frac{3}{4}$
4	14	$5, 4, \frac{7}{2}, 3, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, 2, \frac{7}{4}, \frac{5}{3}, \frac{3}{2}, \frac{4}{3}, 1$
5	14	$6, 5, \frac{9}{2}, 4, \frac{7}{2}, \frac{10}{3}, \frac{13}{4}, 3, \frac{11}{4}, \frac{8}{3}, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, 2, \frac{7}{4}, \frac{5}{3}, \frac{3}{2}, 1$

$d \longrightarrow d + 1$

Theorem (Garzón Mora, 2025)

If the Fine core normals contain a postitive circuit, then the corresponding projection preserves η^F.

Theorem (Garzón Mora, 2025)

Let P be a d-dimensional rational polytope and $\mathrm{Pyr}(P)$ *a lattice pyramid over P. Then*

$$\eta^F(\mathrm{Pyr}(P)) = \min\left[\frac{1}{2}, \frac{\eta^F(P)}{\eta^F(P)+1}\right]$$

First Classification Results

Theorem (Garzón Mora, 2025)

Let P be a 2-dimensional lattice polytope. Then

$$\eta^F(P) = \frac{k}{2} \quad \text{or} \quad \frac{k}{3}$$

for some $k \in \mathbb{Z}_{>0}$.

Theorem (Garzón Mora, 2025)

Let P be a 3-dimensional lattice polytope. Then

$$\eta^F(P) = \frac{k}{q}$$

for some $k \in \mathbb{Z}_{>0}$ and $q \in \{3, 4, 5, 7, 11, 13, 17, 19\}$.

Theorem (Garzón Mora, 2025)

Let P be a d-dimensional lattice polytope. Then

$$\eta^F(P) \in \left\{ d+1, d, d - \frac{1}{2}, d-1, \ldots \right\}$$

Question

$$\mathcal{S}^F := \bigcup_{d \geq 1} \Big(\mathcal{S}^F(d) - d - 1 \Big) \subset \mathbb{Q}_{\leq 0}$$

starts off with a discrete part $0, -1, -3/2, -2, \ldots$.

- *discrete beyond* -2*?*
- *Where is the first accumulation point?*
- *At what point does it become dense?*

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