

Infinite-dimensional Geometry:
Theory and Applications

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Transverse Symplectic Foliation Structure for Thermodynamics- Informed Neural Network and Lie- Groups Machine Learning

Frédéric BARBARESCO
THALES - 13/02/2025

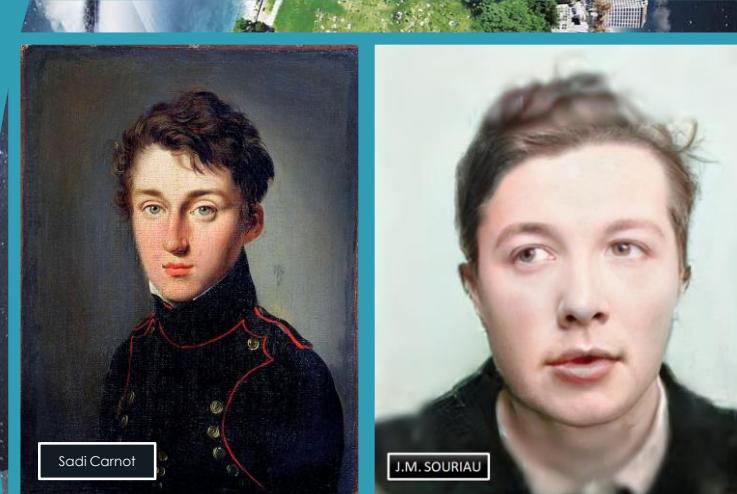
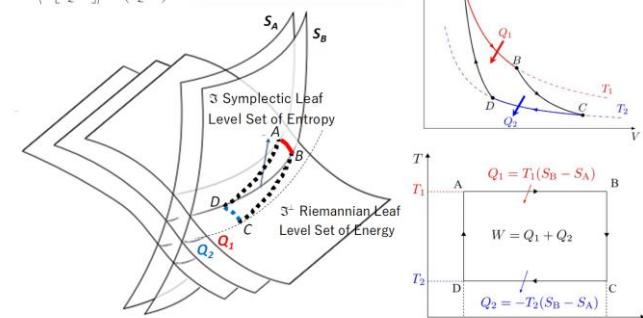
« Every mathematician knows
that it is impossible to
understand an elementary
course in thermodynamics. »
Vladimir Arnold

ENTROPY DEFINITION: Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.

$$\{S, H\}_{\hat{\Theta}}(Q) = 0$$

$$\beta \in \text{Ker } \hat{\Theta}_\beta \Rightarrow \left\langle Q, \left[\frac{\partial S}{\partial Q}, Z \right] \right\rangle + \Theta \left(\frac{\partial S}{\partial Q}, Z \right) = 0$$

$$ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta \left(\frac{\partial S}{\partial Q}, Q \right) = 0$$



Talk dedicated to Yvonne Choquet-Bruhat who passed away at 101 Years Old this week. Topical contributions to Geometry in Physics



PhD supervised by Jean Leray and André Lichnérowicz

A Lady Mathematician in this Strange Universe: Memoirs
Yvonne Choquet-Bruhat
Member of the American and the French Academies of Sciences

ANALYSIS, MANIFOLDS AND PHYSICS
Part I: Basics
Revised Edition
NORTH-HOLLAND

TRAVAUX EN COURS
D. Bernard et Y. Choquet-Bruhat
Physique quantique et géométrie
Formulation mathématique cohérente des phénomènes quantiques
Colloque Géométrie et Physique de 1985 en l'honneur de André Lichnérowicz

TRAVAUX EN COURS
D. Bernard et Y. Choquet-Bruhat
Géométrie différentielle
Géométrie différentielle—Variétés complexes
Feuilletages riemanniens
Colloque Géométrie et Physique de 1985 en l'honneur de André Lichnérowicz

Day in honor of
YVONNE CHOQUET-BRUHAT's
100th birthday

Conclusion

ORGANIZERS:
Laure Saint-Raymond, IHES
& Jérémie Szeftel, Laboratoire Jacques-Louis Lions

Jean-Pierre Bourguignon,
ancien directeur de l'IHES

<https://www.youtube.com/playlist?list=PLx5f8lelFRgHJ5KIRpxRKaqXiHnVzFzCQ>

INSTITUT DE FRANCE
Académie des sciences
5 avril 2016
Yvonne CHOQUET-BRUHAT

Rencontres avec Einstein

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Conférence - Y CHOQUET BRUHAT - Rencontres avec Einstein - Académie des sciences

Produced by Weconnect!

https://www.youtube.com/watch?v=_XCcsjjx5S8

Lille Carnot 2024: www.carnotlille2024.com



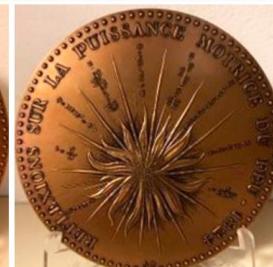
Celebration of 200 years since Sadi Carnot's
Réflexions sur la puissance motrice du Feu,
1824–2024 & les Carnots

Hosted by History of Physics and Applied
Science & Technologies Team (HOPAST) at
IEMN, France - Patronage by French
Académie des Sciences

Official Website URL: www.carnotlille2024.com

Talk: Symplectic Foliation Model of Sadi
Carnot's Thermodynamics: from
Carathéodory's seminal idea to Souriau's Lie
Groups Thermodynamics OPEN

dimensional Geometry: Theory and ApplicationTM



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1824-2024 Bicentenary of Thermodynamics Invention by Sadi Carnot by Raffaele Pisano (Book edited by Springer): www.carnotlille2024.com

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OPEN

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Building a future we can all trust

Energy, Power, Environment. A Symposium on the Occasion of the Bicentenary of the Publication of Sadi Carnot's 'Réflexions sur la puissance motrice du feu'



Maison Française d'Oxford

Institut français de recherche à l'étranger

<https://mfo.web.ox.ac.uk/event/energy-power-environment-symposium-occasion-bicentenary-publication-sadi-carnots-reflexions>



Convened by **Robert Fox** (University of Oxford)

In his *Réflexions sur la puissance motrice du feu*, a modest volume published in 1824, Sadi Carnot laid some of the most important foundations of modern thermodynamics and power technology. In certain respects, the book was not an easy read, and it had faults. The result was a masterpiece, though arguably a flawed one, and it remained almost unread until it was discovered and reconfigured by William Thomson (the future Lord Kelvin) and Rudolf Clausius from the late 1840s. Given the importance that physical scientists and engineers now attach to such fundamental ideas as the Carnot cycle, the second principle of thermodynamics, and the notion of entropy – all of which have roots in Carnot's thinking – why was the *Réflexions* neglected for so long? The contributions to this workshop explore the origins, content, and long-term impact of Carnot's ideas, including the legacy of debates about energy sources and implications for climate change that continue to engage us today.

Pascal Marty (Maison française d'Oxford). Introduction and welcome

First session. Chair: **Maxine Berg** (University of Warwick)

Raffaele Pisano (University of Lille): 'Reading Lazare & Sadi

Carnot's manuscripts: a foundational programme for mechanical and heat machines'

Robert Fox (University of Oxford): 'The technological environment of Sadi Carnot'

Crosbie Smith (University of Kent): 'Interpretations of the *Réflexions* by William Thomson and Rudolf Clausius'

Second session. Chair: **Erica Charters** (University of Oxford)

Ben Marsden (University of Aberdeen): 'Reflections on perpetual motion, alternatives to steam power, and energy physics'

Iwan Morus (Aberystwyth University): '"That vast granary of force": how to power an imperial future'

Jean-Baptiste Fressoz (Ecole des Hautes Études en Sciences Sociales-Centre de Recherches Historiques): 'No transition: a new history of energy'

Rob Iliffe (University of Oxford) Concluding remarks

September
16th-18th
2024

École polytechnique
Palaiseau, France

Sadi Carnot's Legacy

*Celebrating the 200th anniversary
of the 2nd law of thermodynamics*



SADI CARNOT'S LEGACY - "CELEBRATING THE 200TH ANNIVERSARY OF THE 2ND LAW OF THERMODYNAMICS"

When Sadi Carnot published his "Reflections on the motive power of fire" in 1824, there was no sign that one of the greatest scientific revolutions was about to take place, in a world then dominated by mechanics and optics. Yet, by bringing a conceptual analysis to the practical problem of the steam engine, Sadi Carnot wrote the birth certificate of thermodynamics, and, in particular, its second principle.

Today, thermodynamics has branched out into a multitude of fields and applications, from industrial processes to microscopic systems, and continues to renew our view of science.

Since its origins, thermodynamics has raised as many questions as it has answered.

To celebrate the bicentenary of the "Réflexions", this colloquium aims to bring together members of the thermodynamics community and to invite them to take a critical look at modern thermodynamics and the open questions it raises. The colloquium will be structured around pedagogical presentations introducing the various fields of the discipline. Poster sessions will allow participants to share their work.

Sadi Carnot's Legacy by Christophe Goupiel



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Sadi Carnot's Legacy

Celebrating the 200th anniversary
of the 2nd law of thermodynamics



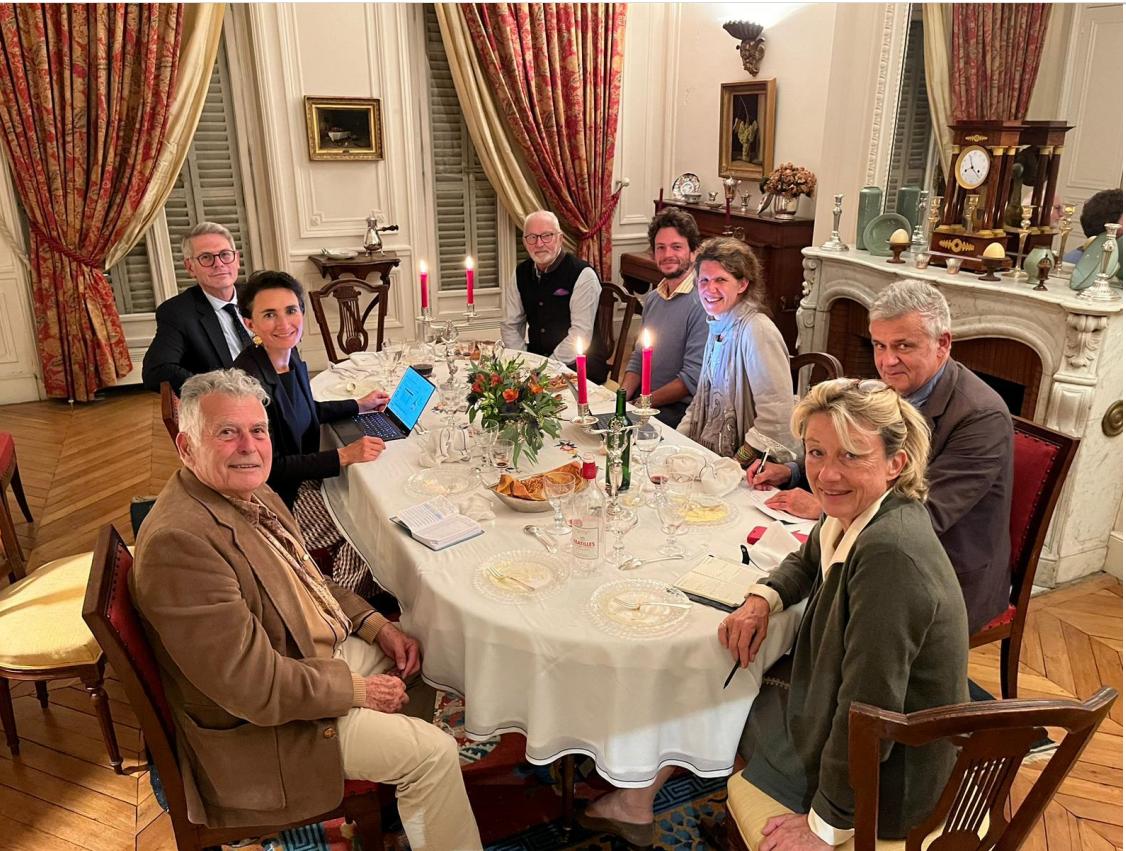
Carnot's Family Project to place Sadi Carnot in Panthéon between his father Lazare Carnot and is nephew Sadi Carnot, the French Republic President

Projet Nicolas Léonard
Sadi Carnot au Panthéon

1824 - 2024 : 200 ans de l'ouvrage *Réflexions
sur la puissance motrice du feu et sur les
machines propres à développer cette puissance*



<https://sadi-carnot-physicien.fr/>



Sadi Carnot « La Légende »



Sadi Carnot by Michel Serres
« Brûler » in « La légende des Sciences »
La Légende des Sciences – Brûler
<https://www.youtube.com/watch?v=J5JSWebXms0>

"La Théorie du Feu" Hommage à Sadi Carnot

<https://www.youtube.com/watch?v=uNRGDYDygt4>

(Couplet 1)

Il naquit à Paris, un jour de juin,
Sadi Carnot, au destin incertain.
Fils d'un révolutionnaire, grand mathématicien,
Le feu dans le cœur, la science en chemin.

À vingt-sept ans, il écrit son grand livre,
Des machines à vapeur, il voulait le livre.
"Réflexions sur le feu, la chaleur, le moteur",
Ses idées jaillissent comme une lueur.

(Refrain)

Thermodynamique, il en est le pionnier,
Avant même que le terme soit inventé.
Un moteur thermique, voilà ce qu'il a tracé,
Les bases d'une science pour l'avenir, gravées.

New 2024 edition of « Académie Française » Dictionary

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MOT À RECHERCHER



1 ^{re}	2 ^e	3 ^e	4 ^e	5 ^e	6 ^e	7 ^e	8 ^e	9 ^e édition
1694	1718	1740	1762	1798	1835	1878	1935	ACTUELLE

Dictionnaire de l'Académie française, 9^e édition (actuelle)

thermodynamique

THERMODYNAMIQUE nom féminin et adjetif

XIX^e siècle. Composé de *thermo-* et de *dynamique*.

1. **N. f.** Partie de la physique étudiant les systèmes macroscopiques du point de vue énergétique lorsqu'ils sont à l'état d'équilibre ou connaissent une évolution, que celle-ci soit spontanée ou causée par des échanges d'énergie ou de matière avec le milieu extérieur. *En étudiant le lien entre la chaleur et la production de travail mécanique, Nicolas Sadi Carnot posa les bases de la thermodynamique.*



Lord Kelvin 200th Birthday in Glasgow

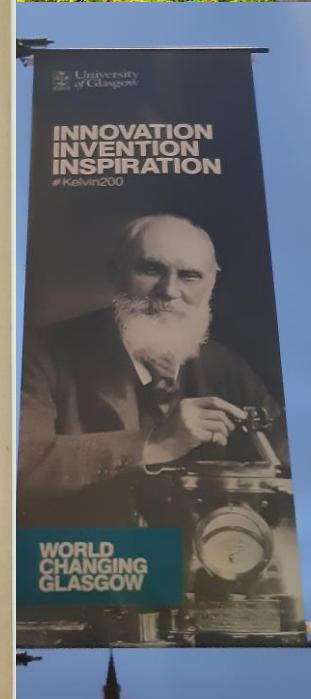
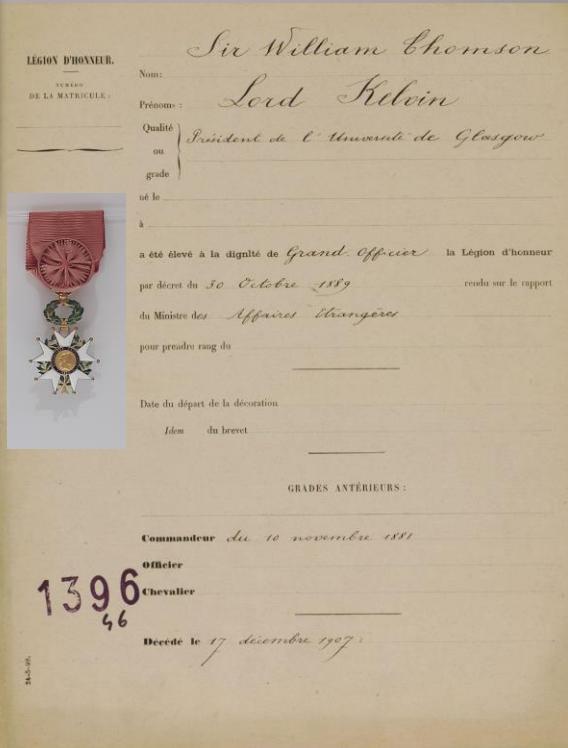
<https://www.gla.ac.uk/explore/lordkelvin200/>



"Nothing in the whole range of Natural Philosophy is more remarkable than the establishment of general laws by such a process of reasoning"
Sir William Thomson, Lord Kelvin about Sadi Carnot

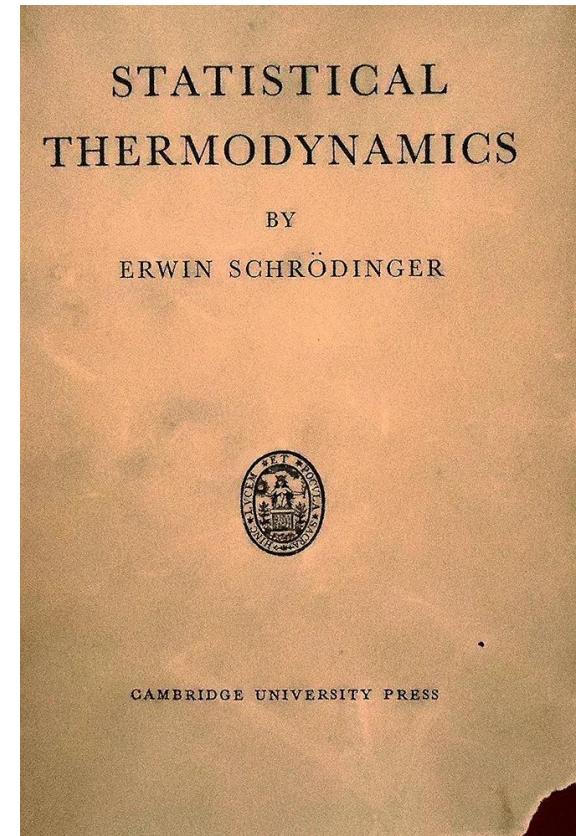
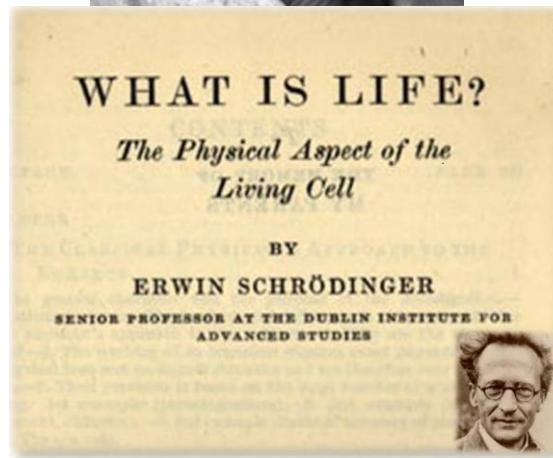
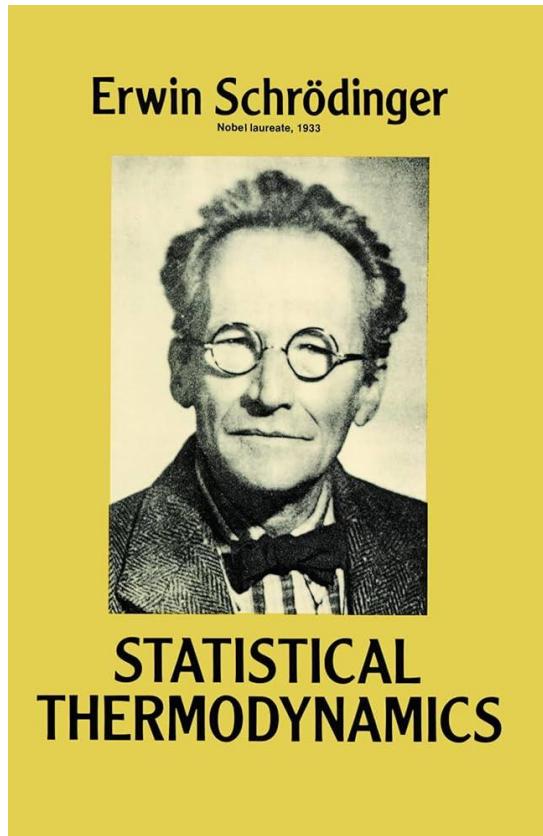
France Legion of Honour Commander's badge 1881

France Legion of Honour Grand Officer's star 1889



Statistical Thermodynamics by Erwin Schrödinger

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« There is nothing more in physical theories than symmetry groups except the mathematical construction which allows precisely to show that there is nothing more » - Jean-Marie Souriau



1. Elementary Symplectic Structures of Thermodynamics
2. Constantin Carathéodory: From the Pfaffian forms to Thermodynamics Axiomatization, & Adiabatic Foliation by Frankel
3. Information Geometry & Natural Gradient for Learning
4. Jean-Marie Souriau: Symplectic Foliation via Moment Map and Koszul-Fisher Metric from Information Geometry for Lie Groups Thermodynamics
5. ENTROPY as Casimir Function in Coadjoint Representation Constant on Symplectic Leaves & Density of Maximum Entropy on Lie Groups
6. Lars Onsager & Baptiste Coquinet: Symplectic Foliation Model of METRIPLECTIC FLOW
7. Transverse Symplectic Foliation Structure
8. Liouville Complete Integrability and Information Geometry
9. Thermodynamics-Informed Neural Network and Symplectic Foliation Structure
10. Conclusion

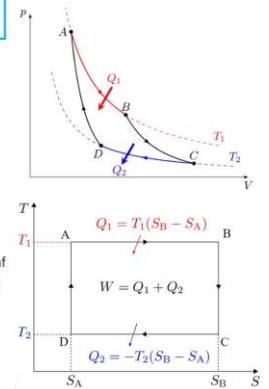
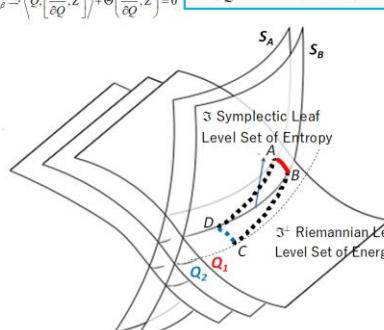
Elementary Symplectic Structures of Thermodynamics

ENTROPY DEFINITION: Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.

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$$ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta \left(\frac{\partial S}{\partial Q} \right) = 0$$



Capture of Symmetries Symmetry Group Equivariance in Physics

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Reference:

> <https://arxiv.org/abs/2203.06153>

Submitted to the Proceedings of the US Community Study
on the Future of Particle Physics (Snowmass 2021)

Symmetry Group Equivariant Architectures for Physics A Snowmass 2022 White Paper

Alexander Bogatskiy¹, Sanmay Ganguly², Thomas Kipf³, Risi Kondor⁴, David W. Miller^{*4}, Daniel Murnane⁵, Jan T. Ofermann⁴, Mariel Pettee^{†5}, Phiala Shanahan⁶, Chase Shimmin⁷, and Savannah Thais⁸

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²ICExPP, University of Tokyo

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⁴University of Chicago

⁵Lawrence Berkeley National Laboratory

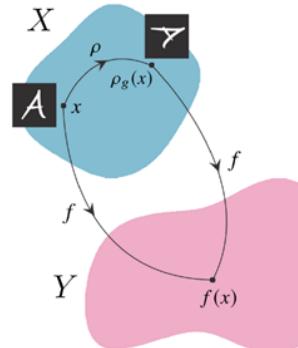
⁶Massachusetts Institute of Technology

⁷Yale University

⁸Princeton University

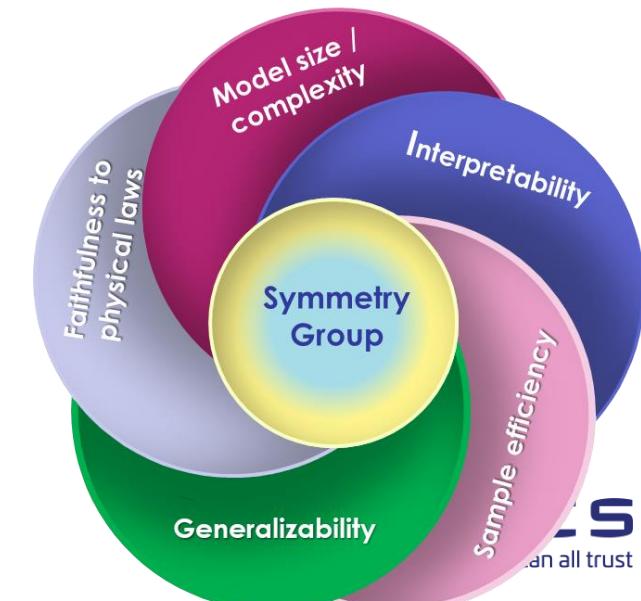
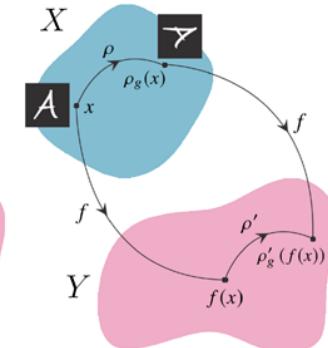
Invariance

$$f(\rho_g(x)) = f(x)$$



Equivariance

$$f(\rho_g(x)) = \rho'_g(f(x))$$



Representation Theory & (Co-adjoint) Orbits Method: A. Kirillov

Grundlehren der mathematischen Wissenschaften 220
A Series of Comprehensive Studies in Mathematics

A. A. Kirillov

Elements of the Theory of Representations

MERITS VERSUS DEMERITS

- 1. Universality: the method works for Lie groups of any type over any field.
- 2. The rules are visual, easy to memorize and illustrate by a picture.
- 3. The method explains some facts which otherwise look mysterious.
- 4. It provides a great number of symplectic manifolds and Poisson commuting families of functions.
- 5. The method introduces two new fundamental notions: coadjoint orbits and moment maps.
- 1. The recipes are not accurately and precisely formulated.
- 2. Sometimes they are wrong and need corrections or modifications.
- 3. It could be difficult to transform this explanation into a rigorous proof.
- 4. Most completely integrable dynamical systems were discovered earlier by other methods.
- 5. The description of coadjoint orbits and their structure is sometimes not an easy problem.



Springer-Verlag Berlin Heidelberg New York

§ 15. The Method of Orbits

At the basis of the method of orbits lies the following “experimental fact”: the theory of infinite-dimensional representations of every Lie group is closely connected with a certain special finite-dimensional representation of this group. This representation acts in the dual space \mathfrak{g}^* of the Lie algebra \mathfrak{g} of the group under study. We will call it a *co-adjoint* or briefly a *K-representation*.¹

Orbits of a Lie group in the space of a *K*-representation are symplectic manifolds. They can be interpreted as phase spaces of a Hamiltonian mechanical system for which the given Lie group is the group of symmetries. In 15.2, we shall give a classification of all homogeneous symplectic manifolds with a given group of symmetries.

It turns out that unitary irreducible representations of the group G are connected with orbits of this group in the *K*-representation. The construction of the representation in an orbit is given in 15.3.

This is a generalization of the procedure of quantization that is used in quantum mechanics. This point of view is explained in more detail in 15.4.

The author sees the significance of the method of orbits not only in the specific theorems obtained by this method, but also in the great collection of simple and intuitive heuristic rules that give the solution of the basic questions of the theory of representations. With the passage of time, these rules will be elevated to the level of strict theorems, but already now their value is indisputable.

We shall show in 15.5 how the operations of restriction to a subgroup and induction from this subgroup can be described with the aid of the natural projection $p: \mathfrak{g}^* \rightarrow \mathfrak{h}^*$, where \mathfrak{h} is the Lie algebra of the subgroup H .

As we shall see in 15.6, generalized characters of irreducible unitary representations admit a simple expression in the form of an integral over the corresponding orbit. In many cases, this allows us to write an explicit expression for the Plancherel measure.

Finally, in 15.7 we show that infinitesimal characters of irreducible unitary representations of a group G can be computed as values of G -invariant polynomials on the corresponding orbits.

15.1. The Co-Adjoint Representation of a Lie Group

Let G be a Lie group, \mathfrak{g} its Lie algebra, and \mathfrak{g}^* the dual space to \mathfrak{g} . The group G acts in \mathfrak{g}^* with the aid of the adjoint representation Ad (see 6.3) and in \mathfrak{g}^* with the aid of the co-adjoint representation, or, briefly, the *K*-representation. If the Lie algebra \mathfrak{g} is realized in the form of the algebra of left-invariant vector-fields on G , then it is natural to realize \mathfrak{g}^* in the form of the space of left-invariant differential forms of the first order on G . The *K*-representation of the group G acts in the space of 1-forms by right translations.

A. KIRILLOV

ÉLÉMENTS DE LA THÉORIE DES REPRÉSENTATIONS



ÉDITIONS MIR • MOSCOU

THALES
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A.A. Kirillov by Jacques Dixmier

Brèves remarques sur l'œuvre de A. A. Kirillov

Jacques Dixmier

La thèse de Kirillov, parue en 1962, a suscité immédiatement beaucoup d'intérêt.

Soit G un groupe de Lie nilpotent simplement connexe. Soient $\mathfrak{g} = \text{Lie}(G)$, \mathfrak{g}^* l'espace vectoriel dual de \mathfrak{g} , dans lequel G opère par la représentation coadjointe. Soit \widehat{G} l'ensemble des orbites coadjointes. Soit $\widehat{\mathcal{G}}$ l'ensemble des représentations unitaires irréductibles de G (la plupart sont de dimension infinie). Alors Kirillov construit une bijection canonique $\mathfrak{g}^*/G \rightarrow \widehat{\mathcal{G}}$; ainsi, les représentations unitaires irréductibles de G sont paramétrées par les orbites coadjointes. En outre, quantité de notions naturelles concernant les représentations s'interprètent géométriquement en termes d'orbites coadjointes: restriction à un sous-groupe, induction unitaire, produit tensoriel, mesure de Plancherel, topologie de \widehat{G} . Détaillois seulement un résultat, d'une élégance extraordinaire: soient $\pi \in \widehat{G}$ et Ω l'orbite associée; soit μ la mesure G -invariante canonique sur Ω (c'est une distribution tempérée sur \mathfrak{g}^*); soit μ' sa transformée de Fourier, distribution tempérée sur \mathfrak{g} ; transportant μ' sur G par l'application exponentielle, on obtient le caractère-distribution de π !

Kirillov s'est vite convaincu, et il a convaincu la communauté mathématique, que cette "méthode des orbites" devait être applicable à des groupes bien plus généraux que les groupes nilpotents.

Il n'a pas hésité à aborder le cas des groupes de Lie connexes quelconques. Evidemment, des difficultés considérables ont surgi immédiatement. Néanmoins, Kirillov a indiqué une voie d'accès, qui ensuite a été largement utilisée.

Depuis plus de trente ans, Kirillov applique la méthode des orbites aux groupes de Lie-Cartan de dimension infinie. Par exemple, soit M une variété C^∞ compacte. Soit G le groupe des difféomorphismes de M , ou l'un des sous-groupes obtenus en prenant les éléments de G qui conservent une forme volume, ou une structure symplectique, ou une structure de contact. On considère l'algèbre de Lie \mathfrak{g} et son dual \mathfrak{g}^* , tous deux définis en tenant compte de la topologie. Aux orbites coadjointes sont associées des représentations unitaires irréductibles de G , du moins dans certains cas, par exemple si l'orbite est de dimension finie ou de codimension finie. Le groupe fondamental de ces orbites peut être non commutatif, par exemple être un groupe symétrique. Kirillov envisage parfois des groupes encore plus généraux (par exemple, le groupe $\text{Diff}_H(E)$ où E est un fibré principal de groupe structural H), et parfois au contraire étudie de manière très approfondie

des cas particuliers importants; par exemple, soient S le cercle, $\text{Diff}_+(S)$ le groupe des difféomorphismes de S conservant l'orientation; alors les orbites coadjointes pour ce groupe sont liées aux fonctions holomorphes univalentes dans le disque unité, et à certaines équations différentielles linéaires; Kirillov définit sur ces orbites des structures complexes kähleriennes invariantes, d'où des représentations de l'algèbre de Virasoro et du groupe de Virasoro-Bott.

Les démonstrations précédentes ont amené Kirillov à s'intéresser, dans plusieurs articles, à des questions de pure géométrie différentielle. Soient M une variété C^∞ , E un fibré en droites sur M , $\Gamma(E)$ l'ensemble des sections C^∞ de E . Un crochet de Lie $[,]$ sur $\Gamma(E)$ est dit local si $[s_1, s_2]$ est continue en (s_1, s_2) et si $\text{supp}[s_1, s_2] \subset \text{supp } s_1 \cap \text{supp } s_2$. Kirillov classefie ces crochets. Il détermine aussi, dans des cas généraux, quels sont les opérateurs multidifférentiels invariants par difféomorphismes (tels que la dérivation extérieure, ou le crochet ordinaire de deux champs de vecteurs).

Plus récemment, Kirillov a utilisé avec succès la méthode des orbites pour des groupes très différents, les $G_n(k)$ ($n = 1, 2, \dots$; k corps commutatif). (On note $G_n(k)$ le groupe des matrices $n \times n$ unipotentes triangulaires supérieures à éléments dans k .) La théorie est particulièrement poussée lorsque k est un corps fini. Elle amène à introduire une suite remarquable des polynômes en une variable.

Voici un autre thème longuement étudié par Kirillov en collaboration avec Gelfand. Soient G un groupe algébrique complexe, \mathfrak{g} son algèbre de Lie, U l'algèbre enveloppante de \mathfrak{g} . D le corps enveloppant de U . Soit $D_{n,k}$ le corps engendré par des indéterminées $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_k$, où tous les crochets sont nuls sauf $[x_i, y_j] = 1$. Alors Gelfand et Kirillov conjecturent en 1966 que D est isomorphe à un corps $D_{n,k}$. Dans les années suivantes, ils démontrent des cas particuliers de plus en plus nombreux de cette conjecture. Bien après, la conjecture sera reconnue comme fausse, mais elle aura suscité un grand nombre de travaux intéressants. Surtout, dès l'article de 1966, Gelfand et Kirillov introduisent diverses notions de dimensions non commutatives; l'une d'entre elles, promise à un grand avenir, sera appelée dimension de Gelfand-Kirillov.

Il n'est pas surprenant que Kirillov ait étudié, pour elles-mêmes, les algèbres de Lie de dimension infinie. Dans un série d'articles, il a mis en évidence des cas où la "croissance" de ces algèbres est strictement intermédiaire entre la croissance polynomiale et la croissance exponentielle et il a construit des identités remarquables vérifiées dans l'algèbre de Lie des champs de vecteurs sur \mathbb{R} , ou des champs hamiltoniens sur \mathbb{R}^2 .

La place me manque pour analyser les nombreux articles isolés abordant des thèmes sur lesquels Kirillov n'est pas revenu par la suite. Mentionnons tout de même 3 de ces articles: 1) il a complété sur un point très important l'étude (par Gelfand-Graev) des représentations unitaires irréductibles de $\text{SL}_2(k)$ (k corps localement compact non connexe). 2) Soient \mathcal{H} un espace hilbertien, G le groupe des opérateurs unitaires u dans \mathcal{H} tels que $1 - u$ soit compact. Alors Kirillov a déterminé les représentations unitaires irréductibles de G . 3) Kirillov a résolu un problème essentiel de géométrie intégrale posé par Gelfand. Ces études n'utilisent

pas la méthode des orbites.

Kirillov a écrit plusieurs livres. Deux d'entre eux ont été traduits en français et en anglais: 1) *Eléments de la théorie des représentations*, 2) *Théorèmes et problèmes d'analyse fonctionnelle*. Il a d'autre part rédigé beaucoup d'articles d'exposition: chacun d'eux est en réalité un court livre. La clarté des exposés fait que ces écrits ont été largement lus, et notamment, avec profit, par des chercheurs débutants.

C'est l'occasion de signaler que Kirillov a eu de nombreux élèves de thèse. Les organisateurs de ce colloque ont eu la bonne idée de faire imprimer sur un T-shirt (offert à Kirillov) la liste de ces élèves: 57 noms, un record !

Kirillov a participé, en 1967, à la fondation d'un journal célèbre: *Functional Analysis and its Applications*. Il a, pendant 4 ans, été vice-président de la Société Mathématique de Moscou. Il est membre de comités éditoriaux de plusieurs journaux. Il a suscité la traduction en russe de nombreux livres et articles édités à l'ouest.

Dans sa conversation, Kirillov mentionne le soutien constant qu'il a reçu de son épouse (qui travaille en informatique). Il est fier de l'œuvre mathématique très connue de son fils.

Ce colloque honore donc un grand mathématicien: grand mathématicien par les voies qu'il a ouvertes, grand mathématicien par l'influence qu'il exerce depuis des années.

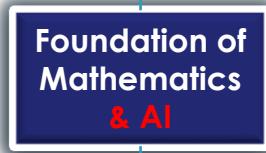
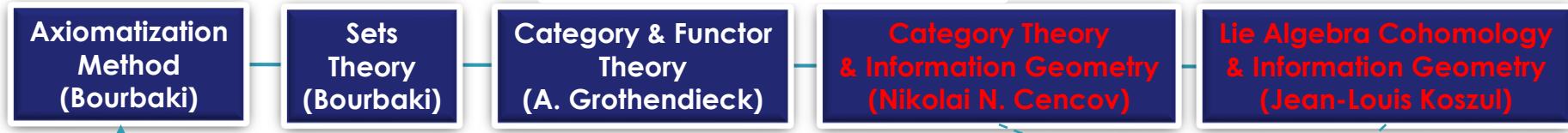
Jacques Dixmier
11 bis rue du Val de Grâce
F-75005 Paris, France

"Kirillov's thesis, published in 1962, immediately aroused a great deal of interest... Kirillov quickly convinced himself, and he convinced the mathematical community, that this "orbit method" should be applicable to groups much more general than nilpotent groups." – Jacques Dixmier

Mathematical Foundations of AI

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CATEGORY THEORY



SYMPLECTIC & FOLIATION STRUCTURES



AI Mathematical Foundations: Symplectic Foliations

Lie Groups & Lie Algebra

Representation Theory

Lie Algebra Cohomology, Co-adjoint Orbits, Cocycle, Quivers

Symplectic Geometry & Poisson Geometry

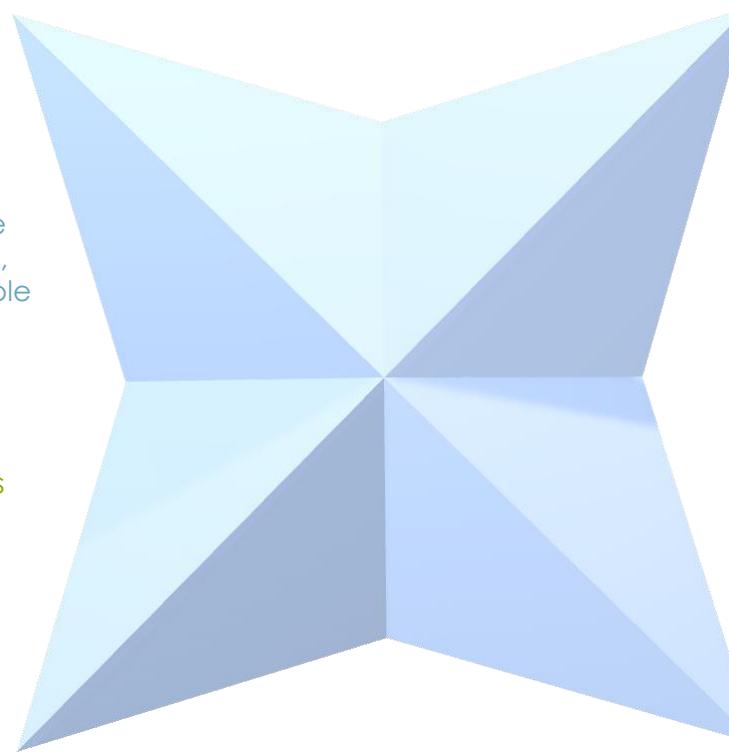
Symplectic Manifold associated to Lie Group Co-adjoint Orbit by KKS 2 Form, Moment Map, Souriau's Maxwell Principle

Integrable System

complete integrability, Liouville integrability, Action-Angles Coordinates, Symplectic Integrators

Foliation Theory

Coisotropic Polar, Bifibration, Bifoliation, Transverse Poisson Structure, Symplectic Foliation, Riemannian Foliation, webs



Information Geometry

Statistical Manifold, Koszul-Fisher Metric, Natural Gradient, Natural Langevin Dynamics, Coordinate Free Neural Network, Hessian Structures, Flat connexion without torsion

Metriplectic Flow

GENERIC Flow, Poisson Bracket, Metric Flow Bracket, 1st & 2nd Thermodynamics Principles, Onsager-Casimir Relations, Dissipation, Transverse Foliations of Entropy and Energy levels sets

Calculus of Variations

Geometrization of Noether Theorem, Poincaré-Cartan Integral Invariant, Euler-Poincaré Equation

Souriau's Lie Groups Thermodynamics

Covariant Gibbs Density, Planck Temperature in Lie Algebra, Heat in dual Lie Algebra, Entropy as Casimir function, Souriau-Fisher Metric

Symplectic Integrator

- Capture of symmetries (Geometry-Informed)
- Capture Noether Invariants via Souriau's Moment Map as Energy, Angular Momentum (Physics-Informed)

$$\frac{dF}{dt} = \{F, H\}$$

non-dissipative

(*) The time evolution of Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}$$

is a symplectomorphism, meaning that it conserves the symplectic 2-form $dp \wedge dq$. A numerical scheme is a symplectic integrator if it also conserves this 2-form and equation are reduced to:

$$z = (p, q) \Rightarrow \dot{z} = \{z, H(z)\}$$

>Metriplectic Integrator

- Capture 1st principle (Energy preservation, Moment map)
- Capture 2nd principle (Entropy Production)

$$\frac{dF}{dt} = \{F, H\} + (F, S)$$

dissipative

$$F = H + S$$

(*) 1st principle: Preservation of Energy

$$\frac{dH}{dt} = 0$$

2nd Principle Entropy production

$$\frac{dS}{dt} \geq 0$$

Metriplectic Flow on Symplectic Foliation & Transverse Metric Foliation

In whole or in part
reserved.

Foliation Leaves =
Level Sets of Energy

METRIC FOLIATION

Foliation Leaves =
Level Sets of Entropy

SYMPLECTIC FOLIATION

1st Principle of
Thermodynamics
Preservation of Energy

2nd Principle of
Thermodynamics
Entropy Production

METRIPLECTIC FLOW

$$\frac{dF}{dt} = \{F, H\} + (F, S)$$

Non-dissipative
Entropy = Constant

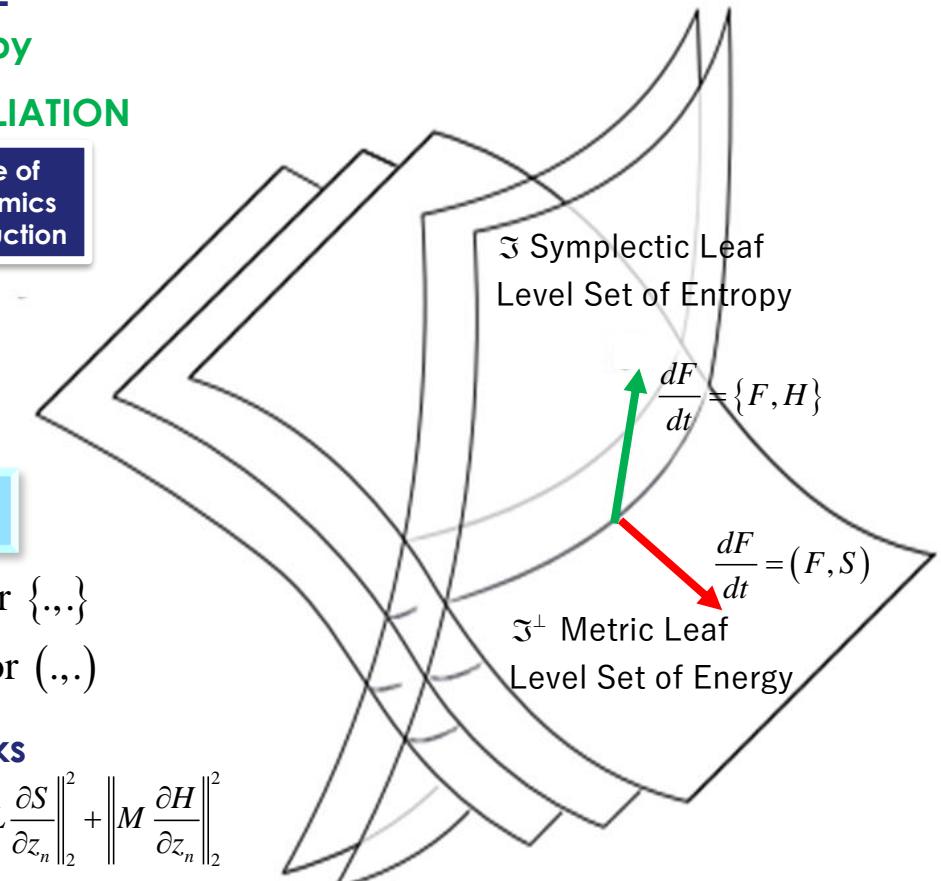
Dissipative
Energy = Constant

$\{F, S\} = 0 \quad \forall F : S$ Entropy Casimir Function for $\{.,.\}$

$(F, H) = 0 \quad \forall F : H$ Energy Casimir Function for $(.,.)$

Thermodynamics-Informed Neural Networks

$$\Lambda = \frac{1}{N_{batch}} \sum_{n=0}^{N_{batch}} (\lambda \Lambda_n^{data} + \Lambda_n^{deg}) \quad \Lambda_n^{data} = \left\| \frac{dz^{GT}}{dt} - \frac{dz^{net}}{dt} \right\|_2^2 \quad \Lambda_n^{deg} = \left\| L \frac{\partial S}{\partial z_n} \right\|_2^2 + \left\| M \frac{\partial H}{\partial z_n} \right\|_2^2$$



GEOMETRIC DEFINITION OF ENTROPY

ENTROPY DEFINITION: Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.

$$\{S, H\}_{\tilde{\Theta}}(Q) = 0$$

$$\beta \in Ker \tilde{\Theta}_\beta \Rightarrow \left\langle Q, \left[\frac{\partial S}{\partial Q}, Z \right] \right\rangle + \tilde{\Theta} \left(\frac{\partial S}{\partial Q}, Z \right) = 0$$

$$ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta \left(\frac{\partial S}{\partial Q} \right) = 0$$

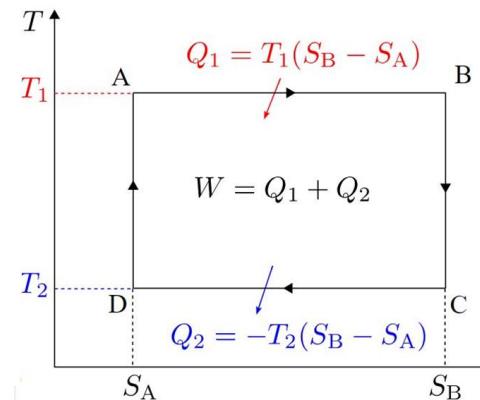
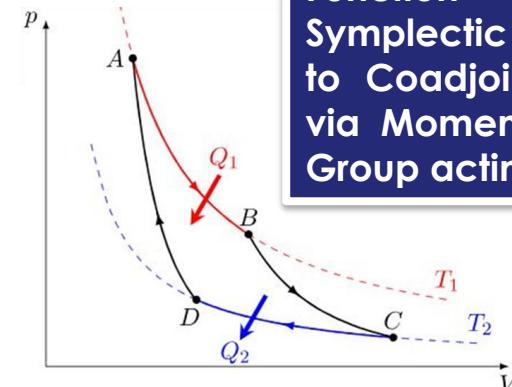
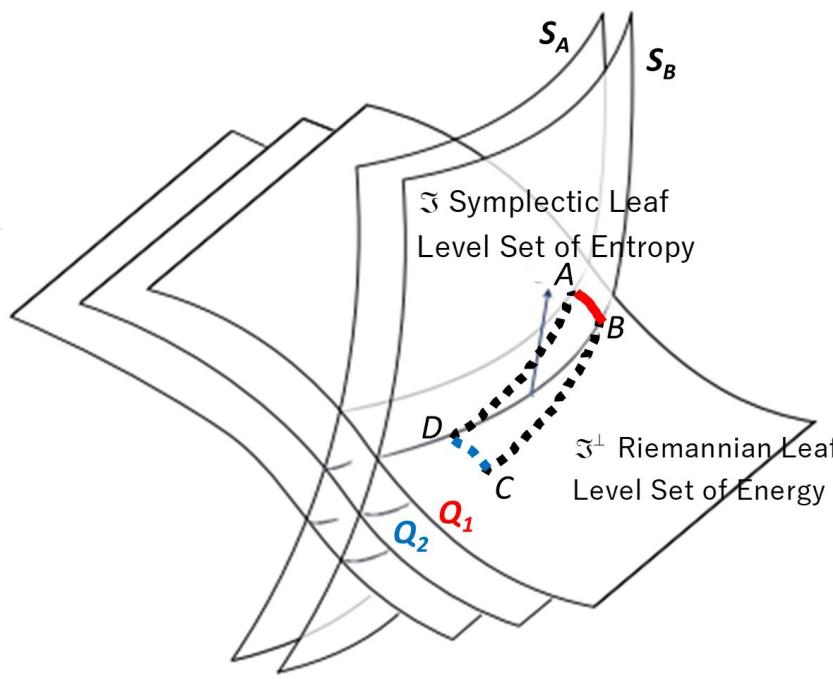
Sadi Carnot Cycle on Souriau's Symplectic Leaves

$$\langle Q, [\beta, Z] \rangle + \tilde{\Theta}(\beta, Z) = 0$$

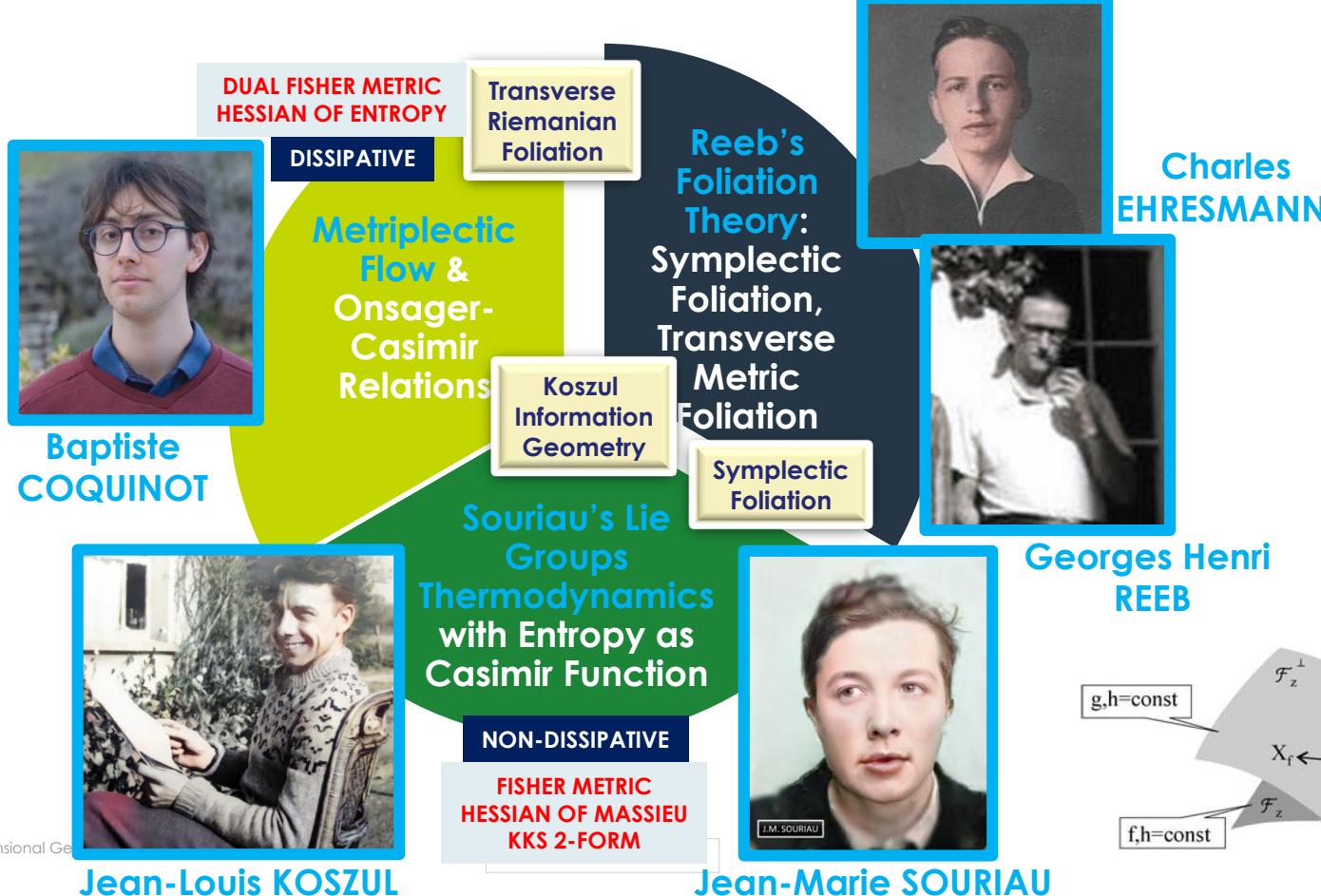
$$\{S, H\}_{\tilde{\Theta}}(Q) = 0$$

$$\beta \in \text{Ker } \tilde{\Theta}_\beta \Rightarrow \left\langle Q, \left[\frac{\partial S}{\partial Q}, Z \right] \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, Z \right) = 0$$

$$ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$



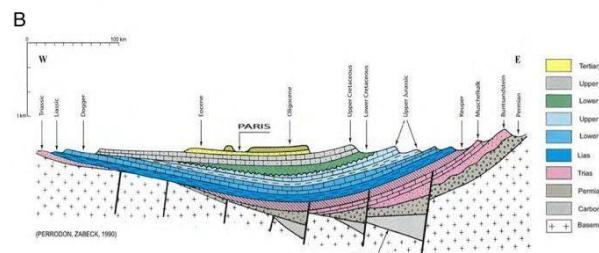
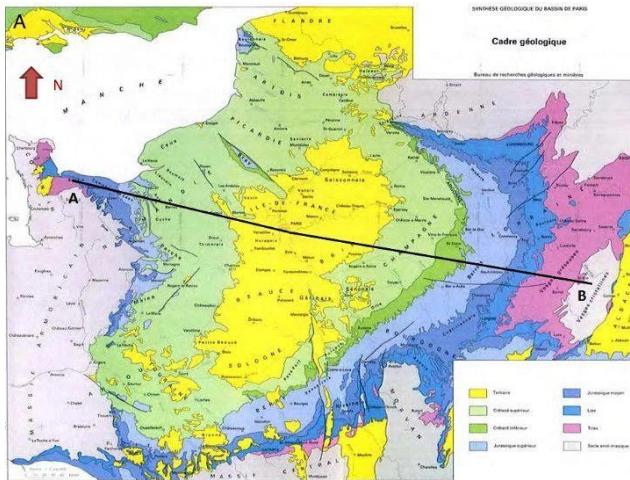
Thermodynamics Triptych : symplectic & transverse metric foliations



« Nord-Bassin Parisien » Seminar on « Geometric Structures of Dissipation »

« Nord-Bassin Parisien » Seminar 2020's

- Paris-Metz-Valenciennes-Lille-Louvain-Liège-Bruxelles



« Sud-Rhodanien » Seminar 1980's

- Lyon, Marseille, Montpellier



« L'oubli et la mémoire sont également inventifs »
J. L. Borges, Le Rapport de Brodie

Moment Map Geometry & Libermann Foliation: Condevaux, Dazord, Molino

Séminaire Sud-Rhodanien

CHAPITRE V

GEOMETRIE DU MOMENT

M. Condevaux, P. Dazord, P. Molino

Introduction

Le but de cet exposé est double : d'une part définir le cadre naturel dans lequel se généralise l'étude du moment d'une action hamiltonienne ; d'autre part donner une démonstration nouvelle, plus directement inspirée par l'intuition géométrique que les démonstrations connues, des théorèmes de convexité du moment d'Atiyah-Guillemin-Sternberg-Kirwan [A] [G-S]₁, [G-S]₂, [K].

L'idée de base de la première partie est la suivante : la dualité symplectique permet à la fois de faire correspondre à un feuilletage symplectiquement complet au sens de P. Libermann [Lib], un feuilletage orthogonal, et d'associer à l'action hamiltonienne d'un groupe de Lie un moment au sens de J.M. Souriau [So]. Une théorie générale du moment, englobant ces deux situations comme cas particuliers, devra rendre compte de la correspondance établie par dualité symplectique entre certains types de feuilletages singuliers de P. Stefan [St] et H.. Süssmann [Su] et certaines Γ -structures d'A. Haefliger [H]. C'est ce qui nous amène à introduire la notion de Γ -structure Poissonienne ; une telle structure est le moment généralisé d'un feuilletage de P. Stefan "symplectiquement complet". On montre que les deux cas particuliers évoqués rentrent bien dans ce cadre ; en fait, plus généralement, toute action symplectique d'un groupe de Lie admet en ce sens un moment généralisé. Par un procédé élémentaire de "développement de Darboux", on définit pour une telle action symplectique un moment réduit, à valeurs dans un cylindre muni d'une structure de Poisson affine ; il semble que l'étude de ces différentes notions puisse conduire à quelques problèmes intéressants.

En ce qui concerne les théorèmes de convexité du moment, on traite d'abord le cas des tores à partir de l'idée suivante : la description élémentaire des modèles locaux conduit immédiatement à un résultat de convexité locale : si Γ^k agit de manière hamiltonienne sur (M, σ) , on en déduit facilement que l'espace \hat{M} des composantes connexes des préimages des points par le moment $J : M \rightarrow \mathbb{R}^k$ a une structure naturelle de variété affine plate à bords et coins affines localement convexe au voisinage du bord. L'appli-

CHAPITRE II

Γ -STRUCTURES POISSONNIENNES ET FEUILLETAGES DE LIBERMANN

P. DAZORD & P. MOLINO

INTRODUCTION -

Les feuilletages de Libermann sont la généralisation, dans le cadre des feuilletages de Stefan, des feuilletages étudiés dans le cas régulier par P. Libermann [8] sous le nom de feuilletages symplectiquement complets et connus des mécaniciens sous le nom de "système complet d'intégrales premières". Ceci justifie le nom donné à cette catégorie de feuilletages

L'étude de ces feuilletages a été clarifiée par l'introduction de la notion de Γ -structure poissonienne qui est l'extension maximale de la notion de moment au sens de J.M. Souriau [10]. (cf. Chapitre V "Géométrie du Moment"). La notion de feuilletage de Libermann apparaît alors comme duale de celle de Γ -structure poissonienne. C'est le point de vue adopté ici.

Dans un dernier paragraphe on utilise la théorie des feuilletages de Libermann pour donner une preuve du théorème de structure locale de A. Weinstein [11] pour les variétés de Poisson.

DUALITE SYMPLECTIQUE, FEUILLETAGES ET GEOMETRIE DU MOMENT

P. MOLINO

Abstract

One gives the links between the notions of singular foliations, Γ -structures and momentum mapping in the context of symplectic geometry.

Introduction

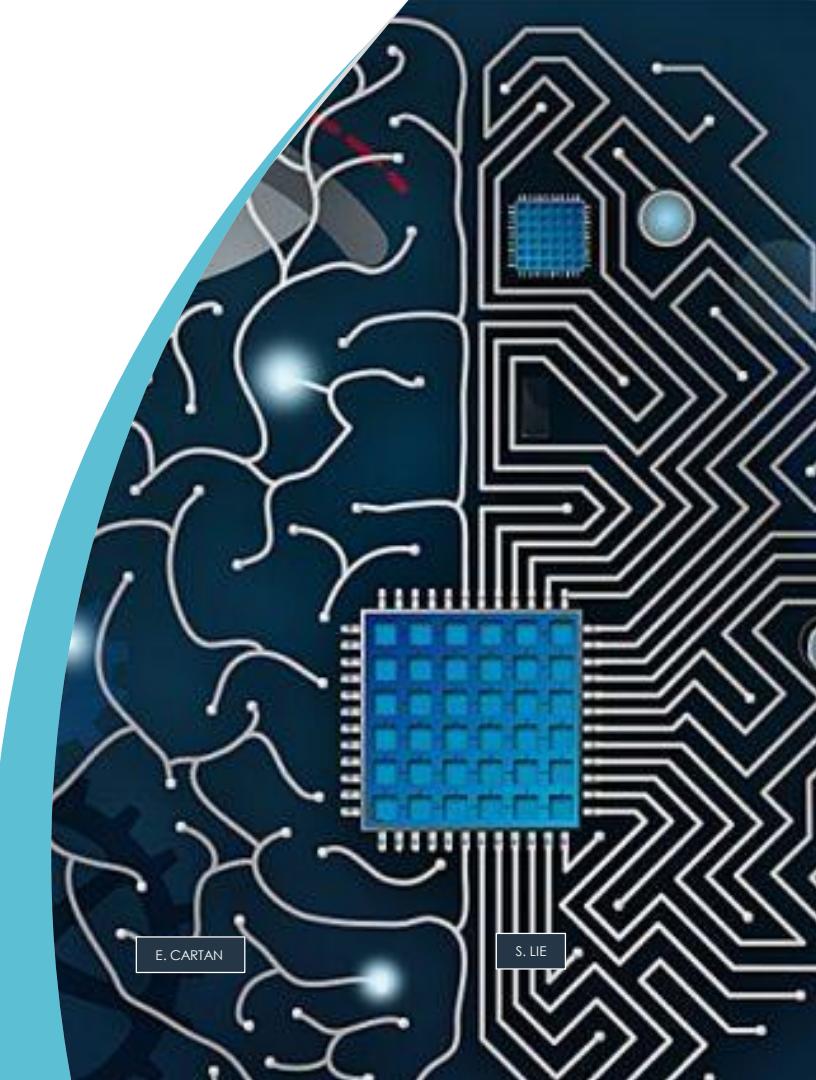
Dans le premier paragraphe de ce papier, après quelques rappels élémentaires de géométrie symplectique, on introduit la notion de *moment généralisé* mise au point en collaboration avec M. Condevaux et P. Dazord [3] dans le cadre du Séminaire Sud-Rhodanien de Géométrie. Essentiellement, c'est la traduction en termes de dualité symplectique entre Γ -structures de Haefliger (ici, les Γ -structures Poissonniennes) et feuilletages singuliers de Stefan (ici les feuilletages de Stefan-Libermann), de la notion de moment d'une action hamiltonienne due à J. M. Souriau [16].

Le second paragraphe est une contribution à la méthode de réduction symplectique [13]: (M, ω, H) étant un système hamiltonien, on considère une sous-variété Σ de M , qui est une sous-variété de niveau d'une famille d'intégrales premières. On suppose que ω induit sur Σ une forme ω_Σ de rang constant; mais au lieu de supposer le feuilletage caractéristique C_Σ simple, on suppose seulement qu'il admet une métrique *bundle-like* [15] complète. On peut alors introduire le *moment généralisé transverse* du feuilletage singulier défini par les adhésions des feuilles, et poursuivre la réduction en munissant l'espace Σ/\bar{C}_Σ des adhésions de feuilles d'une structure de Poisson.

Un cas particulier intéressant est celui où la variété Σ est simplement connexe; on peut alors définir un vrai *moment transverse structural* $J_\Sigma : \Sigma \rightarrow \mathbb{R}^k$, où k est la dimension de l'algèbre de Lie structurale [14]. On peut d'ailleurs prouver, si Σ est compacte, un théorème de convexité du moment structural du type du théorème de convexité du moment pour les actions hamiltoniennes de tores [2] [7].

Dans toute la suite, les structures considérées sont de classe C^∞ .

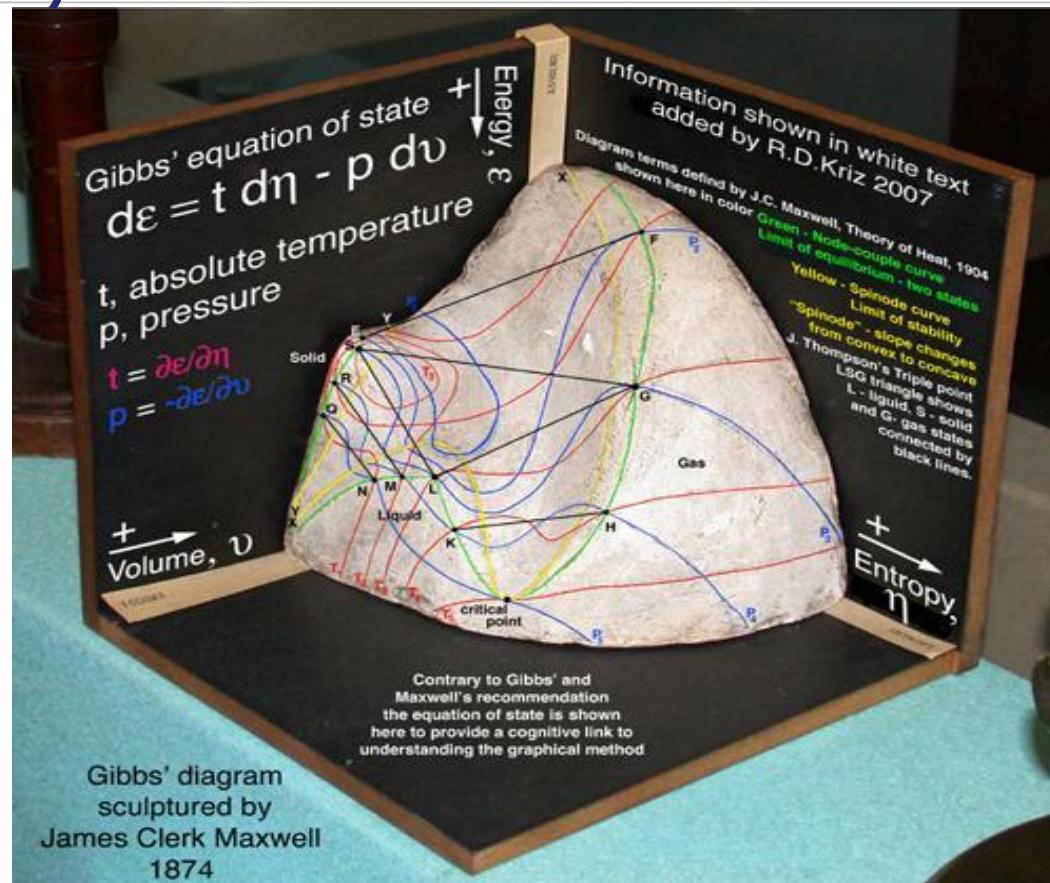
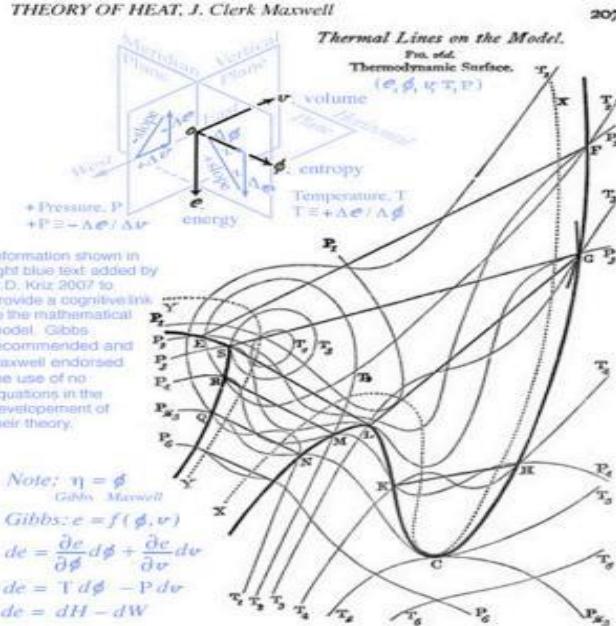
Seminal Idea of Gibbs and Carathéodory



Geometric Theory of Heat: Josiah Willard Gibbs Diagrams sculpted by James Clerk Maxwell (1874)

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THEORY OF HEAT. J. Clerk Maxwell



Sadi Carnot

[...] Very little has been studied until now of the **changes in temperature occurring in bodies as a result of movement**; However, this class of phenomena deserves the attention of observers. **When bodies are in motion, especially when motive power is consumed or produced, remarkable changes occur in the distribution of heat and perhaps in its quantity.** We will provide a small number of facts, where this phenomenon develops with the most evidence.

(Sadi Carnot 1824, Extracts from unpublished notes by Sadi Carnot on mathematics, physics and other subjects. Gallica, Paris)



Seminal Gibbs States from Moment Map of SE(3)

- As observed by Charles-Michel Marle, **Josiah Willard Gibbs** in chapter IV of his book "Elementary Principles in Statistical Mechanics, developed with Especial Reference to the Rational Foundation of Thermodynamics" published in 1902, considered **generalization of Gibbs states built with the moment map of the product of the one-dimensional group of translations in time and the three-dimensional group of rotations in space for the study of systems contained in a rotating vessel**, referring to a paper by Maxwell published in 1878. We can read in the book of Gibbs:

[...] The consideration of the above case of statistical equilibrium may be made the foundation of the theory of the thermodynamic equilibrium of rotating bodies, a subject which has been treated by Maxwell in his memoir On Boltzmann's theorem on the average distribution of energy in a system of material points (Gibbs 1902)

- **Jacques Hadamard** made a review of this book in 1906 (Hadamard 1906) and wrote:

[...] This book is not one of those that one analyzes hastily; but, on the other hand, the questions it deals with have been greatly agitated in recent times; the ideas defended by Gibbs have been the subject of much controversy; the reasoning with which he supported them has also been criticized. It seems interesting to me to study his work in the light of these controversies and by discussing these criticisms (Hadamard 1906)

Seminal Carathéodory's axiomatization of Thermodynamics



[...] In any arbitrary neighborhood of an arbitrarily given initial point there is a state that cannot be arbitrarily approximated by adiabatic changes of state.
(Carathéodory 1909)

Constantin Carathéodory was born in Germany in Berlin in 1873 but was the son of a Turkish ambassador of Greek origin. From 1875 to 1895, Constantin Carathéodory lived in Bruxelles in Belgium, and studied at Belgian École Militaire. After a period in Greece, in London and Egypt, he returns to Berlin in 1900 to study mathematics.

In 1904 in Göttingen, he defended his PhD on special Euler–Lagrange equations, with H. Minkowski,. He was successively professor in Göttingen, Bonn and Hannover. From 1905 to 1910. From Brussels, Carathéodory wrote to Born in 1907 about his definition of the concepts “amount of heat” and “reversibility” derived from the Carnot principle. He was back in Munich in 1924 to work on variational calculus applied to optics

Pfaffian forms by Gaston Darboux

Sur le problème de Pfaff

Gaston Darboux (1822)

[...] The method that Pfaff made known in 1814, in the Memoirs of the Academy of Berlin, for the integration of a partial differential equation with any number of independent variables, was long neglected: the beautiful discoveries of Jacobi and de Cauchy have alone attracted the attention of geometers who deal with this theory. ... In the first Part, I study the reduced forms, and I show that the integration of the first Pfaff system is sufficient and immediately gives the reduced form when it comes to the differential expression corresponding to a partial differential equation. In the second Part, I study the relations between reduced forms, and I demonstrate in particular three propositions which serve as a basis for Mr. Lie's group theory.



Johann Friedrich
Pfaff

Pfaff problem and Pfaffian form attracted the attention of Clebsch (1862), Darboux (1882), Lie and Cartan (1899), among other as explained in the book of Edouard Goursat.

Gaston Darboux



Carathéodory's axiomatization of Thermodynamics

The main Axiom of Constantin Carathéodory of the 2nd law of Thermodynamics is written in 1909 paper:

[...] In jeder beliebigen Umgebung eins vorgeschriebenen Anfangszustandes gibt es Zustände, die durch adiabatische Zustandsänderungen nicht beliebig approximiert werden können [In the neighborhood of any equilibrium state of a system (of any number of thermodynamic coordinates), there exists states that are inaccessible by reversible adiabatic processes]. (Carathéodory 1909)

By analyzing the Pfaffian form, Carathéodory showed that if no heat exchange occurs, a system's adiabatic paths can be described mathematically, and implies the existence of an integrating factor related to temperature, leading to the definition of entropy. The integrating factor transforms the Pfaffian form into an exact differential, dS , which defines the change in entropy.

- For the first law, under any adiabatic change of state, the external work A by the change in energy is equal to zero and if we denote the initial ε and final $\bar{\varepsilon}$ energy:

$$\bar{\varepsilon} - \varepsilon + A = 0$$

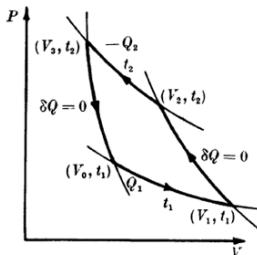
- For the second law, under all adiabatic changes of state that start from any given initial state, certain final states are not attainable and “unattainable” final states can be found in any neighborhood of the initial state.

Wolfgang Pauli Lectures on “Thermodynamics and the Kinetic Theory of Gases” – Rereading of Carathéodory Axiomatization

› Heat conduction and Internal Friction are two main causes of dissipation

› [...] If the second law were untrue, that is, if heat conduction were reversible, then work could be obtained from heat without compensating changes. Since it is only possible to convert work into heat without compensating changes, is the preferred direction in which actual processes take place. We can now also determine which of two temperature t_1 and t_2 is the higher, namely, that one from which heat can flow to the other without compensating changes.

(Pauli 1973)



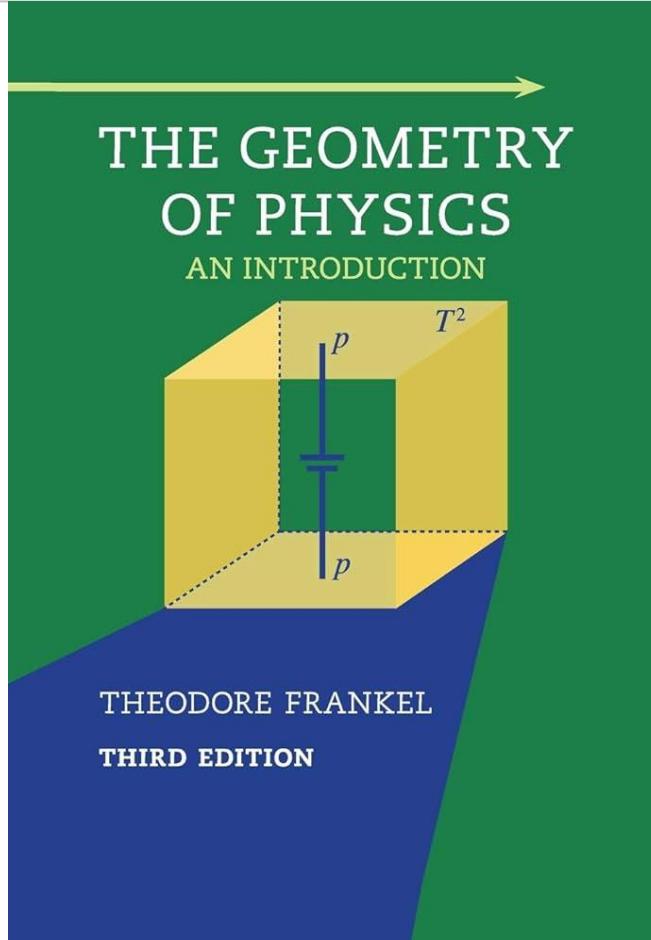
Wolfgang Pauli Lectures on Physics, Volume 3
“Thermodynamics and the Kinetic Theory of Gases”, 2nd chapter dedicated to the 2nd Law:
[...] The second law distinguishes heat from the other forms of energy. It indicates a direction in time and makes apparent that heat is a disordered form of energy. (Pauli 1973)

Constantin Carathéodory axiomatization of thermodynamics interpreted by foliations by Theodore Frankel

- Theodore Frankel (Frankel 2012) was one of first to interpret Carathéodory model with theory of foliations and Frobenius's theorem. In chapter entitled '**Heuristic Thermodynamics via Caratheodory (Can one go adiabatically from some state to any nearby state?)**' of his book "**The Geometry of Physics: An Introduction**", Frankel identified entropy as a **global adiabatic foliation of state space, guaranteed by the holonomy of entropy.**
- Theodore Frankel then rewrite Carathéodory results in a modern geometric framework
Caratheodory's Theorem: Let θ^1 be a continuously differentiable nonvanishing 1-form on an M^n , and suppose that $\theta=0$ is not integrable; thus at some $x_0 \in M^n$ we have $\theta \wedge d\theta \neq 0$. Then there is a neighborhood U of x_0 such that any $y \in U$ can be joined to x_0 by a piecewise smooth path that is always tangent to the distribution.
- Frankel concluded that the adiabatic distribution $Q^1 = 0$ is integrable and that there are locally defined a local entropy function S and τ on the state space M^{n+1} such that $Q^1 = \lambda dS$ that admit a local integrating factor τ .

Constantin Carathéodory axiomatization of thermodynamics interpreted by foliations by Theodore Frankel

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[...] If, for example, **the foliation defined by $Q=0$ has leaves that wind densely (as in a torus) then there is no way that a global function S can exist, since such an S must be constant on each maximal leaf.** It is easy to see, however, that Kelvin's second law of thermodynamics rules out the possibility of not only dense adiabatic leaves, but even leaves that "double back"! For in the proof that "Kelvin implies Caratheodory," we saw that two states related by heating at constant volume cannot be joined by a quasi-static adiabatic. This says that no $\pi^{-1}(v)$ can meet a maximal adiabatic leaf twice. It might be thought that the space M^{n+1} and the adiabatic foliation must then be of a completely trivial nature. (**Frankel 1972, p.33**)

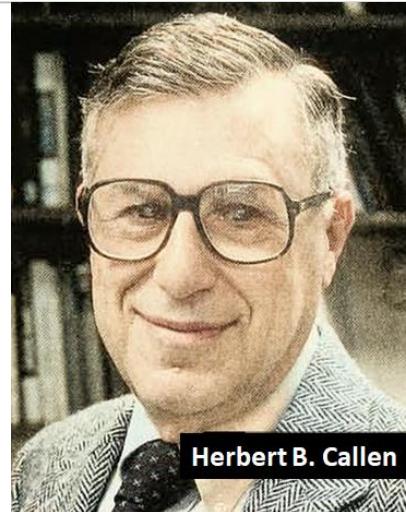
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Thermodynamics as a Science of Symmetry by Herbert B. Callen

► [...] every continuous symmetry of a system implies a conservation theorem, and vice versa ... The most primitive class of symmetries is the class of continuous spacetime transformations. The (presumed) invariance of physical laws under time translation implies the conservation of energy. **Symmetry under spatial translation implies conservation of momentum, and rotational symmetry implies conservation of angular momentum.** (Callen, p.425)

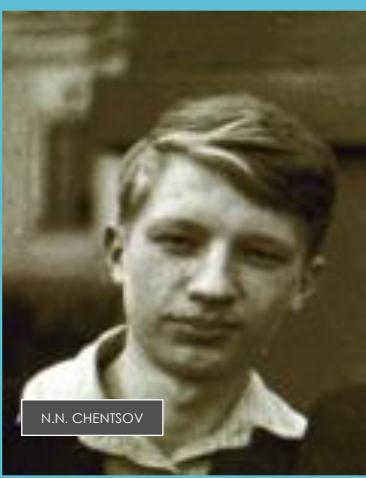
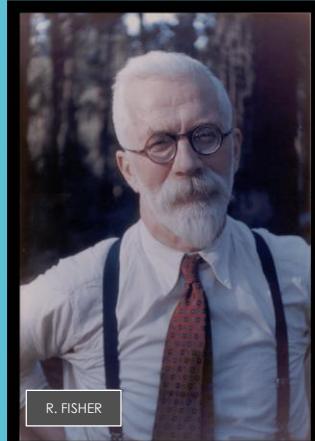
[...] The most immediately evident conserved coordinate is, of course, the energy (time-translation symmetry). **Its relevance as a thermodynamic coordinate under-lies the "first law" of thermodynamics. Time-translation, spatial translation, and spatial rotation symmetries are interrelated in a single class of continuous space-time symmetries.** The symmetry interpretation of thermodynamics immediately suggests, then, that **energy, linear momentum, and angular momentum should play fully analogous roles in thermodynamics.** The equivalence of these roles is rarely evident in conventional treatments, which appear to grant the energy a mis-leadingly unique status. **The momentum and the angular momentum are generally suppressed by restricting the theory to systems at rest, constrained by external "clamps."** Nevertheless, it is evident that in principle **the linear momentum does appear in the formalism in a form fully equivalent to the energy**, for relativistic considerations imply that the energy in one frame appears partially as linear momentum in another frame. Similarly, the angular momentum is only occasionally introduced explicitly into thermodynamic formalisms (as in astrophysical applications to rotating galaxies); it appears, for instance, in the "Boltzmann factor," $\exp(-\beta E - \beta \Lambda \cdot L)$, additively and symmetrically with the energy. To stress these facts we might well amend the first law to read that "**the extended first law of thermodynamics is the symmetry of the laws of physics under space and time translations and under spatial rotation.**". (Callen 1974 p.427)

- Callen HB (1973) A Symmetry Interpretation of Thermodynamics. In: Domingos JDD (eds.). Foundations of Continuum Thermodynamics, Instituto de Alta Cultura-Núcleo de Estudos de Engenharia Mecânica, pp. 61–79.
- Callen, HB (1974) Thermodynamics as a Science of Symmetry, Foundations of Physics 4(4):423–443.
- Callen HB (1985) Thermodynamics and An Introduction to Thermostatistics, 2nd edition, Wiley, New York.



Herbert B. Callen

Information Geometry & Natural Gradient for Learning





Maurice Fréchet



ENS PARIS 1900

Maurice Fréchet, papier de 1943 et cours de l'IHP de l'Hiver 1939

M. Fréchet, Sur l'extension de certaines évaluations statistiques au cas de petits échantillons,
Revue de l'Institut International de Statistique, Vol. 11, No. 3/4 (1943), pp. 182-205

SUR L'EXTENSION DE CERTAINES EVALUATIONS
STATISTIQUES AU CAS DE PETITS ECHANTILLONS
par Maurice Fréchet.

Introduction.

Ce mémoire¹⁾ est consacré à l'extension au cas de petits échantillons de la méthode de détermination empirique d'un paramètre basée sur le principe de la moindre dispersion et à sa comparaison avec les méthodes basées sur le principe de la valeur dominante et sur celui de la plus grande plausibilité.

Si nous nous en étions tenus aux démonstrations, nous aurions pu abréger sensiblement ce mémoire. Mais il nous a paru nécessaire d'entrer dans plus de détails qu'on ne le fait généralement, afin de séparer plus nettement des déductions mathématiques, les hypothèses et les conventions sur lesquelles elles reposent et dont le choix, aussi plausible que possible, n'a cependant rien de nécessaire.

Notations. — Soient X_1, X_2, \dots, X_n , n valeurs prises par une variable aléatoire X au cours de n épreuves indépendantes. On se limitera, dans la suite, au cas où la loi de répartition de X peut s'exprimer par une probabilité élémentaire δdx et où, de plus, la densité de probabilité à un point x est une fonction d'une forme connue $f(x, \theta)$, dépendant d'un paramètre dont la valeur vraie θ_0 est inconnue.

On se propose d'évaluer θ_0 connaissant d'une part, la forme $f(x, \theta)$ de δ et, d'autre part, les résultats de n épreuves qui ont donné les valeurs numériques x_1, x_2, \dots, x_n à X_1, X_2, \dots, X_n . Sous cette forme stricte, le problème ne peut être résolu par de simples déductions mathématiques.

Il s'agit donc de fixer certaines conventions plausibles qui assigneront à tout „échantillon” de n valeurs x_1, x_2, \dots, x_n de X une valeur déterminée t , laquelle sera prise comme valeur empirique de la valeur vraie θ_0 . t est donc une certaine fonction convenablement choisie de x_1, x_2, \dots, x_n .

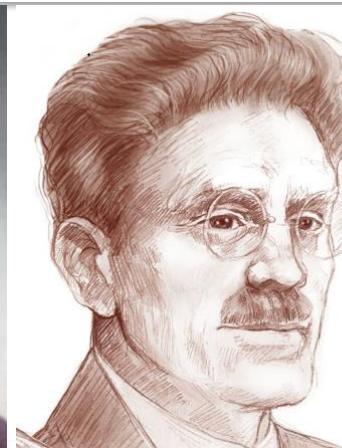
(1) $t = H_n(x_1, \dots, x_n).$

On voit que chaque échantillon détermine t , de sorte que

(2) $T_n = H_n(X_1, \dots, X_n)$

est une variable aléatoire dont chaque échantillon détermine une valeur. (Quand, dans nos raisonnements, n sera fixe, nous écrirons, pour simplifier, H au lieu de H_n et T au lieu de T_n).

¹⁾ Le contenu de ce mémoire a formé une partie de notre cours de statistique mathématique à l'Institut Henri Poincaré pendant l'hiver 1939–1940. Il constitue l'un des chapitres du deuxième cahier (en préparation) de nos „Leçons de Statistique Mathématique”, dont le premier cahier „Introduction: Exposé préliminaire de Calcul des Probabilités” (119 pages in-quarto, dactylographiées) vient de paraître au „Centre de Documentation Universitaire”, Tournous et Constans, Paris.



Manuscrit perdu du cours de statistique mathématique à l'Institut Henri Poincaré pendant l'Hiver 1939-1940 !

¹⁾ Le contenu de ce mémoire a formé une partie de notre cours de statistique mathématique à l'Institut Henri Poincaré pendant l'hiver 1939–1940. Il constitue l'un des chapitres du deuxième cahier (en préparation) de nos „Leçons de Statistique Mathématique”, dont le premier cahier „Introduction: Exposé préliminaire de Calcul des Probabilités” (119 pages in-quarto, dactylographiées) vient de paraître au „Centre de Documentation Universitaire”, Tournous et Constans, Paris.

Maurice Fréchet 1943 Seminal Paper (Clairaut Equation)

Fréchet, M. Sur l'extension de certaines évaluations statistiques au cas de petits échantillons. Revue de l'Institut International de Statistique 1943, 11, 182–205.

Etude des densités distinguées. Appelons (provisoirement, dans ce mémoire) **densité distinguée**, toute densité de probabilité $f(x, \theta)$ telle que la fonction

$$(46) \quad \theta + \frac{\frac{\partial L f(x, \theta)}{\partial \theta}}{\int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial \theta} f(x, \theta) \right]^2 \frac{dx}{f(x, \theta)}}$$

soit indépendante de θ .

Pour ces densités distinguées, on va pouvoir déterminer la fonction minimisante $H'(X_1, \dots, X_n)$ et étendre au cas des petits échantillons la comparaison des méthodes d'estimation faites par divers auteurs dans le cas des grands échantillons. Il vaut donc la peine de chercher la forme générale de $f(x, \theta)$ pour cette catégorie de variables.

de θ . En appelant $h(x)$ cette fonction, on voit qu'on a l'identité de la forme

$$(47) \quad \lambda(\theta) \frac{\partial}{\partial \theta} L f(x, \theta) = h(x) - \theta$$

où $\lambda(\theta) > 0$. On peut considérer $\frac{1}{\lambda(\theta)}$ comme la dérivée seconde d'une fonction $\mu(\theta)$; d'où $\frac{\partial}{\partial \theta} L f(x, \theta) = \mu''(\theta) [h(x) - \theta]$.

Par suite $L f(x, \theta) - \mu' \theta [h(x) - \theta] - \mu(\theta)$ est une quantité indépendante de θ que nous pouvons représenter par $l(x)$.

Ainsi toute densité distinguée, $f(x, \theta)$, est de la forme

$$(48) \quad f(x, \theta) = e^{\mu' \theta [h(x) - \theta] + \mu(\theta) + l(x)}$$

(52bis)

$$\lambda \mu'' = 1.$$

Incidemment, puisque, d'après (52), $\lambda(\theta)$ est positif, il en résulte aussi que $\mu'' \left(= \frac{1}{\lambda(\theta)} \right)$ est aussi positif. **Métrique de Fisher**

On peut d'ailleurs préciser d'une manière plus directe que par (50), le choix des fonctions $\mu(\theta)$, $h(x)$, $l(x)$: on peut prendre arbitrairement $h(x)$ et $l(x)$ ¹⁾ et alors $\mu(\theta)$ est déterminé par (50) ou même mieux par une formule explicite. En effet, (50) peut s'écrire

$$e^{\theta \mu' - \mu} = \int_{-\infty}^{+\infty} e^{\mu' \theta h(x) + l(x)} dx.$$

Donnons-nous alors arbitrairement $h(x)$ et $l(x)$ et soit s une variable arbitraire: la fonction

$$\int_{-\infty}^{+\infty} e^{s h(x) + l(x)} dx \quad 1)$$

sera une fonction positive connue que nous pourrons représenter par $e^{\Psi(s)}$. On voit alors que $\mu(\theta)$ sera défini par

$$\theta \mu' - \mu = \psi(\mu') \quad \text{ou}$$

(55)

$$\mu = \theta \mu' - \psi(\mu') \quad \text{Legendre-Clairaut}$$

c'est-à-dire une équation de Clairaut. La solution $\mu' = \text{constante}$ réduirait $f(x, \theta)$, d'après (48) à une fonction indépendante de θ , cas où le problème n'aurait plus de sens. μ est donc donné par la solution singulière de (55), qui est unique et s'obtient en éliminant s entre $\mu = \theta s - \psi(s)$ et $\theta = \psi'(s)$ ou encore entre

$$e^{\theta s - \mu} = \int_{-\infty}^{+\infty} e^{s h(x) + l(x)} dx \text{ et}$$

$$(55\text{bis}) \quad \int_{-\infty}^{+\infty} e^{s h(x) + l(x)} [h(x) - \theta] dx = 0.$$

Si l'on veut, $\mu(\theta)$ est donné par la relation

$$e^{-\mu} = e^{-\theta s} \int_{-\infty}^{+\infty} e^{s h(x) + l(x)} dx$$

où s est donné en fonction de θ par la relation implicite (55bis).

Axiomatization of Information Geometry via Category Theory

Nikolai Nikolaevich Chentsov in 60's

Math. Operationsforsch. Statist., Ser. Statistics, Vol. 9 (1978) No. 2, 267–276

Algebraic Foundation of Mathematical Statistics²

N. N. Čencov¹.

It results from the following sentence which is a non-symmetric analogy to PYPHAGOR's theorem for the information deviation [10], [6].

Theorem 6. *Let the probability measures R be dominated by $\{P_s\}$, which form the exponential family. If there is such distribution $P_\sigma \in \{P_s\}$ that*

$$\int_{\Omega} \left[\ln \frac{dP_{s'}}{dP_{s''}}(\omega) \right] (R - P_\sigma)(d\omega) = 0, \quad (21)$$

whatever $P_{s'}, P_{s''} \in \{P_s\}$ are, then

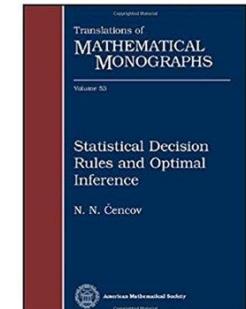
$$I(R; P_s) = I(R; P_\sigma) + I(P_\sigma; P_s), \quad \forall P_s. \quad (22)$$

Corollary. *The measure P_σ is the I -nearest to R point of the exponential family.*

Recently a number of proposals for the arising non-symmetric PYPHAGORIC information geometry has been recovered by CSISZAR [11].



Nikolai
Nikolaevich
Chentsov
(1930-1992)



Fisher Metric and Fréchet-Darmois (Cramer-Rao) Bound

| Cramer-Rao –Fréchet-Darmois Bound has been introduced by Fréchet in 1939 and by Rao in 1945 as inverse of the Fisher Information Matrix: $I(\theta)$

$$R_{\hat{\theta}} = E \left[(\theta - \hat{\theta})(\theta - \hat{\theta})^+ \right] \geq I(\theta)^{-1} \quad [I(\theta)]_{i,j} = -E \left[\frac{\partial^2 \log p_\theta(z)}{\partial \theta_i \partial \theta_j^*} \right]$$

| Rao has proposed to introduce an invariant metric in parameter space of density of probabilities (axiomatised by N. Chentsov):

$$ds_\theta^2 = \text{Kullback - Divergence}(p_\theta(z), p_{\theta+d\theta}(z))$$

$$ds_\theta^2 = - \int p_\theta(z) \log \frac{p_{\theta+d\theta}(z)}{p_\theta(z)} dz$$

$$ds_\theta^2 \underset{\text{Taylor}}{\approx} \sum_{i,j} g_{ij} d\theta_i d\theta_j^* = \sum_{i,j} [I(\theta)]_{i,j} d\theta_i d\theta_j^* = d\theta^+ . I(\theta) . d\theta$$

$$\begin{aligned} w &= W(\theta) \\ \Rightarrow ds_w^2 &= ds_\theta^2 \end{aligned}$$

Distance Between Gaussian Density with Fisher Metric

Fisher Matrix for Gaussian Densities:

$$I(\theta) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix} \quad \text{avec} \quad E\left[\left(\theta - \hat{\theta}\right)\left(\theta - \hat{\theta}\right)^T\right] \geq I(\theta)^{-1} \quad \text{et} \quad \theta = \begin{pmatrix} m \\ \sigma \end{pmatrix}$$

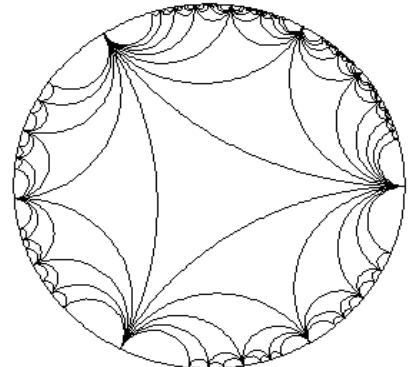
► Fisher matrix induced the following differential metric :

$$ds^2 = d\theta^T \cdot I(\theta) \cdot d\theta = \frac{dm^2}{\sigma^2} + 2 \cdot \frac{d\sigma^2}{\sigma^2} = \frac{2}{\sigma^2} \left[\left(\frac{dm}{\sqrt{2}} \right)^2 + (d\sigma)^2 \right]$$

► Poincaré Model of upper half-plane and unit disk

$$z = \frac{m}{\sqrt{2}} + i \cdot \sigma \quad \omega = \frac{z - i}{z + i} \quad (|\omega| < 1)$$

$$\Rightarrow ds^2 = 8 \cdot \frac{|d\omega|^2}{(1 - |\omega|^2)^2}$$



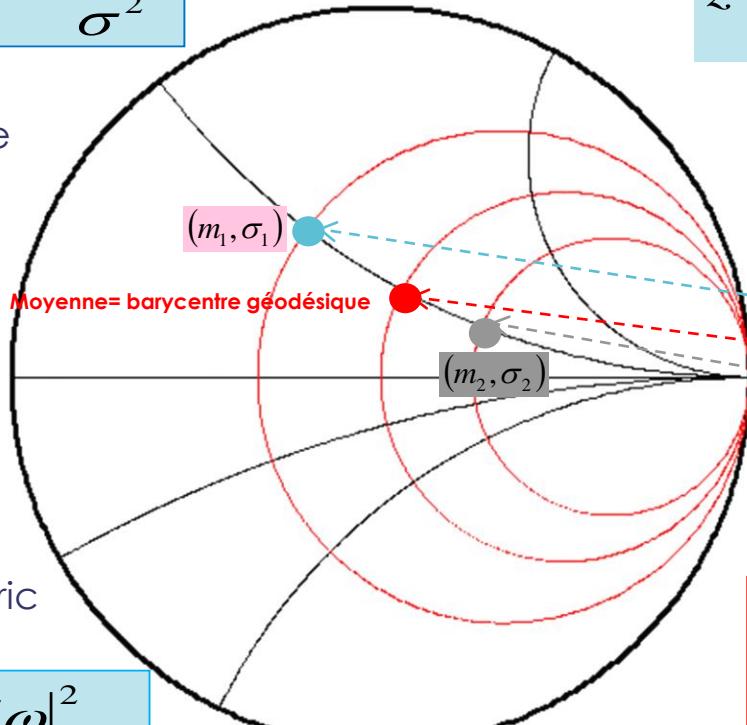
1 monovariate gaussian = 1 point in Poincaré unit disk

$$ds^2 = \frac{dm^2}{\sigma^2} + 2 \cdot \frac{d\sigma^2}{\sigma^2}$$

$$z = \frac{m}{\sqrt{2}} + i \cdot \sigma$$

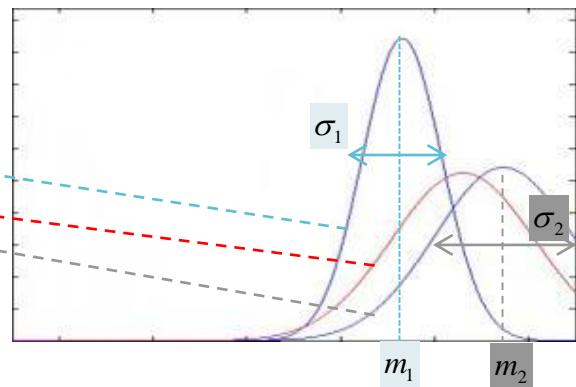
$$\omega = \frac{z - i}{z + i} \quad (\|\omega\| < 1)$$

Fisher Metric in
Poincaré Half-Plane



Poincaré-Fisher metric
In Unit Disk

$$ds^2 = 8 \cdot \frac{|d\omega|^2}{(1 - |\omega|^2)^2}$$



$$d^2(\{\{m_1, \sigma_1\}, \{m_2, \sigma_2\}\}) = 2 \cdot \left(\log \frac{1 + \delta(\omega^{(1)}, \omega^{(2)})}{1 - \delta(\omega^{(1)}, \omega^{(2)})} \right)^2$$

$$\text{with } \delta(\omega^{(1)}, \omega^{(2)}) = \left| \frac{\omega^{(1)} - \omega^{(2)}}{1 - \omega^{(1)} \omega^{(2)*}} \right|$$

Information Geometry & Natural Gradient

- This simple gradient descent has a first drawback of using the same non-adaptive learning rate for all parameter components, and a second drawback of non invariance with respect to parameter re-encoding inducing different learning rates. **S.I. Amari** has introduced the **natural gradient** to preserve this invariance to be insensitive to the characteristic scale of each parameter direction. The gradient descent could be corrected by $I(\theta)^{-1}$ where I is the **Fisher information matrix** with respect to parameter θ , given by:

$$I(\theta) = \begin{bmatrix} g_{ij} \end{bmatrix}$$

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta}$$

$$\text{with } g_{ij} = \left[-E_{y \approx p(y/\theta)} \left[\frac{\partial^2 \log p(y/\theta)}{\partial \theta_i \partial \theta_j} \right] \right]_{ij} = \left[E_{y \approx p(y/\theta)} \left[\frac{\partial \log p(y/\theta)}{\partial \theta_i} \frac{\partial \log p(y/\theta)}{\partial \theta_j} \right] \right]_{ij}$$

Natural Gradient & Stochastic Gradient: Natural Langevin Dynamics

| Natural Langevin Dynamics: Natural Gradient with Langevin Stochastics descent

- To regularize solution and avoid over-fitting, Stochastic gradient is used, as Langevin Stochastic Gradients
- **Yann Ollivier** (FACEBOOK FAIR, previously CNRS LRI Orsay) and **Gaëtan Marceau-Caron** (MILA, previously CNRS LRI Orsay and THALES LAS/ATM & TRT PhD) have proposed to coupled **Natural Gradient** with **Langevin Dynamics: Natural Langevin Dynamics (Best SMF/SEE GSI'17 paper)**

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial \left(l_t(y_t)^T - \frac{1}{N} \log \alpha(\theta_{t-1}) \right)}{\partial \theta} + \sqrt{\frac{2\eta_t}{N}} I(\theta_{t-1})^{-1/2} N(0, I_d)$$



- The resulting natural Langevin dynamics combines the advantages of Amari's natural gradient descent and Fisher-preconditioned Langevin dynamics for large neural networks

Natural Gradient of Information Geometry for LLM (Large Language Model) by GOOGLE LLC

FAdam: Adam is a natural gradient optimizer using diagonal empirical Fisher information

Dongseong Hwang
Google LLC
Mountain View, CA, USA
dongseong@google.com

Abstract

This paper establishes a mathematical foundation for the Adam optimizer, elucidating its connection to natural gradient descent through Riemannian and information geometry. We rigorously analyze the diagonal empirical Fisher information matrix (FIM) in Adam, clarifying all detailed approximations and advocating for the use of log probability functions as loss, which should be based on discrete distributions, due to the limitations of empirical FIM. Our analysis uncovers flaws in the original Adam algorithm, leading to proposed corrections such as enhanced momentum calculations, adjusted bias corrections, adaptive epsilon, and gradient clipping. We refine the weight decay term based on our theoretical framework. Our modified algorithm, **Fisher Adam (FAdam)**, demonstrates superior performance across diverse domains including LLM, ASR, and VQ-VAE, achieving state-of-the-art results in ASR.

$$\theta_{t+1} = \theta_t - \eta F^{-1} \nabla \mathcal{L}(\theta)$$

LibriSpeech WERs	dev	test	dev-other	test-other	avg
Adam (w2v-BERT paper [44])	1.30	2.60	1.40	2.70	2.00
Adam	1.30	2.54	1.33	2.59	1.93
FAdam	1.29	2.49	1.34	2.49	1.90

Table 1: LibriSpeech WERs

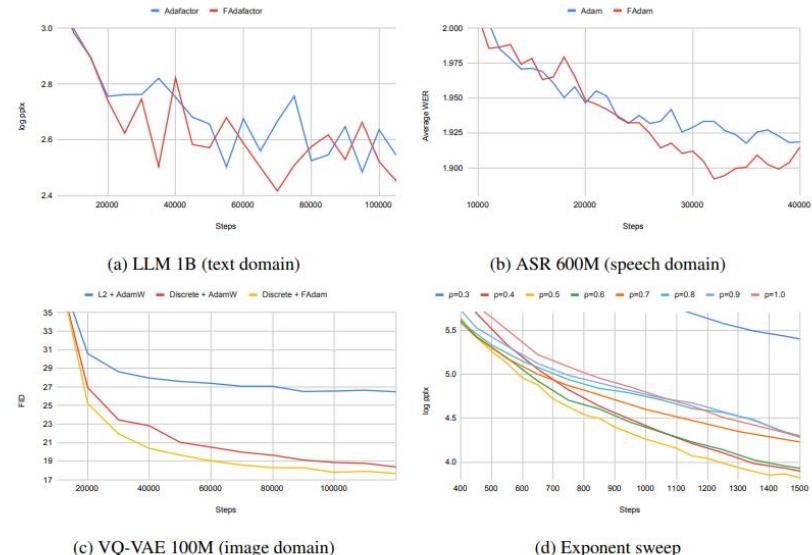


Figure 1: Comparison of FAdam and Adam performance. (a) Eval loss (log pplx) on 1B LLMs, presenting FAdafactor outperforms Adafactor. (b) Average WER on LibriSpeech using 600M Conformer models, presenting FAdam outperforms Adam. (c) FID of ImageNet generation using 100M VQ-VAE models, presenting FAdam outperforms AdamW. (d) Comparison of FIM exponents on a 1B LLM, showing 0.5 (square root) as the optimal choice.



Jean-Marie Souriau: Symplectic Foliation via Moment Map and Koszul-Fisher Metric from Information Geometry for Lie Groups Thermodynamics



SOURIAU 2019

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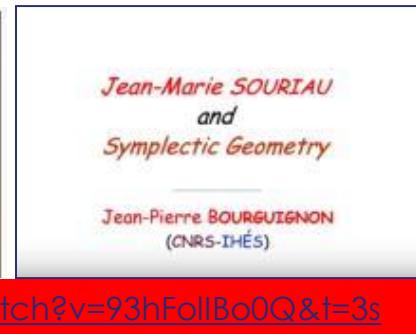
SOURIAU 2019

Internet website : <http://souriau2019.fr>

In 1969, 50 years ago, Jean-Marie Souriau published the book "**Structure des système dynamiques**", in which using the ideas of J.L. Lagrange, he formalized the "**Geometric Mechanics**" in its modern form based on **Symplectic Geometry**

Chapter IV was dedicated to "Thermodynamics of Lie groups" (ref André Blanc-Lapierre)

Testimony of **Jean-Pierre Bourguignon** at Souriau'19 (IHES director of the European ERC)



<https://www.youtube.com/watch?v=93hFoliBo0Q&t=3s>

SOURIAU 2019

Conference May 27-31 2019, Paris-Diderot University

<https://www.youtube.com/watch?v=beM2pUK1H7o>

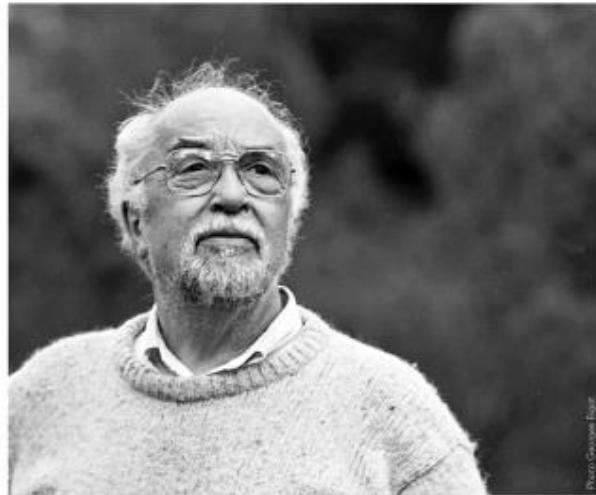


Photo Georges Egor

JEAN-MARIE SOURIAU

In 1969, the groundbreaking book of Jean-Marie Souriau appeared "Structure des Systèmes Dynamiques". We will celebrate, in 2019, the jubilee of its publication, with a conference in honour of the work of this great scientist.

Symplectic Mechanics, Geometric Quantization, Relativity, Thermodynamics, Cosmology, Diffeology & Philosophy

Frédéric Barbaresco
Daniel Bennequin
Jean-Pierre Bourguignon
Pierre Cartier
Dan Christensen
Maurice Courbage
Thibault Damour
Paul Donato
Paolo Giordano
Seinp Güer
Patrick Iglesias-Zemmour
Isbel Karshon
Jean-Pierre Magnot
Yvette Kosmann-Schwarzbach
Marc Lachièze-Rey
Martin Pinsonnault
Elisa Prato
Urs Schreiber
Jean-Jacques Souriau (inventor)
Robert Triv
Jordan Watts
Emin Wu
San Ma Ngai
Alan Weinstein

80|Prime



Jean-Marie Souriau @ ENS Ulm 1942

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Jacques Dixmier



Dixmier Jacques

Algèbres
enveloppantes

Jean-Marie Souriau



Souriau Jean Marie

- Structure des systèmes dynamiques

René Deheuvels



Deheuvels René

- Formes quadratiques et groupes classiques
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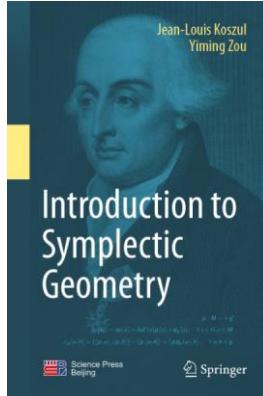
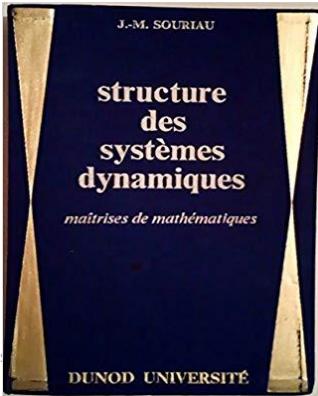
<i>Bauchet Paul</i>	<i>Boileux Paul</i>	<i>Cabannes Paul</i>	<i>Casal Paul</i>
<i>Catala Jean-Jacques</i>	<i>Deheuvels René</i>	<i>Lefebvre Paul</i>	<i>Lommeren Jacques</i>
<i>Fourrier Pierre</i>	<i>Macias Marcel</i>	<i>Richard Paul</i>	<i>Rinel Jeanne</i>
<i>Saupot Paul</i>	<i>Soumaré Jean-Marie</i>	<i>Souby Michel</i>	
<i>Valentin François</i>	<i>Vialin Michel</i>		

Machine Learning on Lie Groups

Lie Groups Machine Learning based on Souriau Lie-Groups Thermodynamics

- We will introduce "**Lie Groups Machine Learning**" [1] that extends statistics and machine learning on Lie Groups based on representation theory and Lie algebra cohomology.
- From Jean-Marie Souriau "**Lie Groups Thermodynamics**" [3] initiated in the framework of symplectic model of statistical mechanics, new geometric statistical tools have been developed to define:
 - maximum Entropy densities of probability on Lie Groups for supervised methods
 - Fisher metric extension from Information Geometry for unsupervised methods.

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« Il n'y a rien de plus dans les théories physiques que les groupes de symétrie si ce n'est la construction mathématique qui permet précisément de montrer qu'il n'y a rien de plus » - Jean-Marie Souriau

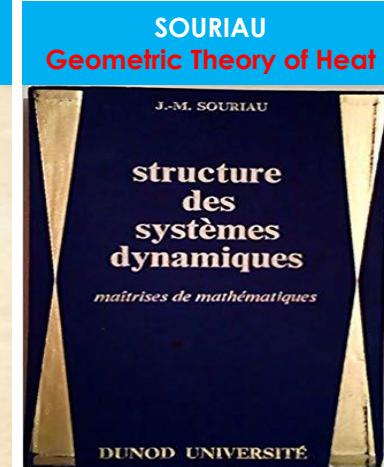
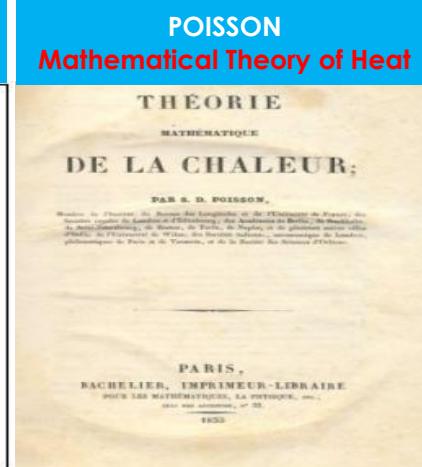
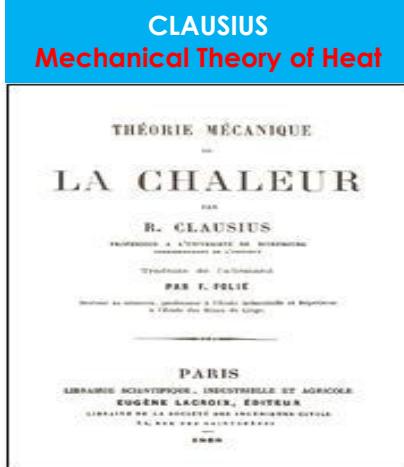
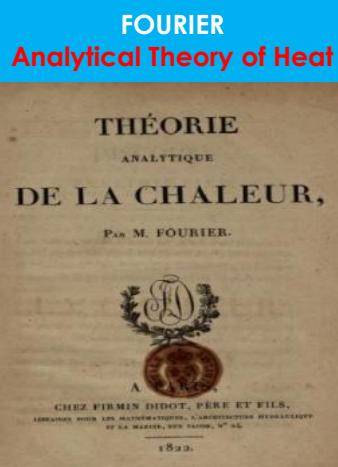
Souriau's Moment Map =
Geometrization of Noether Theorem

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Souriau Lie Groups Thermodynamics: General Equations

Ces formules sont universelles, en ce sens qu'elles ne mettent pas en jeu la variété symplectique U - mais seulement le groupe G , son cocycle symplectique f et les couples Θ, Q . Peut-être cette "thermodynamique des groupes de Lie" a-t-elle un intérêt mathématique.

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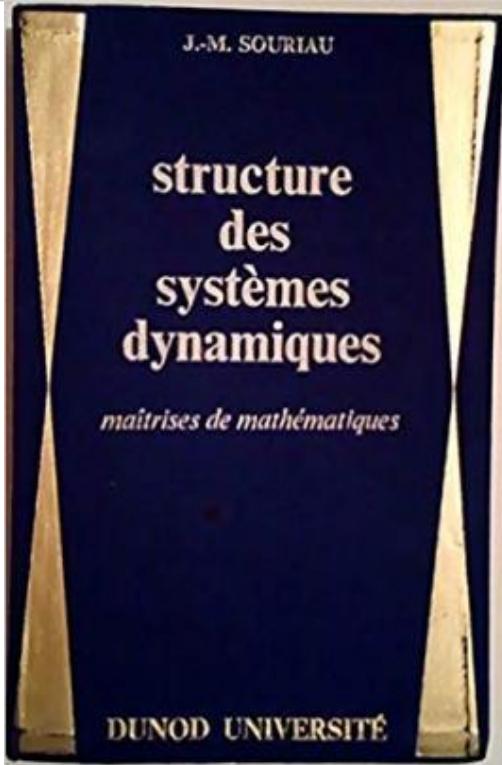
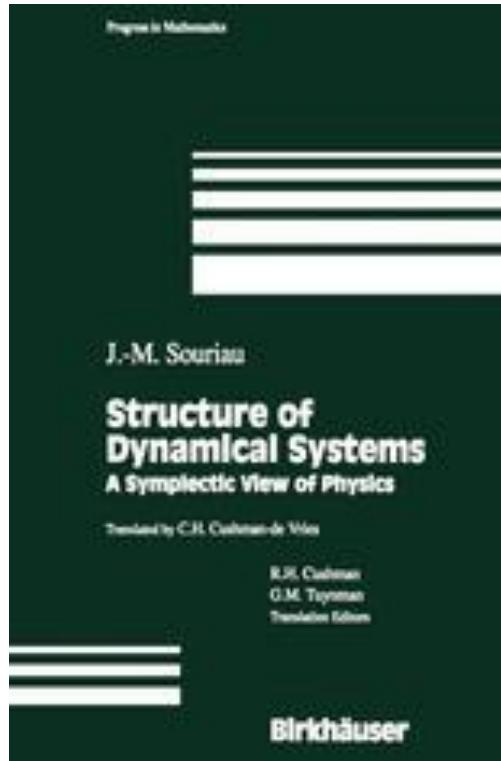
SYMPLECTIC THEORY OF HEAT



ESI Seminar "Infinite-Dimension"

J.M. Souriau Book « Structure des systèmes dynamiques », 1969

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http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm
<http://www.springer.com/us/book/9780817636951>

- | Introduction of symplectic geometry in mechanics
- | Invention of the “moment map”
- | Geometrization of Noether’s theorem
- | Barycentric decomposition theorem
- | The total mass of an isolated dynamic system is the class of cohomology of the default of equivariance for the moment map
- | Lie Groups Thermodynamics (Chapter IV)

Souriau SSD Chapter IV: Gibbs Equilibrium is not covariant with respect to Dynamic Groups of Physics

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MÉCANIQUE STATISTIQUE COVARIANTE

Le groupe des translations dans le temps (7.9) est un sous-groupe du groupe de Galilée ; mais ce n'est pas un sous-groupe invariant, ainsi que le

montre un calcul trivial. Si un système dynamique est conservatif dans un repère d'inertie, il en résulte qu'il peut ne plus être conservatif dans un autre. La formulation (17.24) du principe de Gibbs doit donc être élargie, pour devenir compatible avec la relativité galiléenne.

Nous proposons donc le principe suivant :

(17.77) Si un système dynamique est invariant par un sous-groupe de Lie G' du groupe de Galilée, les équilibres naturels du système constituent l'ensemble de Gibbs du groupe dynamique G' .

Soit \mathcal{G}' l'algèbre de Lie G' ; on sait que \mathcal{G}' est une sous-algèbre de Lie de celle de G , notée \mathcal{G} ; un équilibre du système sera caractérisé par un élément Z de \mathcal{G}' , donc de \mathcal{G} ; on pourra écrire

$$(17.78) \quad Z = \begin{bmatrix} j(\omega) & \beta & \gamma \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{bmatrix}$$

en utilisant les notations (13.4) ; Z parcourt l'ensemble Ω défini en (16.219) ; à chaque valeur de Z est associé un élément M du dual \mathcal{G}'^* de \mathcal{G}' , valeur moyenne du moment μ ; on peut appliquer les formules (16.219), (16.220), qui généralisent les relations thermodynamiques (17.26), (17.27), (17.28). On voit que c'est Z (17.78) qui généralise la « température » ; le théorème d'isothermie (17.32) s'étend immédiatement : l'équilibre d'un système composé de plusieurs parties sans interactions s'obtient en attribuant à chaque composante un équilibre correspondant à la même valeur de Z ; l'entropie s , le potentiel de Planck z et le moment moyen M sont additifs. W

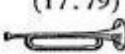
J.M. Souriau, Structure des systèmes dynamiques,
Chapitre IV « Mécanique Statistique »



Trompette de Souriau

Lorsque le fait qu'on rencontre est en opposition avec une théorie régnante, il faut accepter le fait et abandonner la théorie, alors même que celle-ci, soutenue par de grands noms, est généralement adoptée

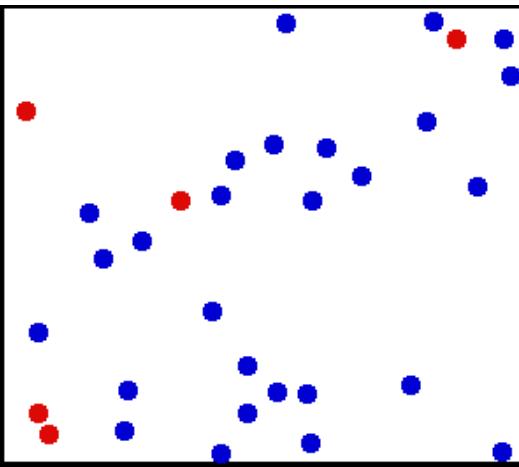
- Claude Bernard "Introduction à l'Étude de la Médecine Expérimentale"



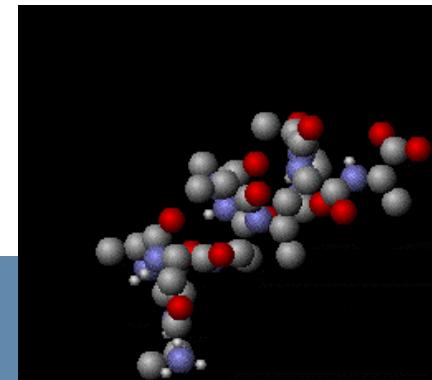
Symplectic Model of Souriau Lie Groups Thermodynamics

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- | Souriau Geometric (Planck) Temperature is an element of Lie Algebra of Dynamical Group (Galileo/Poincaré groups) acting on the system. Heat is an element of the dual of Lie algebra.
- | Generalized Entropy is Legendre Transform of minus logarithm of Laplace Transform and is Casimir Function on Symplectic leaves (obtained by coadjoint orbit via moment map)
- | Fisher(-Souriau) Metric is a Geometric Calorific Capacity (hessian of Massieu Potential)



Souriau formalism is fully covariant, with no special coordinates (covariance of Gibbs density wrt Dynamical Groups)



Jean-Marie Souriau Seminal Paper - 1974

Statistical Mechanics, Lie Group and Cosmology - 1st part: Symplectic Model of Statistical Mechanics Jean-Marie Souriau

Abstract: The classical notion of Gibbs' canonical ensemble is extended to the case of a symplectic manifold on which a Lie group has a symplectic action ("dynamic group"). The rigorous definition given here makes it possible to extend a certain number of classical thermodynamic properties (temperature is here an element of the Lie algebra of the group, heat an element of its dual), notably inequalities of convexity. In the case of non-commutative groups, particular properties appear: the symmetry is spontaneously broken, certain relations of cohomological type are verified in the Lie algebra of the group. Various applications are considered (rotating bodies, covariant or relativistic statistical Mechanics). [These results specify and complement a study published in an earlier work (*), which will be designated by the initials SSD].

(*) Souriau, J.-M., Structure des systèmes dynamique. Dunod, collection Dunod Université, Paris 1969.
http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

Souriau, J.-M., Mécanique statistique, groupes de Lie et cosmologie, Colloques Internationaux

C.N.R.S., n°237 – Géométrie symplectique et physique mathématique, pp.59-113, 1974

English translation by F. Barbaresco:

https://www.academia.edu/42630654/Statistical_Mechanics_Lie_Group_and_Cosmology_1_st_part_Symplectic_Model_of_Statistical_Mechanics

Symplectic Geometry & Foliations Theory Epic

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Joseph-Louis Lagrange



Gaston Darboux



Elie Cartan



Erich Kähler



Hermann Weyl



Carl-Ludwig Siegel



Jürgen K. Moser



Loo-Keng Hua



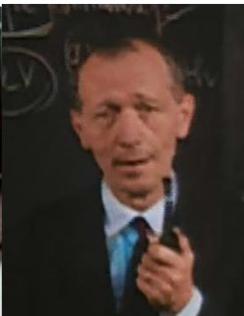
Vladimir Arnold



Lee Hwa-Chung



Charles Ehresmann



Georges Reeb



André Lichnerowicz



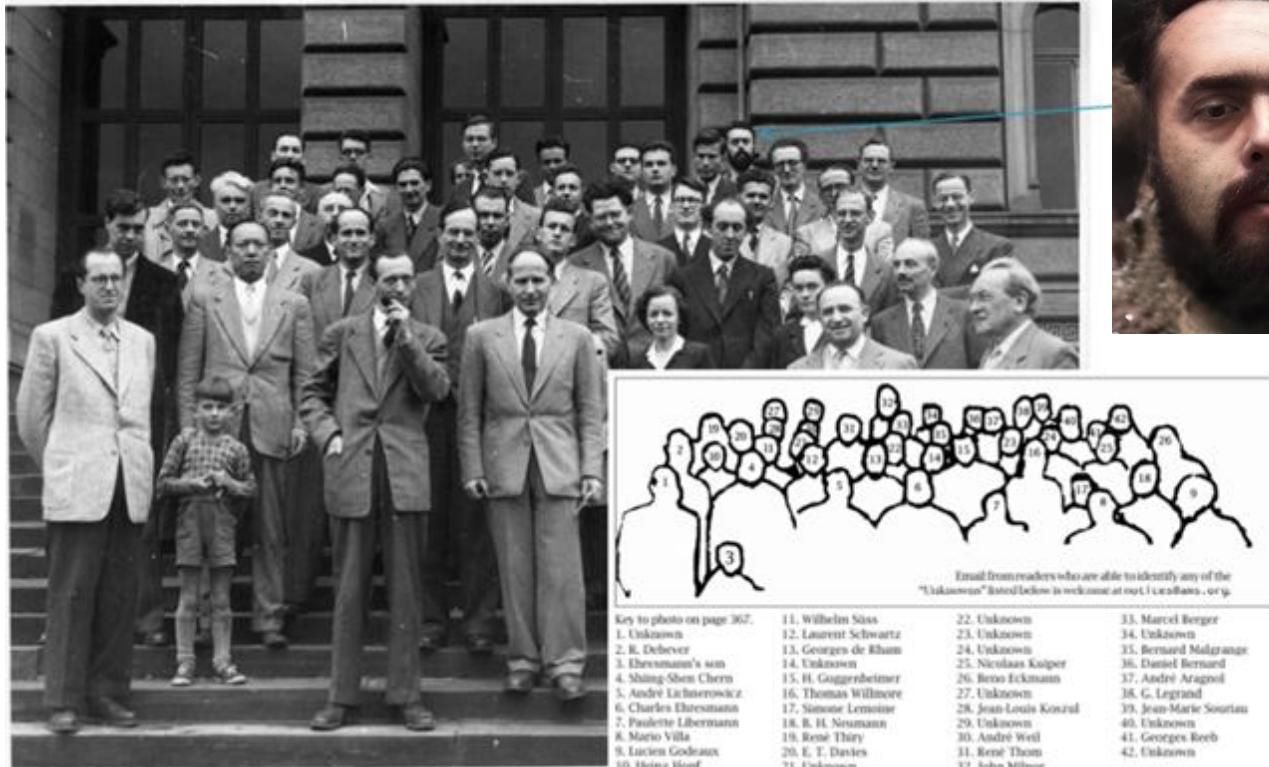
Jean-Marie Souriau

J.M. SOURIAU

F. Barbaresco, *Esthétique structurelle des feuilletages symplectiques du mouvement, de la chaleur et de l'information : la quête romantique de Jean-Marie Souriau de Carthage à Massilia ou le triptyque de la Nature des choses (De Rerum Natura) « Esthétisme – Structure – Mouvement »*, ENS/IRCAM MAMUPHI seminar, to be published by Spartacus Editor for MAMUPHI anniversary <https://spartacus-idh.com/>
MAMUPHI video: <https://www.youtube.com/watch?v=dWyWXubGfXA>
MAMUPHI Slides: <http://www.entretemps.asso.fr/2022-2023/Barbaresco.pdf>
https://www.academia.edu/112471996/Structural_aesthetics_of_the_symplectic_foliations_of_movement_heat_and_information_the_romantic_quest_of_Jean_Marie_Souriau_from_Carthage_to_Massilia_or_the_trptych_of_the_Nature_of_things_De_Rerum_Natura_Aesthetics_Structure_Movement_

Strasbourg 1953

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LII — GÉOMÉTRIE DIFFÉRENTIELLE — CNRS

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DU
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

LII

GÉOMÉTRIE DIFFÉRENTIELLE

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13, Quai Anatole-France, Paris 7^e
1953

PUBLICATIONS SCIENTIFIQUES
DE L'UNIVERSITÉ D'ALGER

SÉRIE A SCIENCES

MATHÉMATIQUES

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IMPRIMERIE DURAND - CHARTRES - 9, RUE FULBERT

Lagrange 2-form rediscovered by Jean-Marie Souriau

- Rewriting equations of classical mechanics in phase space

$$m \frac{d^2 r}{dt^2} = F \longrightarrow m \frac{dv}{dt} = F \quad \text{et} \quad v = \frac{dr}{dt}$$

$\begin{pmatrix} r \\ v \end{pmatrix}$

- Souriau rediscovered that Lagrange had considered the evolution space:

$$\begin{cases} m\delta v - F\delta t = 0 \\ \delta r - v\delta t = 0 \end{cases}$$

$$y = \begin{pmatrix} t \\ r \\ v \end{pmatrix} \in V$$

- A dynamic system is represented by a foliation. This foliation is determined by an antisymmetric covariant 2nd order tensor σ , called the Lagrange (-Souriau) form, a bilinear operator on the tangent vectors of V .

$$\sigma(\delta y)(\delta' y) = \langle m\delta v - F\delta t, \delta' r - v\delta' t \rangle - \langle m\delta' v - F\delta' t, \delta r - v\delta t \rangle$$

$\delta y = \begin{pmatrix} \delta t \\ \delta r \\ \delta v \end{pmatrix}$

$\delta' y = \begin{pmatrix} \delta' t \\ \delta' r \\ \delta' v \end{pmatrix}$

- In the Lagrange-Souriau model, σ is a 2-form on the evolution space V , and the differential equation of motion implies: $\delta y \in \varepsilon$

$$\sigma(\delta y)(\delta' y) = 0, \quad \forall \delta' y$$

$$\sigma(\delta y) = 0 \quad \text{ou} \quad \delta y \in \ker(\sigma)$$

Souriau Work Roots: François Gallissot Theorem

- > **Gallissot Theorem:** There are 3 types of differential forms generating the equations of a material point motion, **invariant by the action of the Galileo group**

$$A : \begin{cases} s = \frac{1}{2m} \sum_{i=1}^3 (mdv_i - F_i dt)^2 \\ e = \frac{m}{2} \sum_{j=1}^3 (dx_j - v_j dt)^2 \end{cases}$$

F. GALLISSOT, Les formes extérieures en Mécanique (*Thèse*), Durand, Chartres, 1954.

$$B : f = \sum_1^3 \delta_{ij} (dx_i - v_i dt) (mdv_j - F_j dt) \text{ with } \delta_{ij} \text{ krönecker symbol}$$

$$C: \omega = \sum_1^3 \delta_{ij} (mdv_i - F_i dt) \wedge (dx_j - v_j dt)$$

- > $d\omega = 0$ constrained the Pfaff form $\delta_{ij} F_i dx_j$ to be closed and to be reduced to the differential of U : $C \Rightarrow \omega = m \delta_{ij} dv_i \wedge dx_j - dH \wedge dt$ with $H = T - U$ and $T = 1/2 \sum_i m(v_i)^2$

- > It proves that ω has an exterior differential $d\omega$ generating **Poincaré-Cartan Integral invariant:**

$$d\omega = \sum_{i=1}^3 mv_i dx_j - Hdt$$

François Gallissot Work in 1952 based on Elie and Henri Cartan works

LES FORMES EXTÉRIEURES EN MÉCANIQUE

par F. GALLISSOT.

1952

INTRODUCTION

La mécanique des systèmes paramétriques développée traditionnellement d'après les idées de Lagrange s'est toujours heurtée à des difficultés notables lorsqu'elle a désiré aborder les questions de frottement entre solides (impossibilité et indétermination) ou la notion générale de liaison (asservissement de M. Béghin), d'autre part la forme lagrangienne des équations du mouvement ne nous donne aucune indication sur la nature du problème de l'intégration.

Dans ces célèbres leçons sur les invariants intégraux Élie Cartan a montré que toutes les propriétés des équations différentielles de la dynamique des systèmes holonomes résultaient de l'existence de l'invariant intégral $\int \omega$, $\omega = p_i dq^i - H dt$. Ainsi à tout système holonome dont les forces dérivent d'une fonction de forces est associé une forme ω , les équations du mouvement étant les caractéristiques de la forme extérieure $d\omega$. Au cours de ces dix dernières années, sous l'influence des topologistes s'est édifiée sur des bases qui semblent définitives la théorie des formes extérieures sur les variétés différentiables. Il est alors naturel de se demander si la mécanique classique ne peut pas bénéficier largement de ce courant d'idées, si elle ne peut pas être construite en plaçant à sa base une forme extérieure de degré deux, si grâce à la notion de variétés, la notion de liaison ne peut pas être envisagée sous un angle plus intelligible, si les indéterminations et impossibilités qui paraissent paradoxaux dans le cadre lagrangien n'ont pas une explication naturelle, enfin s'il n'est pas possible de considérer sous un jour nouveau le problème de l'intégration des équations du mouvement, ces dernières étant engendrées par une forme Ω de degré deux.

S'affranchir
de la servitude
des coordonnées

$$i(E)\Omega = 0$$

Pour atteindre ces divers objectifs il m'a semblé utile de reprendre dans le chapitre 1 l'étude des bases logiques sur lesquelles est édifiée la mécanique galiléenne. Je montre ainsi dans le § 1 que lorsqu'on se propose de trouver des formes génératrices des équations du mouvement d'un point matériel invariantes dans les transformations du groupe galiléen, la forme la plus intéressante est une forme extérieure de degré deux définie sur une variété $V, \equiv E \otimes E^*, T$ (E , espace euclidien, T droite numérique temporelle)⁽¹⁾. Dans le § 11 on montre qu'à tout système paramétrique holonome à n degrés de liberté est associé une forme Ω de degré deux de rang $2n$ définie sur une variété différentiable dont les caractéristiques sont les équations du mouvement⁽²⁾. Cette forme s'exprime si l'on veut au moyen de $2n$ formes de Pfaff et de dt , la forme hamiltonienne n'étant qu'un cas particulier simple. Dans le § 3 j'indique sommairement comment on peut s'affranchir de la servitude des coordonnées dans l'étude des systèmes dynamiques et le rôle important joué par l'opérateur $i(\cdot)$ antidérivation de M. H. Cartan⁽³⁾, le champ caractéristique E de la forme Ω étant défini par la relation $i(E)\Omega = 0$.

(1) M. KRAVTCHEKO a présenté cette conception au VIII^e Congrès de Mécanique.

(2) Dès 1946 M. LICHNEROWICZ au *Bulletin des Sciences Mathématiques* tome LXX, p. 90 a déjà introduit les formes extérieures pour la formation des équations des systèmes holonomes et linéairement non holonomes.

(3) M. H. CARTAN, *Colloque de Topologie*, Bruxelles, 1950. Masson, Paris, 1951.

F. GALLISSOT, *Les formes extérieures en Mécanique (Thèse)*, Durand, Chartres, 1954.

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Interior/Exterior Products and Lie derivative

- $i_V \omega$ is the $(p-1)$ -form on X obtained by inserting $V(x)$ as the first argument of ω :

Interior product : $i_V \omega(v_2, \dots, v_p) = \omega(V(x), v_2, \dots, v_p)$

- $\theta \wedge \omega$ is the $(p+1)$ -form on X where ω is a p -form and θ is a 1-form on X :

Exterior product : $\theta \wedge \omega(v_0, \dots, v_p) = \sum_{i=0}^p (-1)^i \theta(v_i) \omega(v_0, \dots, \hat{v}_i, \dots, v_p)$

(where the hat indicates a term to be omitted).

- $L_V \omega$ is a p -form on X , and $L_V \omega = 0$ if the flow of V consists of symmetries of ω .

Lie derivative : $L_V \omega(v_1, \dots, v_p) = \left. \frac{d}{dt} e^{tV^*} \omega(v_1, \dots, v_p) \right|_{t=0}$

Exterior derivative and E.Cartan, H. Cartan & S. Lie formulas

- > $d\omega$ is the $(p+1)$ -form on X defined by taking the ordinary derivative of ω and then antisymmetrizing:

Exterior derivative : $d\omega(v_0, \dots, v_p) = \sum_{i=0}^p (-1)^i \frac{\partial \omega}{\partial x}(v_i)(v_0, \dots, \hat{v}_i, \dots, v_p)$

$$p=0, [d\omega]_i = \partial_i \omega ; p=1, [d\omega]_{ij} = \partial_i \omega_j - \partial_j \omega_i ; p=2, [d\omega]_{ijk} = \partial_i \omega_{jk} + \partial_j \omega_{ki} + \partial_k \omega_{ij}$$

- > The properties of the exterior and Lie Derivative are the following:

$$L_V \omega = di_V \omega + i_V d\omega \quad (\text{E. Cartan})$$

$$i_{[U,V]} \omega = i_V L_U \omega - L_U i_V \omega \quad (\text{H. Cartan})$$

$$L_{[U,V]} \omega = L_V L_U \omega - L_U L_V \omega \quad (\text{S. Lie})$$

Souriau Moment Map (1/2)

- Let (X, σ) be a connected symplectic manifold.
- A vector field η on X is called symplectic if its flow preserves the 2-form :

$$L_\eta \sigma = 0$$

- If we use Elie Cartan's formula, we can deduce that :

$$L_\eta \sigma = di_\eta \sigma + i_\eta d\sigma = 0$$

- but as $d\sigma = 0$ then $di_\eta \sigma = 0$. We observe that the 1-form $i_\eta \sigma$ is closed.
- When this 1-form is exact, there is a smooth function $x \mapsto H$ on X with:

$$i_\eta \sigma = -dH$$

- This vector field η is called Hamiltonian and could be defined as a symplectic gradient :

$$\eta = \nabla_{Symp} H$$

Souriau Moment Map (2/2)

$$di_\eta \sigma = 0$$

$$i_\eta \sigma = -dH$$

► We define the Poisson bracket of two functions H, H' by :

$$\{H, H'\} = \sigma(\eta, \eta') = \sigma(\nabla_{Symp} H', \nabla_{Symp} H)$$

with $i_\eta \sigma = -dH$ and $i_{\eta'} \sigma = -dH'$

► Let a Lie group G that acts on X and that also preserve σ .

► A moment map exists if these infinitesimal generators are actually hamiltonian, so that a map exists:

$$\Phi : X \rightarrow \mathfrak{g}^* \quad \text{with} \quad i_{Z_X} \sigma = -dH_Z \quad \text{where} \quad H_Z = \langle \Phi(x), Z \rangle$$

Notation !

I use notation as used by Koszul and Souriau which is not the most classical one

$$Ad_g^* = \left(Ad_{g^{-1}} \right)^*$$

with

$$\langle Ad_g^* F, Y \rangle = \langle F, Ad_{g^{-1}} Y \rangle, \forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$$

Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Lie Group Adjoint Representation

- the adjoint representation of a Lie group Ad_g is a way of representing its elements as linear transformations of the Lie algebra, considered as a vector space

$$Ad_g = (d\Psi_g)_e : \mathfrak{g} \rightarrow \mathfrak{g}$$

$$X \mapsto Ad_g(X) = gXg^{-1}$$

$$Ad_g(X) = \frac{d}{dt} ge^{tX} g^{-1} \Big|_{t=0} \in T_e G$$

$$\Psi : G \rightarrow Aut(G)$$

$$g \mapsto \Psi_g(h) = ghg^{-1}$$

$$ad = T_e Ad : T_e G \rightarrow End(T_e G)$$

$$X, Y \in T_e G \mapsto ad_X(Y) = [X, Y]$$

Lie Group Co-Adjoint Representation

- the coadjoint representation of a Lie group Ad_g^* , is the dual of the adjoint representation (\mathfrak{g}^* denotes the dual space to \mathfrak{g}):

$$\forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*, \text{ then } \langle Ad_g^* F, Y \rangle = \langle F, Ad_{g^{-1}} Y \rangle$$

$$K = Ad_g^* = (Ad_{g^{-1}})^* \quad \text{and} \quad K_*(X) = -(ad_X)^*$$

Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Co-adjoint Orbits as Homogeneous Symplectic Manifold by KKS 2-form

- > A coadjoint orbit:

$$O_F = \{Ad_g^* F, g \in G\} \text{ subset of } \mathfrak{g}^*, F \in \mathfrak{g}^*$$

carry a natural homogeneous symplectic structure by a closed G-invariant 2-form:

$$\sigma_\Omega(K_{*X}F, K_{*Y}F) = B_F(X, Y) = \langle F, [X, Y] \rangle, X, Y \in \mathfrak{g}$$

- > The coadjoint action on O_F is a Hamiltonian G-action with moment map $\Omega \rightarrow \mathfrak{g}^*$

Souriau Fundamental Theorem « *Every symplectic manifold is a coadjoint orbit* » is based on classification of symplectic homogeneous Lie group actions by Souriau, Kostant and Kirillov

$$g \in G$$



$$O_F = \{Ad_g^* F, g \in G, F \in \mathfrak{g}^*\}$$

Lie Group

Coadjoint Orbit

(action of Lie Group on dual Lie algebra)

$$\sigma_\Omega(ad_F X, ad_F Y) = \langle F, [X, Y] \rangle$$

$$X, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$$

Homogeneous Symplectic Manifold

(a smooth manifold with a closed differential 2-form σ , such that $d\sigma=0$, where the Lie Group acts transitively)

To make statistics on Lie Groups, migrate on the symplectic manifold generated by coadjoint orbits to capture symmetries

Bedrock of Information Geometry



Jean-Marie Souriau (ENS 1942)



Jean-Louis Koszul (ENS 1940)

Information Geometry & Machine Learning : Legendre structure

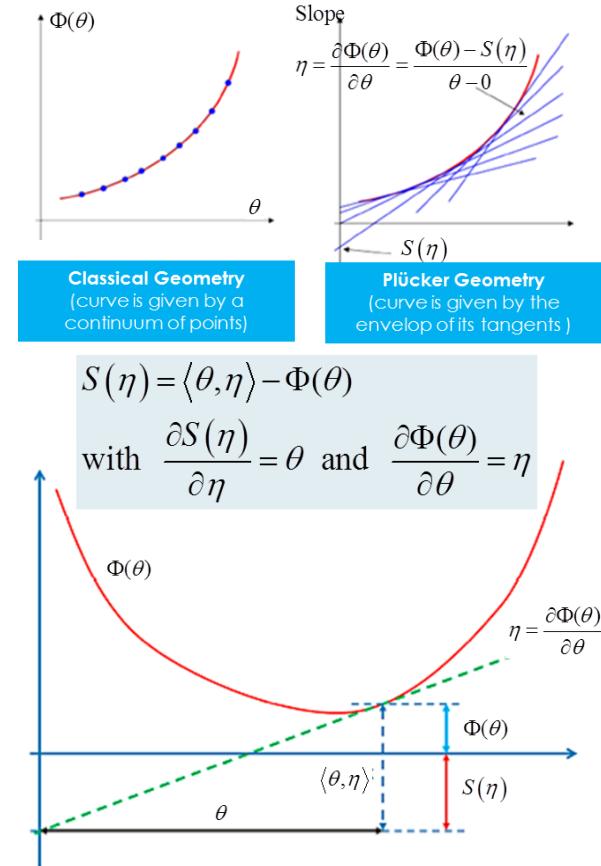
Legendre Transform, Dual Potentials & Fisher Metric

- > S.I. Amari has proved that the Riemannian metric in an exponential family is the **Fisher information matrix** defined by:

$$g_{ij} = - \left[\frac{\partial^2 \Phi}{\partial \theta_i \partial \theta_j} \right]_{ij} \quad \text{with } \Phi(\theta) = -\log \int_R e^{-\langle \theta, y \rangle} d\lambda$$

- > and the dual potential, the **Shannon entropy**, is given by the **Legendre transform**:

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta) \quad \text{with } \eta_i = \frac{\partial \Phi(\theta)}{\partial \theta_i} \quad \text{and} \quad \theta_i = \frac{\partial S(\eta)}{\partial \eta_i}$$



Fisher Metric and Koszul 2 form on sharp convex cones

Koszul-Vinberg Characteristic Function, Koszul Forms

- > **J.L. Koszul** and **E. Vinberg** have introduced an affinely invariant Hessian metric on a sharp convex cone through its **characteristic function**

$$\Phi_{\Omega}(\theta) = -\log \int_{\Omega^*} e^{-\langle \theta, y \rangle} dy = -\log \psi_{\Omega}(\theta) \text{ with } \theta \in \Omega \text{ sharp convex cone}$$

$$\psi_{\Omega}(\theta) = \int_{\Omega^*} e^{-\langle \theta, y \rangle} dy \text{ with Koszul-Vinberg Characteristic function}$$

- > **1st Koszul form α** : $\alpha = d\Phi_{\Omega}(\theta) = -d \log \psi_{\Omega}(\theta)$

- > **2nd Koszul form γ** : $\gamma = D\alpha = Dd \log \psi_{\Omega}(\theta)$

$$(Dd \log \psi_{\Omega}(x))(u) = \frac{1}{\psi_{\Omega}(u)^2} \left[\int_{\Omega^*} F(\xi)^2 d\xi \cdot \int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi)G(\xi) d\xi \right)^2 \right] > 0 \text{ with } F(\xi) = e^{-\frac{1}{2}\langle x, \xi \rangle} \text{ and } G(\xi) = e^{-\frac{1}{2}\langle x, \xi \rangle} \langle u, \xi \rangle$$

- > Diffeomorphism: $\eta = \alpha = -d \log \psi_{\Omega}(\theta) = \int_{\Omega^*} \xi p_{\theta}(\xi) d\xi$ with $p_{\theta}(\xi) = \frac{e^{-\langle \xi, \theta \rangle}}{\int_{\Omega^*} e^{-\langle \xi, \theta \rangle} d\xi}$

- > Legendre transform: $S_{\Omega}(\eta) = \langle \theta, \eta \rangle - \Phi_{\Omega}(\theta)$ with $\eta = d\Phi_{\Omega}(\theta)$ and $\theta = dS_{\Omega}(\eta)$



Jean-Louis Koszul

Koszul-Vinberg Characteristic Function

Koszul-Vinberg Characteristic function

- The name “characteristic function” come from the following link:

Let Ω be a cone in U and Ω^* its dual, for any $\lambda > 0$, $H_\lambda(x) = \{y \in U / \langle x, y \rangle = \lambda\}$

and let $d^{(\lambda)}y$ denote the Lebesgue measure on $H_\lambda(x)$:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle x, y \rangle} dy = \frac{(m-1)!}{\lambda^{m-1}} \int_{\Omega^* \cap H_\lambda(x)} d^{(\lambda)}y$$

- There exist a bijection $x \in \Omega \mapsto x^* \in \Omega^*$, satisfying the relation $(gx)^* = {}^t g^{-1}x^*$ for all $g \in G(\Omega) = \{g \in GL(U) / g\Omega = \Omega\}$ the linear automorphism group of Ω and x^* is:

$$x^* = \int_{\Omega^* \cap H_\lambda(x)} y d^{(\lambda)}y / \int_{\Omega^* \cap H_\lambda(x)} d^{(\lambda)}y$$

- We can observe that x^* is the center of gravity of $\Omega^* \cap H_\lambda(x)$. We have the property that $\psi_\Omega(gx) = |\det(g)|^{-1} \psi_\Omega(x)$ for all $x \in \Omega, g \in G(\Omega)$ and then that $\psi_\Omega(x)dx$ is an invariant measure on Ω .

Koszul-Vinberg Characteristic Function

Koszul-Vinberg Characteristic function

- We can observe that x^* is the center of gravity of $\Omega^* \cap H_\lambda(x)$. We have the property that $\psi_\Omega(gx) = |\det(g)|^{-1} \psi_\Omega(x)$ for all $x \in \Omega, g \in G(\Omega)$ and then that $\psi_\Omega(x)dx$ is an invariant measure on Ω .
- Writing $\partial_a = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}$, one can write:

$$\partial_a \Phi_\Omega(x) = \partial_a (-\log \psi_\Omega(x)) , \quad a \in U, x \in \Omega$$

$$\partial_a \Phi_\Omega(x) = \psi_\Omega(x)^{-1} \int_{\Omega^*} \langle a, y \rangle e^{-\langle x, y \rangle} dy = \langle a, x^* \rangle$$

- Then, the tangent space to the hypersurface $\{y \in U / \psi_\Omega(y) = \psi_\Omega(x)\}$ at $x \in \Omega$ is given by $\{y \in U / \langle x^*, y \rangle = m\}$. For $x \in \Omega, a, b \in U$, the bilinear form $\partial_a \partial_b \log \psi_\Omega(x)$ is symmetric and positive definite, so that it defines an invariant Riemannian metric on Ω .

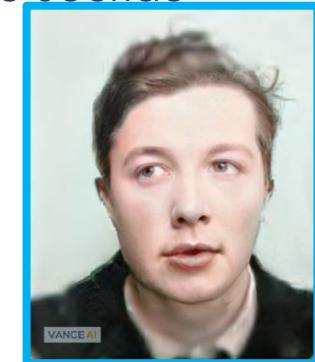
Fisher Metric and Souriau 2-form: Lie Groups Thermodynamics

| Statistical Mechanics, Dual Potentials & Fisher Metric

- In geometric statistical mechanics, **J.M. Souriau** has developed a “**Lie groups thermodynamics**” of dynamical systems where the (maximum entropy) **Gibbs density is covariant** with respect to the action of the Lie group. In the Souriau model, previous structures of information geometry are preserved:

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda \quad U : M \rightarrow \mathfrak{g}^*$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$



Jean-Marie Souriau

- In the Souriau **Lie groups thermodynamics** model, β is a “geometric” (Planck) temperature, element of Lie algebra \mathfrak{g} of the group, and Q is a “geometric” heat, element of dual Lie algebra \mathfrak{g}^* of the group.

Fisher-Souriau Metric as a non-null Cohomology extension of KKS 2 form (Kirillov-Kostant-Souriau 2 form)

| Souriau definition of Fisher Metric is related to the extension of KKS 2-form (Kostant-Kirillov-Souriau) in case of non-null Cohomogy:

Souriau-Fisher Metric

$$I(\beta) = [g_\beta] \text{ with } g_\beta([\beta, Z_1], [\beta, Z_2]) = \tilde{\Theta}_\beta(Z_1, [\beta, Z_2])$$

$$\text{with } \tilde{\Theta}_\beta(Z_1, Z_2) = \tilde{\Theta}(Z_1, Z_2) + \langle Q, [Z_1, Z_2] \rangle$$

Non-null cohomology: aditional term from Souriau Cocycle

Equivariant KKS 2 form

$$\tilde{\Theta}(X, Y) = J_{[X, Y]} - \{J_X, J_Y\} \text{ with } J(x) : M \rightarrow \mathfrak{g}^* \text{ such that } J_X(x) = \langle J(x), X \rangle, X \in \mathfrak{g}$$

$$\tilde{\Theta}(X, Y) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{R} \quad \text{with } \Theta(X) = T_e \theta(X(e)) \quad \tilde{\Theta}(\beta, Z) + \langle Q, [\beta, Z] \rangle = 0$$

$$X, Y \mapsto \langle \Theta(X), Y \rangle$$

$$\beta \in \text{Ker } \tilde{\Theta}_\beta$$

**Souriau Fundamental
Equation of Lie Group Thermodynamics**

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$$Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$$

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Non-equivariance of Coadjoint operator

- Non-equivariance of Coadjoint operator:

$$Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$$

- This is the action of Lie Group on Moment map:

$$J(\Phi_g(x)) = a(g, J(x)) = Ad_g^*(J(x)) + \theta(g)$$

- By noting the action of the group on the dual space of the Lie algebra:

$$G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*, (s, \xi) \mapsto s\xi = Ad_s^* \xi + \theta(s)$$

- Associativity is given by:

$$(s_1 s_2) \xi = Ad_{s_1 s_2}^* \xi + \theta(s_1 s_2) = Ad_{s_1}^* Ad_{s_2}^* \xi + \theta(s_1) + Ad_{s_1}^* \theta(s_2)$$

$$(s_1 s_2) \xi = Ad_{s_1}^* (Ad_{s_2}^* \xi + \theta(s_2)) + \theta(s_1) = s_1 (s_2 \xi) , \quad \forall s_1, s_2 \in G, \xi \in \mathfrak{g}^*$$

Souriau Cocycle

- $\theta(g) \in \mathfrak{g}^*$ is called nonequivariance one-cocycle, and it is a measure of the lack of equivariance of the moment map.

$$\theta(st) = J((st).x) - Ad_{st}^* J(x)$$

$$\theta(st) = [J(s.(t.x)) - Ad_s^* J(t.x)] + [Ad_s^* J(t.x) - Ad_s^* Ad_t^* J(x)]$$

$$\theta(st) = \theta(s) + Ad_s^* [J(t.x) - Ad_t^* J(x)]$$

$$\theta(st) = \theta(s) + Ad_s^* \theta(t)$$

Souriau one-cocycle and compute 2-cocycle

$$\tilde{\Theta}(X, Y) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{R} \quad \text{with } \Theta(X) = T_e \theta(X(e))$$

$$X, Y \mapsto \langle \Theta(X), Y \rangle$$

► We can also compute tangent of one-cocycle θ at neutral element, to compute 2-cocycle Θ :

$$\zeta \in \mathfrak{g}, \quad \theta_\zeta(s) = \langle \theta(s), \zeta \rangle = \langle J(s.x), \zeta \rangle - \langle Ad_s^* J(x), \zeta \rangle$$

$$\theta_\zeta(s) = \langle J(s.x), \zeta \rangle - \langle J(x), Ad_{s^{-1}}\zeta \rangle$$

$$T_e \theta_\zeta(\xi) = \langle T_x J \cdot \xi_p(x), \zeta \rangle + \langle J(x), ad_\xi \zeta \rangle \quad \text{with } \xi_p = X_{\langle J, \zeta \rangle}$$

$$T_e \theta_\zeta(\xi) = X_{\langle J(x), \xi \rangle} [\langle J(x), \zeta \rangle] + \langle J(x), [\xi, \zeta] \rangle$$

$$T_e \theta_\zeta(\xi) = -\{\langle J, \xi \rangle, \langle J, \zeta \rangle\} + \langle J(x), [\xi, \zeta] \rangle = \Theta(\xi)$$

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Souriau Tensor

$$\tilde{\Theta}(X, Y) = J_{[X, Y]} - \{J_X, J_Y\} = -\langle d\theta(X), Y \rangle , \quad X, Y \in \mathfrak{g}$$

$$\tilde{\Theta}([X, Y], Z) + \tilde{\Theta}([Y, Z], X) + \tilde{\Theta}([Z, X], Y) = 0 , \quad X, Y, Z \in \mathfrak{g}$$

► By differentiating the equation on affine action, we have:

$$T_x J(\xi_p(x)) = -ad_{\xi}^* J(x) + \Theta(\xi, .)$$

$$dJ(Xx) = ad_X J(x) + d\theta(X) , \quad x \in M, X \in \mathfrak{g}$$

$$\langle dJ(Xx), Y \rangle = \langle ad_X J(x), Y \rangle + \langle d\theta(X), Y \rangle , \quad x \in M, X, Y \in \mathfrak{g}$$

$$\langle dJ(Xx), Y \rangle = \langle J(x), [X, Y] \rangle + \langle d\theta(X), Y \rangle = \{\langle J, X \rangle, \langle J, Y \rangle\}(x)$$

$$\langle J(x), [X, Y] \rangle - \{\langle J, X \rangle, \langle J, Y \rangle\}(x) = -\langle d\theta(X), Y \rangle$$

Souriau Riemannan Metric and Fisher Metric

- For $\beta \in \Omega$, let g_β be the Hessian form on $T_\beta \Omega \equiv \mathfrak{g}$ with the potential $\Phi(\beta) = -\log \psi_\Omega(\beta)$. For $X, Y \in \mathfrak{g}$, we define:

$$g_\beta(X, Y) = -\frac{\partial^2 \Phi}{\partial \beta^2}(X, Y) = \left(\frac{\partial^2}{\partial s \partial t} \right)_{s=t=0} \log \psi_\Omega(\beta + sX + tY)$$

- The positive definitiveness is given by Cauchy-Schwarz inequality:

$$g_\beta(X, Y) = \frac{1}{\psi_\Omega(\beta)^2} \left\{ \int_M e^{-\langle U(\xi), \beta \rangle} d\lambda(\xi) \cdot \int_M \langle U(\xi), X \rangle^2 e^{-\langle U(\xi), \beta \rangle} d\lambda(\xi) \right. \\ \left. - \left(\int_M \langle U(\xi), X \rangle e^{-\langle U(\xi), \beta \rangle} d\lambda(\xi) \right)^2 \right\}$$

$$= \frac{1}{\psi_\Omega(\beta)^2} \left\{ \int_M \left(e^{-\langle U(\xi), \beta \rangle/2} \right)^2 d\lambda(\xi) \cdot \int_M \left(\langle U(\xi), X \rangle e^{-\langle U(\xi), \beta \rangle/2} \right)^2 d\lambda(\xi) \right. \\ \left. - \left(\int_M e^{-\langle U(\xi), \beta \rangle/2} \cdot \langle U(\xi), X \rangle e^{-\langle U(\xi), \beta \rangle/2} d\lambda(\xi) \right)^2 \right\} \geq 0$$

Souriau Riemannan Metric and Fisher Metric

$$g_\beta(X, Y) = \left\langle -\frac{\partial Q}{\partial \beta}(X), Y \right\rangle \text{ for } X, Y \in \mathfrak{g}$$

> we have for any $\beta \in \Omega$, $g \in G$ and $Y \in \mathfrak{g}$:

$$\left\langle Q(Ad_g \beta), Y \right\rangle = \left\langle Q(\beta), Ad_{g^{-1}} Y \right\rangle + \left\langle \theta(g), Y \right\rangle$$

> Let us differentiate the above expression with respect to g . Namely, we substitute $g = \exp(tZ_1)$, $t \in R$ and differentiate at $t = 0$. Then the left-hand side becomes:

$$\left(\frac{d}{dt} \right)_{t=0} \left\langle Q(\beta + t[Z_1, \beta] + o(t^2)), Y \right\rangle = \left\langle \frac{\partial Q}{\partial \beta}([Z_1, \beta]), Y \right\rangle$$

> and the right-hand side of is calculated as:

$$\left(\frac{d}{dt} \right)_{t=0} \left\langle Q(\beta), Y - t[Z_1, Y] + o(t^2) \right\rangle + \left\langle \theta(I + tZ_1 + o(t^2)), Y \right\rangle$$

Souriau Riemannian Metric and Fisher Metric

➤ Therefore,

$$\left\langle \frac{\partial Q}{\partial \beta}([Z_1, \beta]), Y \right\rangle = \langle d\theta(Z_1), Y \rangle - \langle Q(\beta), [Z_1, Y] \rangle$$

➤ Substituting $Y = -[\beta, Z_2]$ to the above expression:

$$g_\beta([\beta, Z_1], [\beta, Z_2]) = \left\langle -\frac{\partial Q}{\partial \beta}([Z_1, \beta]), [\beta, Z_2] \right\rangle$$

$$g_\beta([\beta, Z_1], [\beta, Z_2]) = \langle d\theta(Z_1), [\beta, Z_2] \rangle + \langle Q(\beta), [Z_1, [\beta, Z_2]] \rangle$$

➤ We define then symplectic 2-cocycle and the tensor:

$$\Theta(Z_1) = d\theta(Z_1)$$

$$\tilde{\Theta}(Z_1, Z_2) = \langle \Theta(Z_1), Z_2 \rangle = J_{[Z_1, Z_2]} - \{J_{Z_1}, J_{Z_2}\}$$

$$\tilde{\Theta}_\beta(Z_1, Z_2) = \langle Q(\beta), [Z_1, Z_2] \rangle + \tilde{\Theta}(Z_1, Z_2) \rightarrow g_\beta([\beta, Z_1], [\beta, Z_2]) = \tilde{\Theta}_\beta(Z_1, [\beta, Z_2])$$

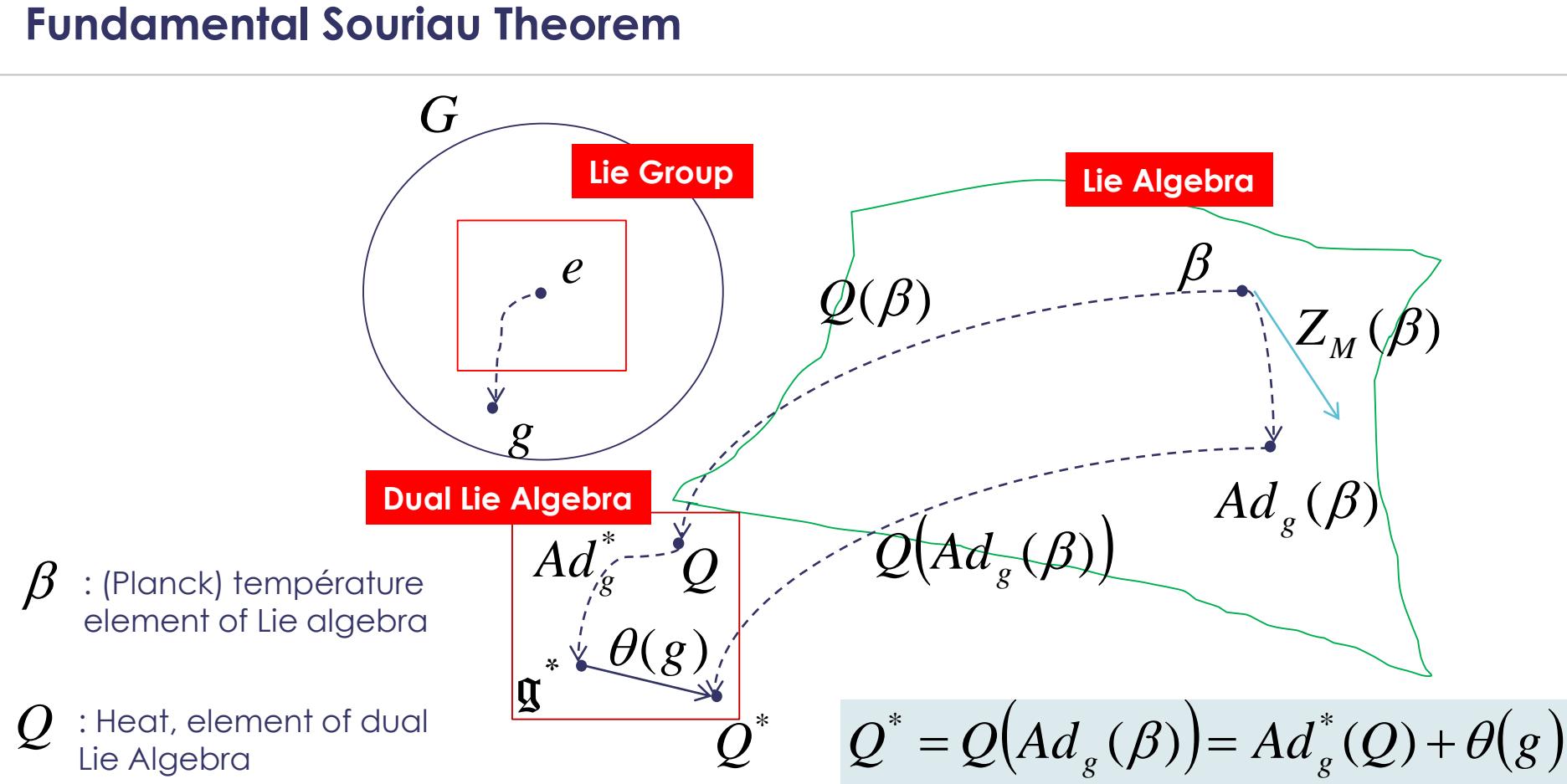
Fundamental Souriau Theorem

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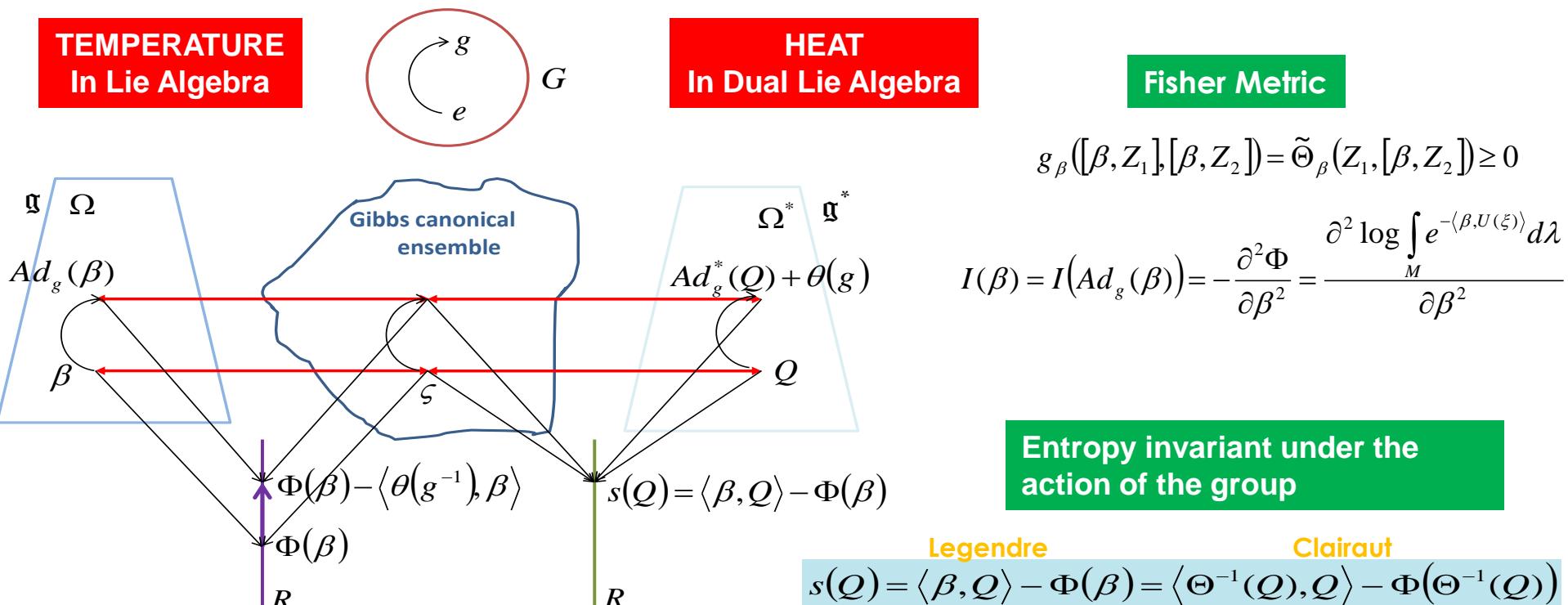
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Souriau-Fisher Metric & Souriau Lie Groups Thermodynamics: Bedrock for Lie Group Machine Learning



Logarithm of Partition Function
(Massieu Characteristic Function)

Entropy

$$Q = \Theta(\beta) = \frac{\partial \Phi}{\partial \beta} \in \mathfrak{g}^*$$

$$\beta = \Theta^{-1}(Q) \in \mathfrak{g}$$

| Symplectic cocycles of the Galilean group: V. Bargmann (Ann. Math. 59, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is one-dimensional.

| Gallileo Lie Group & Algebra

$$\begin{cases} \vec{x}' = R\vec{x} + \vec{u} \cdot t + \vec{w} \\ t' = t + e \end{cases}$$

\vec{x}, \vec{u} and $\vec{w} \in \mathbb{R}^3, e \in \mathbb{R}^+$

$$R \in SO(3)$$

| Bargmann Central extension:

$$\begin{bmatrix} R & \vec{u} & 0 & \vec{w} \\ 0 & 1 & 0 & e \\ -\vec{u}^t R & -\frac{\|\vec{u}\|^2}{2} & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{x}' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \vec{u} & \vec{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{\omega} & \vec{\eta} & \vec{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{cases} \vec{\eta} \text{ and } \vec{\gamma} \in \mathbb{R}^3, \varepsilon \in \mathbb{R}^+ \\ \vec{\omega} \in so(3) : \vec{x} \mapsto \vec{\omega} \times \vec{x} \end{cases}$$

Souriau Gibbs states for Hamiltonian actions of subgroups of the Galilean group

Souriau Gibbs states for one-parameter subgroups of the Galilean group

- **Souriau Result:** Action of the full Galilean group on the space of motions of an isolated mechanical system is not related to any Equilibrium Gibbs state (the open subset of the Lie algebra, associated to this Gibbs state, is empty)
- The **1-parameter subgroup of the Galilean group** generated by β element of Lie Algebra, is the set of matrices

$$\exp(\tau\beta) = \begin{pmatrix} A(\tau) & \vec{b}(\tau) & \vec{d}(\tau) \\ 0 & 1 & \tau\varepsilon \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \begin{cases} A(\tau) = \exp(\tau j(\vec{\omega})) \text{ and } \vec{b}(\tau) = \left(\sum_{i=1}^{\infty} \frac{\tau^i}{i!} (j(\vec{\omega}))^{i-1} \right) \vec{\alpha} \\ \vec{d}(\tau) = \left(\sum_{i=1}^{\infty} \frac{\tau^i}{i!} (j(\vec{\omega}))^{i-1} \right) \vec{\delta} + \varepsilon \left(\sum_{i=2}^{\infty} \frac{\tau^i}{i!} (j(\vec{\omega}))^{i-2} \right) \vec{\alpha} \end{cases}$$
$$\beta = \begin{pmatrix} j(\vec{\omega}) & \vec{\alpha} & \vec{\delta} \\ 0 & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$$

Souriau Thermodynamics of butter churn (device used to convert cream into butter) or “La Thermodynamique de la crémier”

If we consider the case of the centrifuge

$$\vec{\omega} = \omega \vec{e}_z, \vec{\alpha} = 0 \text{ and } \vec{\delta} = 0$$

Rotation speed : $\frac{\omega}{\varepsilon}$

$$f_i(\vec{r}_{i0}) = -\frac{\omega^2}{2\varepsilon^2} \|\vec{e}_z \times \vec{r}_{i0}\|^2$$

with $\Delta = \|\vec{e}_z \times \vec{r}_{i0}\|$ distance to axis z

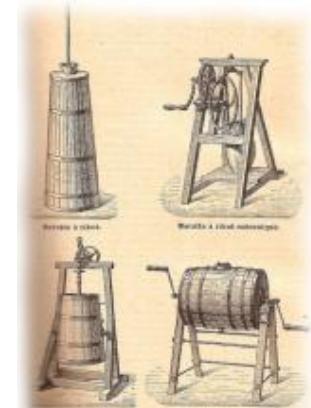
“The angular momentum is imparted to the gas when the molecules collide with the rotating walls, which changes the Maxwell distribution at every point, shifting its origin. The walls play the role of an angular momentum reservoir. Their motion is characterized by a certain angular velocity, and the angular velocities of the fluid and of the walls become equal at equilibrium, exactly like the equalization of the temperature through energy exchanges”. – Roger Balian



$$\rho_i(\beta) = \frac{1}{P_i(\beta)} \exp(-\langle J_i, \beta \rangle) = cst. \exp\left(-\frac{1}{2m_i \kappa T} \|\vec{p}_{i0}\|^2 + \frac{m_i}{2\kappa T} \left(\frac{\omega}{\varepsilon}\right)^2 \Delta^2\right)$$

- the behaviour of a gas made of point particles of various masses in a centrifuge rotating at a constant angular velocity (the heavier particles concentrate farther from the rotation axis than the lighter ones)

$$\frac{\omega}{\varepsilon}$$



Link with Classical Thermodynamics

| We have the reciprocal formula:

$$Q = \frac{\partial \Phi}{\partial \beta}$$

$$\beta = \frac{\partial S}{\partial Q}$$

$$S(Q) = \left\langle \frac{\partial \Phi}{\partial \beta}, \beta \right\rangle - \Phi$$

$$\Phi(\beta) = \left\langle Q, \frac{\partial S}{\partial Q} \right\rangle - S$$

| For Classical Thermodynamics (Time translation only), we recover the definition of Boltzmann-Clausius Entropy:

$$\begin{cases} \beta = \frac{\partial S}{\partial Q} \\ \beta = \frac{1}{T} \end{cases} \Rightarrow dS = \frac{dQ}{T}$$

Souriau Model of Covariant Gibbs Density

Covariant Souriau-Gibbs density

- Souriau has then defined a Gibbs density that is covariant under the action of the group:

$$p_{Gibbs}(\xi) = e^{\Phi(\beta) - \langle U(\xi), \beta \rangle} = \frac{e^{-\langle U(\xi), \beta \rangle}}{\int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega}$$

$$\text{with } \Phi(\beta) = -\log \int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega$$

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\int_M U(\xi) e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega}{\int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega} = \int_M U(\xi) p(\xi) d\lambda_\omega$$

- We can express the Gibbs density with respect to Q by inverting the relation

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \Theta(\beta) . \text{ Then } p_{Gibbs,Q}(\xi) = e^{\Phi(\beta) - \langle U(\xi), \Theta^{-1}(Q) \rangle} \text{ with } \beta = \Theta^{-1}(Q)$$

SOURIAU'S MOMENT MAP

By VLADIMIR ARNOLD

A Poisson action of a group G on a symplectic manifold M defines a mapping of M into the dual space of the Lie algebra of the group

$$P: M \rightarrow \mathfrak{g}^*.$$

That is, we fix a point x in M and consider the function on the Lie algebra which associates to an element a of the Lie algebra the value of the Hamiltonian H_a at the fixed point x :

$$p_x(a) = H_a(x).$$

This p_x is a linear function on the Lie algebra and is the element of the dual space to the algebra associated to x :

$$P(x) = p_x.$$

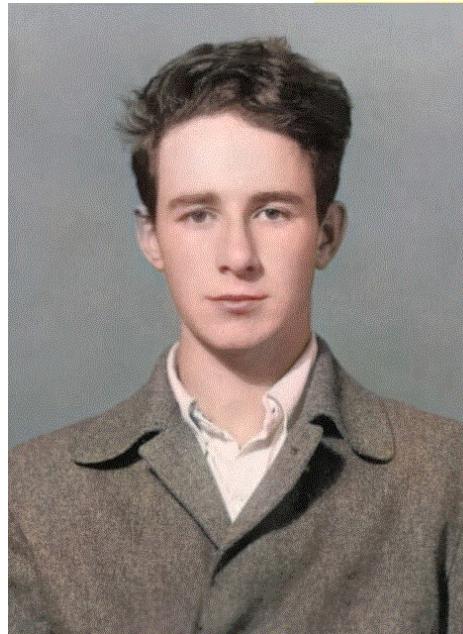
Following Souriau (*Structure des systèmes dynamiques*, Dunod, 1970), we will call the mapping P the *momentum*. Note that the value of the momentum is always a vector in the space \mathfrak{g}^* .

Theorem. Under the momentum mapping P , a Poisson action of a connected Lie group G is taken to the co-adjoint action of G on the dual space \mathfrak{g}^* of its Lie algebra (cf. Appendix 2), i.e., the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ \downarrow P & & \downarrow P \\ \mathfrak{g}^* & \xrightarrow{\text{Ad}_{g^{-1}}^*} & \mathfrak{g}^* \end{array}$$

Corollary. Suppose that a hamiltonian function $H: M \rightarrow \mathbb{R}$ is invariant under the Poisson action of a group G on M . Then the momentum is a first integral of the system with hamiltonian function H .

Graduate Texts in Mathematics



V.I. Arnold

Mathematical
Methods of
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Mechanics

Second Edition

 Springer

3.5 Noether type theorems

SOURIAU'S MOMENT MAP = GEOMETRIZATION OF NOETHER THEOREM

Consider now the “levels” of the moment map¹³ $\mu : W \rightarrow \mathfrak{g}^*$ of a hamiltonian G -action on the symplectic manifold (W, ω) . The most classical form of E. Noether’s theorem seems to be stated nowadays as:

¹³I shall call level ξ the inverse image $\mu^{-1}(\xi)$ even if ξ is not a number.

Theorem 3.5.1 *Let H be a function on W which is invariant by the G -action. Then μ is constant on the trajectories of the hamiltonian vector field X_H .*

Proof. Indeed, if $\gamma(t)$ is a trajectory of X_H , one can write, for any $X \in \mathfrak{g}$:

$$\begin{aligned}\frac{d}{dt} < \mu \circ \gamma(t), X > &= < T_{\gamma(t)}\mu(X_H(\gamma(t))), X > \\ &= < X_H(\gamma(t)), {}^tT_{\gamma(t)}\mu(X) > \\ &= < X_H(\gamma(t)), (i_X\omega)_{\gamma(t)} > \\ &= \omega(X, X_H)_{\gamma(t)} \\ &= -dH_{\gamma(t)}(X)\end{aligned}$$

But H is invariant and X a fundamental vector field of the action, and thus:

$$H(\exp(sX) \cdot \gamma(t)) = H(\gamma(t))$$

which, when differentiated at $s = 0$, is:

$$dH(X(\gamma(t))) = 0.$$



The field X_H is thus tangent to the levels $\mu^{-1}(\xi)$.

Michèle Audin

**The Topology of
Torus Actions on
Symplectic Manifolds**

<https://link.springer.com/book/10.1007/978-3-0348-7221-8>

Birkhäuser

ENTROPY as Casimir Function in Coadjoint Representation Constant on Symplectic Leaves & Density of Maximum Entropy on Lie Groups

Jean-Marie Souriau



Souriau Entropy Invariance

| Casimir Invariant Function in coadjoint representation

- We observe that Souriau Entropy $S(Q)$ defined on coadjoint orbit of the group has a property of invariance :

$$S(Ad_g^\#(Q)) = S(Q)$$

- with respect to Souriau affine definition of coadjoint action:

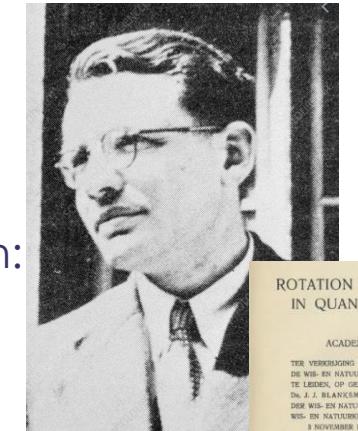
$$Ad_g^\#(Q) = Ad_g^*(Q) + \theta(g)$$

- where $\theta(g)$ is called the Souriau cocycle.

$$Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$$

$$S(Q(Ad_g(\beta))) = S(Q)$$

New Entropy Definition:
Casimir Function in
Coadjoint Representation
Invariant under the action
of the Group



Hendrik Casimir
(Thesis supervised by
Niels Bohr & Paul Ehrenfest)

H.B.G. Casimir, On the Rotation of a Rigid Body in
Quantum Mechanics, Doctoral Thesis, Leiden, 1931.

Entropy Invariance under the action of the Group (1/2)

$$\beta \in \mathfrak{g} \rightarrow Ad_g(\beta) \Rightarrow \Psi(Ad_g(\beta)) = \int_M e^{-\langle U, Ad_g(\beta) \rangle} d\lambda_\omega$$

$$\Psi(Ad_g(\beta)) = \int_M e^{-\langle Ad_{g^{-1}}^* U, \beta \rangle} d\lambda_\omega = \int_M e^{-\langle U(Ad_{g^{-1}}\beta) - \theta(g^{-1}), \beta \rangle} d\lambda_\omega$$

$$\Psi(Ad_g(\beta)) = e^{\langle \theta(g^{-1}), \beta \rangle} \Psi(\beta)$$

$$\theta(g^{-1}) = -Ad_{g^{-1}}^* \theta(g) \Rightarrow \Psi(Ad_g(\beta)) = e^{-\langle Ad_{g^{-1}}^* \theta(g), \beta \rangle} \Psi(\beta)$$

$$\Phi(\beta) = -\log \Psi(\beta)$$

$$\Rightarrow \Phi(Ad_g(\beta)) = \Phi(\beta) - \langle \theta(g^{-1}), \beta \rangle = \Phi(\beta) + \langle Ad_{g^{-1}}^* \theta(g), \beta \rangle$$

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Entropy Invariance under the action of the Group (2/2)

$$S(Q) = \langle Q, \beta \rangle - \Phi(\beta) \Rightarrow S(Q(Ad_g \beta)) = \langle Q(Ad_g \beta), Ad_g \beta \rangle - \Phi(Ad_g \beta)$$

$$\left\{ \begin{array}{l} Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g) \end{array} \right.$$

$$\left\{ \begin{array}{l} \Phi(Ad_g(\beta)) = -\log \Psi(Ad_g(\beta)) = -\langle \theta(g^{-1}), \beta \rangle + \Phi(\beta) \end{array} \right.$$

$$\Rightarrow S(Q(Ad_g \beta)) = \langle Ad_g^*(Q) + \theta(g), Ad_g \beta \rangle + \langle \theta(g^{-1}), \beta \rangle - \Phi(\beta)$$

$$\Rightarrow S(Q(Ad_g \beta)) = \langle Ad_g^*(Q) + \theta(g), Ad_g \beta \rangle - \langle Ad_{g^{-1}}^* \theta(g), \beta \rangle - \Phi(\beta)$$

$$\Rightarrow S(Q(Ad_g \beta)) = \langle Ad_{g^{-1}}^* Ad_g^*(Q) + Ad_{g^{-1}}^* \theta(g), \beta \rangle - \langle Ad_{g^{-1}}^* \theta(g), \beta \rangle - \Phi(\beta)$$

$$Ad_{g^{-1}}^* Ad_g^*(Q) = Q \Rightarrow S(Q(Ad_g \beta)) = \langle Q, \beta \rangle - \Phi(\beta) = S(\beta)$$

Casimir Function and Entropy

- Classically, the Entropy is defined axiomatically as Shannon or von Neumann Entropies without any geometric structures constraints.
- Entropy could be built by Casimir Function Equation:

$$\left(ad_{\frac{\partial S}{\partial Q}}^* Q \right)_j + \Theta \left(\frac{\partial S}{\partial Q} \right)_j = C_{ij}^k ad_{\left(\frac{\partial S}{\partial Q} \right)^i}^* Q_k + \Theta_j = 0$$

$$\tilde{\Theta}(X, Y) = \langle \Theta(X), Y \rangle = J_{[X, Y]} - \{J_X, J_Y\} = -\langle d\theta(X), Y \rangle \quad , \quad X, Y \in \mathfrak{g}$$

$$\Theta(X) = T_e \theta(X(e))$$

$$\theta(g) = Q(Ad_g(\beta)) - Ad_g^*(Q)$$

Souriau Entropy and Casimir Invariant Function

Demo

- if we consider the heat expression $Q = \frac{\partial \Phi}{\partial \beta}$, that we can write $\delta \Phi - \langle Q, \delta \beta \rangle = 0$.
- For each $\delta \beta$ tangent to the orbit, and so generated by an element Z of the Lie algebra, if we consider the relation $\Phi(Ad_g(\beta)) = \Phi(\beta) - \langle \theta(g^{-1}), \beta \rangle$, we differentiate it at $g = e$ using the property that:

$$\tilde{\Theta}(X, Y) = -\langle d\theta(X), Y \rangle, \quad X, Y \in \mathfrak{g}$$

- we obtain : $\langle Q, [\beta, Z] \rangle + \tilde{\Theta}(\beta, Z) = 0$
- From last Souriau equation, if we use the identities $\beta = \frac{\partial S}{\partial Q}$, $ad_\beta Z = [\beta, Z]$ and $\tilde{\Theta}(\beta, Z) = \langle \Theta(\beta), Z \rangle$
- Then we can deduce that: $\left\langle ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta\left(\frac{\partial S}{\partial Q}\right), Z \right\rangle = 0, \quad \forall Z$
- So, Entropy $S(Q)$ should verify:

$$ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0 \quad \{S, H\}_{\tilde{\Theta}}(Q) = 0 \quad \forall H : \mathfrak{g}^* \rightarrow R, \quad Q \in \mathfrak{g}^*$$

$$\{S, H\}_{\tilde{\Theta}}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q}\right) = 0$$

Entropy as Invariant Casimir Function in Coadjoint Representation

NEW GEOMETRIC DEFINITION OF ENTROPY

$$\{S, H\}_{\tilde{\Theta}}(Q) = 0$$

$$ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

$$\{S, H\}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle = -C_{ij}^k Q_k \frac{\partial S}{\partial Q_i} \cdot \frac{\partial H}{\partial Q_j}$$

$$[e_i, e_j] = C_{ij}^k e_k \quad , \quad C_{ij}^k \text{ structure coefficients}$$

$$\{S, H\}_{\tilde{\Theta}}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q}\right) = 0 \quad , \quad \forall H : \mathfrak{g}^* \rightarrow \mathbb{R}, \quad Q \in \mathfrak{g}^*$$

$$\tilde{\Theta}(X, Y) = J_{[X, Y]} - \{J_X, J_Y\} \quad \text{where} \quad J_X(x) = \langle J(x), X \rangle$$

$$\tilde{\Theta}(X, Y) = \langle \Theta(X), Y \rangle \quad \text{with} \quad \Theta(X) = T_e \theta(X(e))$$

$$\theta(g) = Q(Ad_g(\beta)) - Ad_g^*(Q)$$

Souriau Entropy and Casimir Invariant Function

| Demo

$$\{S, H\}_{\tilde{\Theta}}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle + \left\langle \Theta\left(\frac{\partial S}{\partial Q}\right), \frac{\partial H}{\partial Q} \right\rangle = 0$$

$$\{S, H\}_{\tilde{\Theta}}(Q) = \left\langle Q, ad_{\frac{\partial S}{\partial Q}} \frac{\partial H}{\partial Q} \right\rangle + \left\langle \Theta\left(\frac{\partial S}{\partial Q}\right), \frac{\partial H}{\partial Q} \right\rangle = 0$$

$$\{S, H\}_{\tilde{\Theta}}(Q) = \left\langle ad_{\frac{\partial S}{\partial Q}}^* Q, \frac{\partial H}{\partial Q} \right\rangle + \left\langle \Theta\left(\frac{\partial S}{\partial Q}\right), \frac{\partial H}{\partial Q} \right\rangle = 0$$

$$\forall H, \{S, H\}_{\tilde{\Theta}}(Q) = \left\langle ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta\left(\frac{\partial S}{\partial Q}\right), \frac{\partial H}{\partial Q} \right\rangle = 0 \Rightarrow ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

Souriau Entropy and Casimir Invariant Function

Link with Souriau development

> Souriau property: $\beta \in \text{Ker} \tilde{\Theta}_\beta \Rightarrow \langle Q, [\beta, Z] \rangle + \tilde{\Theta}(\beta, Z) = 0$

$$\Rightarrow \langle Q, ad_\beta Z \rangle + \tilde{\Theta}(\beta, Z) = 0 \Rightarrow \langle ad_\beta^* Q, Z \rangle + \tilde{\Theta}(\beta, Z) = 0$$

$$\beta = \frac{\partial S}{\partial Q} \Rightarrow \left\langle ad_{\frac{\partial S}{\partial Q}}^* Q, Z \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, Z\right) = \left\langle ad_{\frac{\partial S}{\partial Q}}^* Q, Z \right\rangle + \left\langle \Theta\left(\frac{\partial S}{\partial Q}\right), Z \right\rangle = 0$$

$$\Rightarrow \left\langle ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta\left(\frac{\partial S}{\partial Q}\right), Z \right\rangle = 0, \forall Z$$

$$\Rightarrow ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

Souriau relation on foliation

- In his 1974 paper, Jean-Marie Souriau has written (without proof):

$$\langle Q, [\beta, Z] \rangle + \tilde{\Theta}(\beta, Z) = 0$$

- To prove this equation, we have to consider the parametrized curve
 $t \mapsto Ad_{\exp(tZ)}\beta$ with $Z \in \mathfrak{g}$ and $t \in R$

- The parameterized curve $Ad_{\exp(tZ)}\beta$ passes, for $t=0$, through the point β , since $Ad_{\exp(0)}$ is the identical map of the Lie Algebra \mathfrak{g} . This curve is in the adjoint orbit of β . So by taking its derivative with respect to t , then for $t=0$, we obtain a tangent vector in β at the adjoint orbit of this point. When Z takes all possible values in \mathfrak{g} , the vectors thus obtained generate all the vector space tangent in β to the orbit of this point:

$$\left. \frac{d\Phi(Ad_{\exp(tZ)}\beta)}{dt} \right|_{t=0} = \left\langle \frac{d\Phi}{d\beta} \left(\left. \frac{d(Ad_{\exp(tZ)}\beta)}{dt} \right|_{t=0} \right) \right\rangle = \langle Q, ad_z \beta \rangle = \langle Q, [Z, \beta] \rangle$$

Souriau relation on foliation

- As we have seen before $\Phi(Ad_g\beta) = \Phi(\beta) - \langle \theta(g^{-1}), \beta \rangle$. If we set $g = \exp(tZ)$, we obtain:
$$\Phi(Ad_{\exp(tZ)}\beta) = \Phi(\beta) - \langle \theta(\exp(-tZ)), \beta \rangle$$
- By derivation with respect to t at $t=0$, we finally recover the equation given by Souriau :

$$\left. \frac{d\Phi(Ad_{\exp(tZ)}\beta)}{dt} \right|_{t=0} = \langle Q, [Z, \beta] \rangle = -\langle d\theta(-Z), \beta \rangle \text{ with } \tilde{\Theta}(X, Y) = -\langle d\theta(X), Y \rangle$$

Fundamental Equation of Geometric Thermodynamic: Entropy Function is an Invariant Casimir Function in Coadjoint Representation

Entropy S

Heat Q , (Planck) Temperature β and Φ Massieu Characteristic Function

$$S : \mathfrak{g}^* \rightarrow R \\ Q \mapsto S(Q)$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta), Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^*, \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

Invariance of Entropy S
Under the action of the Group

New Definition of Entropy S
as Invariant Casimir Function in Coadjoint Representation

$$Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$$

$$ad_{\frac{\partial S}{\partial Q}}^* Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

$$S(Q(Ad_g(\beta))) = S(Q)$$

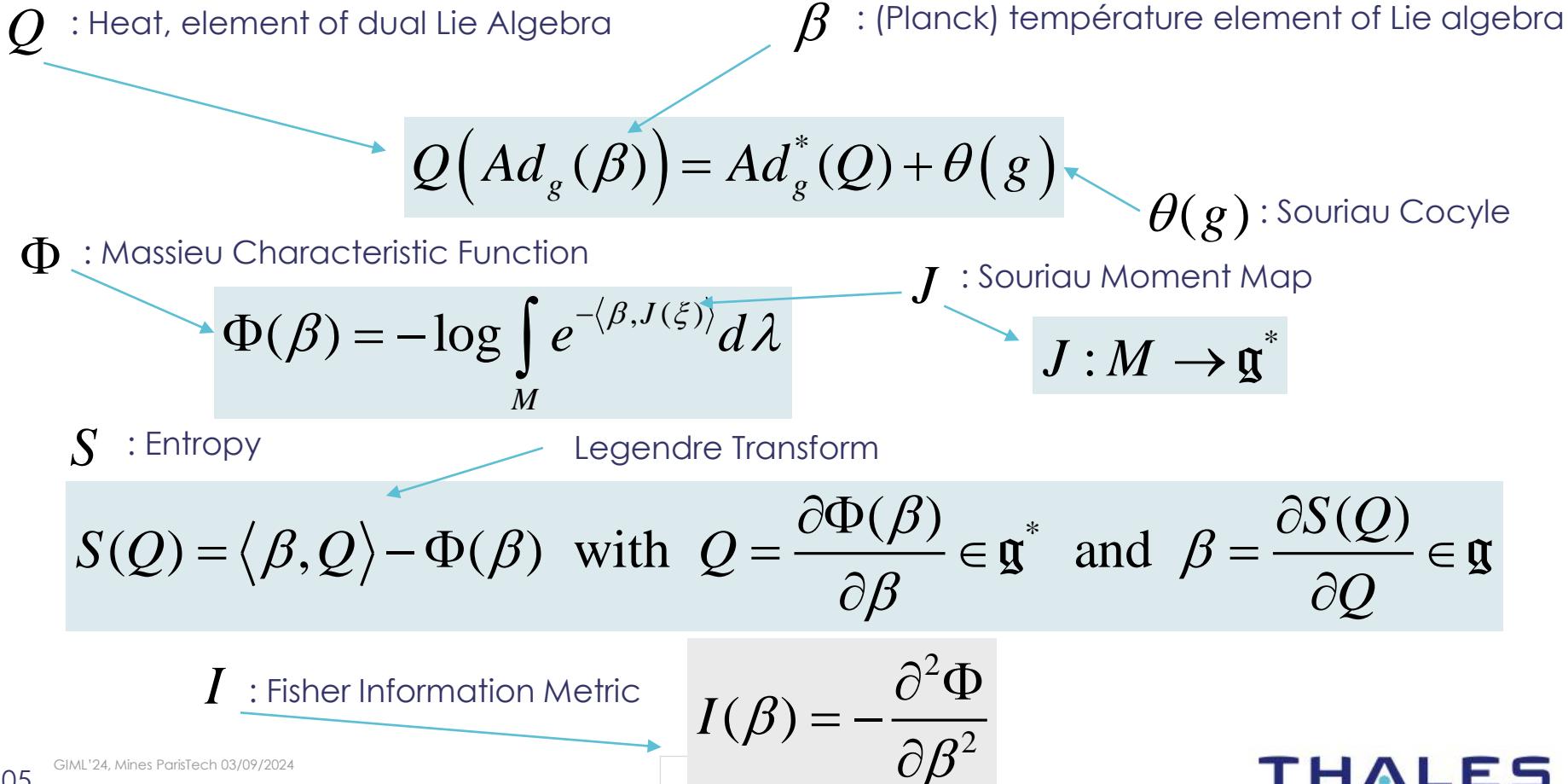
$$\{S, H\}_{\tilde{\Theta}}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q}\right) = 0$$

$$\Theta(X) = T_e \theta(X(e))$$

$$\tilde{\Theta}(X, Y) = \langle \Theta(X), Y \rangle = J_{[X, Y]} - \{J_X, J_Y\} = -\langle d\theta(X), Y \rangle$$

Moment Map J

Lie Groups Thermodynamic Equations and its extension (1/3)



Lie Groups Thermodynamic Equations and its extension (2/3)

Entropy Invariance under the action of the Group !

$$S(Ad_g^{\#}(Q)) = S(Q)$$

$$Ad_g^{\#}(Q) = Ad_g^{*}(Q) + \theta(g)$$

Souriau characteristic of the foliation

$$\langle Q, [\beta, Z] \rangle + \tilde{\Theta}(\beta, Z) = 0$$

Entropy & Poisson Bracket

$$\rightarrow \{S, H\}_{\tilde{\Theta}}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right) = 0$$

Entropy Solution of Casimir Equation

$$ad_{\frac{\partial S}{\partial Q}}^{*} Q + \Theta\left(\frac{\partial S}{\partial Q} \right) = 0$$

$$\Theta(X) = T_e \theta(X(e))$$

$$\theta(g) = Q(Ad_g(\beta)) - Ad_g^{*}(Q)$$

$$\tilde{\Theta}(X, Y) = \langle \Theta(X), Y \rangle = J_{[X, Y]} - \{J_X, J_Y\} = -\langle d\theta(X), Y \rangle$$

THALES

Lie Groups Thermodynamic Equations and its extension (3/3)

Entropy Production

$$dS = \tilde{\Theta}_\beta \left(\frac{\partial H}{\partial Q}, \beta \right) dt$$

2nd principle is related to positivity of Fisher tensor

$$\frac{dS}{dt} = \tilde{\Theta}_\beta \left(\frac{\partial H}{\partial Q}, \frac{\partial S}{\partial Q} \right) \geq 0$$

Metric Tensor related to Fisher Metric

$$\tilde{\Theta}_\beta \left(\frac{\partial H}{\partial Q}, \beta \right) = \tilde{\Theta} \left(\frac{\partial H}{\partial Q}, \beta \right) + \left\langle Q, \left[\frac{\partial H}{\partial Q}, \beta \right] \right\rangle$$

Time Evolution of Heat wrt to Hamiltonian H

$$\frac{dQ}{dt} = ad_{\frac{\partial H}{\partial Q}}^* Q + \Theta \left(\frac{\partial H}{\partial Q} \right)$$

Stochastic Equation

$$dQ + \left[ad_{\frac{\partial H}{\partial Q}}^* Q + \Theta \left(\frac{\partial H}{\partial Q} \right) \right] dt + \sum_{i=1}^N \left[ad_{\frac{\partial H_i}{\partial Q}}^* Q + \Theta \left(\frac{\partial H_i}{\partial Q} \right) \right] \circ dW_i(t) = 0$$

Euler-Poincaré Equation in case of Non-Null Cohomology

$$\frac{dQ}{dt} = ad_{\frac{\partial H}{\partial Q}}^* Q + \Theta \left(\frac{\partial H}{\partial Q} \right)$$

$$Q = \frac{\partial \Phi}{\partial \beta}$$

$$\frac{d}{dt} \frac{\partial \Phi}{\partial \beta} = ad_{\frac{\partial H}{\partial Q}}^* \frac{\partial \Phi}{\partial \beta} + \Theta \left(\frac{\partial H}{\partial Q} \right)$$

$$\left(ad_{\frac{\partial H}{\partial Q}}^* \frac{\partial \Phi}{\partial \beta} \right)_j + \Theta \left(\frac{\partial H}{\partial Q} \right)_j = C_{ij}^k ad_{\left(\frac{\partial H}{\partial Q} \right)^i}^* \left(\frac{\partial \Phi}{\partial \beta} \right)_k + \Theta_j$$

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître » - Henri Poincaré, CRAS, 18 Février 1901

SÉANCE DU LUNDI 18 FÉVRIER 1901,

PRÉSIDENCE DE M. FOUQUÉ.

MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADEMIE.

MÉCANIQUE RATIONNELLE. — Sur une forme nouvelle des équations de la Mécanique. Note de M. **H. POINCARÉ**.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

$$\frac{d}{dt} \frac{dT}{d\eta_s} = \sum c_{ski} \frac{dT}{d\eta_i} \eta_k + \Omega_s.$$

« Elles sont surtout intéressantes dans le cas où U étant nul, T ne dépend que des η » - Henri Poincaré

de Saxcé, G. Euler-Poincaré equation for Lie groups with non null symplectic cohomology. Application to the mechanics. In GSI 2019. LNCS; Nielsen, F., Barbaresco, F., Eds.; Springer: Berlin, Germany, 2019; Volume 11712

OPEN

THALES

Lie-Poisson variational principle

$$\frac{dQ}{dt} = ad_{\frac{\partial H}{\partial Q}}^* Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

➤ This Lie-Poisson equation is equivalent to this **Lie-Poisson variational principle**:

$$\delta \int_0^\tau \left(\left\langle Q(t), \frac{\partial H}{\partial Q}(t) \right\rangle - H(Q(t)) \right) dt = 0 \text{ where}$$

$$\begin{cases} \frac{\partial H}{\partial Q} = g^{-1} \dot{g} \in \mathfrak{g}, g \in G \\ \frac{\partial^2 H}{\partial Q^2} \delta Q = \delta \left(\frac{\partial H}{\partial Q} \right), \eta = g^{-1} \delta g \\ \left\langle Q, \delta \left(\frac{\partial H}{\partial Q} \right) \right\rangle = \left\langle Q, \dot{\eta} + \left[\frac{\partial H}{\partial Q}, \eta \right] \right\rangle + \tilde{\Theta} \left(\frac{\partial H}{\partial Q}, \eta \right) \end{cases}$$

Lie-Poisson variational principle

> Proof of Lie-Poisson variational principle:

$$\begin{aligned} \delta \int_0^\tau \left(\left\langle Q(t), \frac{\partial H}{\partial Q}(t) \right\rangle - H(Q(t)) \right) dt &= \int_0^\tau \left(\left\langle \delta Q, \frac{\partial H}{\partial Q} \right\rangle + \left\langle Q, \delta \left(\frac{\partial H}{\partial Q} \right) \right\rangle - \left\langle \delta Q, \frac{\partial H}{\partial Q} \right\rangle \right) dt = 0 \\ &= \int_0^\tau \left(\left\langle Q, \frac{d\eta}{dt} + \left[\frac{\partial H}{\partial Q}, \eta \right] \right\rangle + \tilde{\Theta} \left(\frac{\partial H}{\partial Q}, \eta \right) \right) dt = \int_0^\tau \left(\left\langle Q, \frac{d\eta}{dt} \right\rangle + \left\langle Q, ad_{\frac{dH}{dQ}} \eta \right\rangle + \left\langle \Theta \left(\frac{\partial H}{\partial Q} \right), \eta \right\rangle \right) dt \\ &\stackrel{\substack{\text{Int.} \\ \text{by} \\ \text{parts}}}{=} \int_0^\tau \left\langle -\frac{dQ}{dt} + ad_{\frac{dH}{dQ}}^* Q + \Theta \left(\frac{\partial H}{\partial Q} \right), \eta \right\rangle dt + \langle Q, \eta \rangle \Big|_0^\tau = 0 \end{aligned}$$

Koszul Poisson Cohomology and Entropy Characterization

| Poisson Cohomology was introduced by A. Lichnerowicz and J.L. Koszul.

| Koszul Cohomology and seminal work of Elie Cartan. Koszul made reference to seminal E. Cartan paper

> “*Elie Cartan does not explicitly mention $\Lambda(g')$ [the complex of alternate forms on a Lie algebra], because he treats groups as symmetrical spaces and is therefore interested in differential forms which are invariant to both by the translations to the left and the translations to the right, which corresponds to the elements of $\Lambda(g')$ invariant by the prolongation of the coadjoint representation. Nevertheless, it can be said that by 1929 an essential piece of the cohomological theory of Lie algebras was in place.*” – Jean-Louis Koszul

Koszul Poisson Cohomology and Entropy Characterization

- Y. Vorob'ev and M.V. Karasev have suggested cohomology classification in terms of closed forms and de Rham Cohomology of coadjoint orbits Ω (called Euler orbits by authors), symplectic leaves of a Poisson manifold N .
- Let $Z^k(\Omega)$ and $H^k(\Omega)$ be the space of closed k-forms on Ω and their de Rham cohomology classes.
- Considering the base of the fibration of N by these orbits as N/Ω , they have introduced the smooth mapping

$$Z^k[\Omega] = C^\infty(N/\Omega \rightarrow Z^k(\Omega)) \text{ and } H^k[\Omega] = C^\infty(N/\Omega \rightarrow H^k(\Omega))$$

- The elements of $Z^k(\Omega)$ are closed forms on Ω , depending on coordinates on N/Ω
- Then $H^0[\Omega] = \text{Casim}(N)$ is the set of Casimir functions on N , of functions which are constant on all Euler orbits.
- **Entropy is then characterized by zero-dimensional de Rham Cohomology.**
- The center of Poisson algebra induced from the symplectic structure is the zero-dimensional de Rham cohomology group, the Casimir functions.

Poincaré Unit Disk and $SU(1,1)$ Lie Group

> The group of complex unimodular pseudo-unitary matrices $SU(1,1)$:

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} / |a|^2 - |b|^2 = 1, \quad a, b \in \mathbb{C} \right\}$$

> the Lie algebra $\mathfrak{g} = \mathfrak{su}(1,1)$ is given by:

$$\mathfrak{g} = \left\{ \begin{pmatrix} -ir & \eta \\ \eta^* & ir \end{pmatrix} / r \in \mathbb{R}, \eta \in \mathbb{C} \right\}$$

with the following bases $(u_1, u_2, u_3) \in \mathfrak{g}$:

$$u_1 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

with the commutation relation:

$$[u_3, u_2] = u_1, [u_3, u_1] = u_2, [u_2, u_1] = -u_3$$

Poincaré Unit Disk and $SU(1,1)$ Lie Group

- Dual base on dual Lie algebra is named

$$(u_1^*, u_2^*, u_3^*) \in \mathfrak{g}^*$$

- The dual vector space $\mathfrak{g}^* = \mathfrak{su}^*(1,1)$ can be identified with the subspace of $\mathfrak{sl}(2, C)$ of the form:

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / x, y, z \in \mathbb{R} \right\}$$

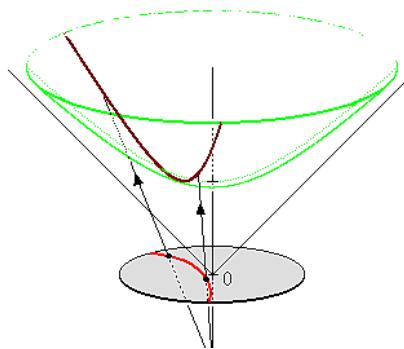
- Coadjoint action of $g \in G$ on dual Lie algebra $\xi \in \mathfrak{g}^*$ is written $g \cdot \xi$

Coadjoint Orbit of $SU(1,1)$ and Souriau Moment Map

$$J(z) = r \left(\frac{z + z^*}{(1 - |z|^2)} u_1^* + \frac{z - z^*}{i(1 - |z|^2)} u_2^* + \frac{1 + |z|^2}{(1 - |z|^2)} u_3^* \right) \in O(ru_3^*), z \in D$$

- J is linked to the natural action of G on D (by fractional linear transforms) but also the coadjoint action of G on $O(ru_3^*) = G / K$
- J^{-1} could be interpreted as the stereographic projection from the two-sphere S^2 onto $\mathbb{C} \cup \infty$:

The coadjoint action of $G = SU(1,1)$ is the upper sheet $x_3 > 0$ of the two-sheet hyperboloid



Charles-Michel Marle, Projection stéréographique et moments, hal-02157930, version 1, Juin 2019

L115 $\left\{ \xi = x_1 u_1^* + x_2 u_2^* + x_3 u_3^* : -x_1^2 - x_2^2 + x_3^2 = r^2 \right\}$

Moment Map for $SU(1,1)$

Invariant Moment Map

- > The associated moment map $J : D \rightarrow su^*(1,1)$ defined by $J(z).u_i = J_i(z, z^*)$, maps D into a coadjoint orbit in $su^*(1,1)$.
- > Then, we can write the moment map as a matrix element of $su^*(1,1)$:

$$J(z) = J_1(z, z^*)u_1^* + J_2(z, z^*)u_2^* + J_3(z, z^*)u_3^*$$

$$J(z) = r \begin{pmatrix} 1+|z|^2 & -2\frac{z^*}{1-|z|^2} \\ \frac{1-|z|^2}{1+|z|^2} & 1-\frac{|z|^2}{1-|z|^2} \\ 2\frac{z}{1-|z|^2} & -\frac{1+|z|^2}{1-|z|^2} \end{pmatrix} \in \mathfrak{g}^*$$

$$J(z) = r \left(\frac{z+z^*}{(1-|z|^2)} u_1^* + \frac{z-z^*}{i(1-|z|^2)} u_2^* + \frac{1+|z|^2}{(1-|z|^2)} u_3^* \right), z \in D$$

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / x, y, z \in \mathbb{R} \right\}$$

Poisson Bracket for SU(1,1)

- Since the unit disk is Kählerian, it is symplectic and so can be given a phase space structure and interpretation. This Poisson Bracket could be written in terms of the Poincare disk coordinates as:

$$\{f, g\} = \frac{(1 - |z|^2)^2}{2i} \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial z^*} - \frac{\partial f}{\partial z^*} \frac{\partial g}{\partial z} \right)$$

- It is possible to define new coordinates (q, p) that are canonical in the sense that:

$$\{f, g\} = \left(\frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \right) \text{ with coordinates given by: } \frac{q + ip}{2} = \frac{z}{\sqrt{1 - |z|^2}}$$

- The Metriplectic equation is then given by:

$$\frac{\partial f}{\partial t} = \{f, H\} + (f, H) = \frac{(1 - |z|^2)^2}{2i} \left(\frac{\partial f}{\partial z} \frac{\partial H}{\partial z^*} - \frac{\partial f}{\partial z^*} \frac{\partial H}{\partial z} \right) + (f, H)$$

Souriau Lie Groups Thermodynamics Model for $SU(1,1)$ and Unit Disk

$$J(z) = J_1(z, z^*) u_1^* + J_2(z, z^*) u_2^* + J_3(z, z^*) u_3^*$$

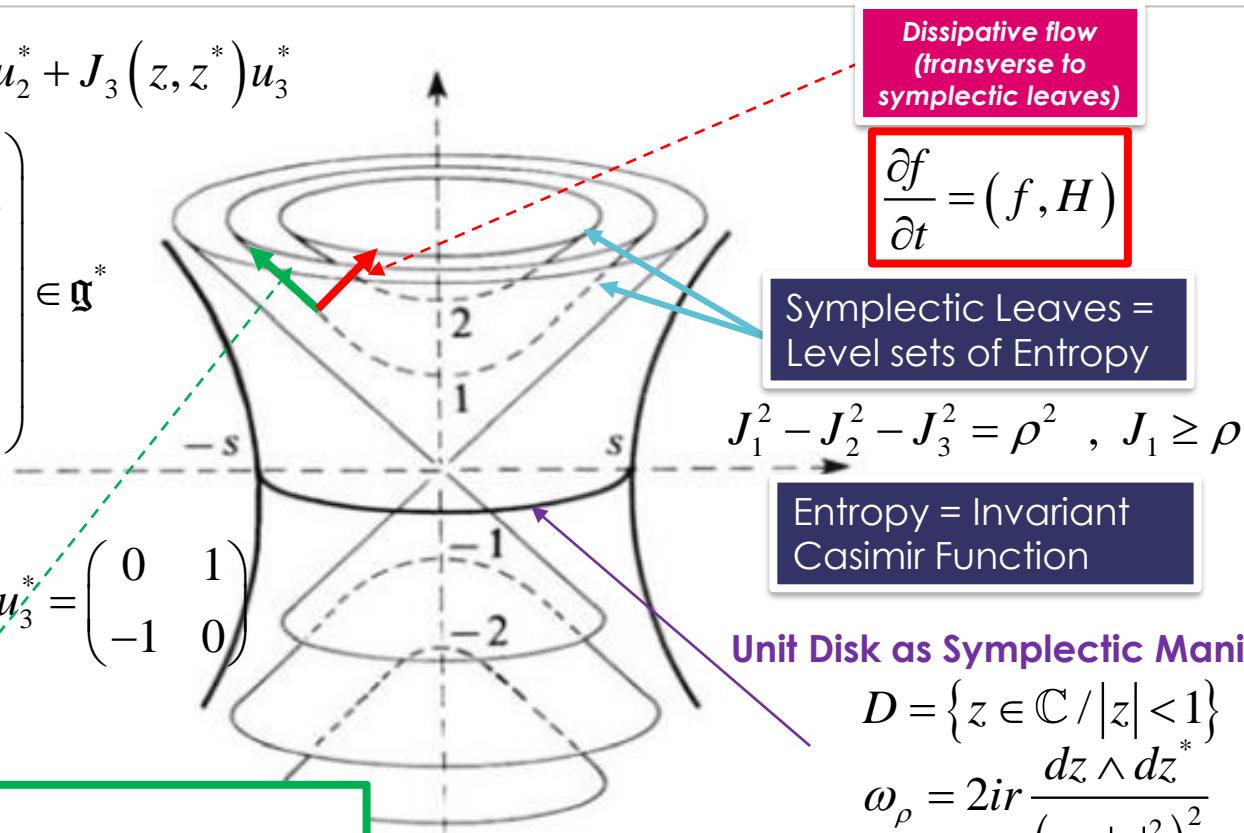
$$J(z) = \rho \begin{pmatrix} \frac{1+|z|^2}{1-|z|^2} & -2\frac{z^*}{1-|z|^2} \\ 2\frac{z}{1-|z|^2} & -\frac{1+|z|^2}{1-|z|^2} \end{pmatrix} \in \mathfrak{g}^*$$

$$J_1 u_1^* + J_2 u_2^* + J_3 u_3^* \in su^*(1,1)$$

$$u_1^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u_2^* = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_3^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Non Dissipative flow
(along the symplectic leaves)

$$\frac{\partial f}{\partial t} = \{f, H\} = \frac{(1-|z|^2)^2}{2i} \left(\frac{\partial f}{\partial z} \frac{\partial H}{\partial z^*} - \frac{\partial f}{\partial z^*} \frac{\partial H}{\partial z} \right)$$



$$D = \{z \in \mathbb{C} / |z| < 1\}$$

$$\omega_\rho = 2ir \frac{dz \wedge dz^*}{(1-|z|^2)^2}$$

THALES

Souriau Gibbs density for $SU(1,1)$

Covariant Gibbs density

► We can write the covariant Gibbs density in the unit disk given by moment map of the Lie group $SU(1,1)$ and geometric temperature in its Lie algebra $\beta \in \Lambda_\beta$:

$$p_{Gibbs}(z) = \frac{e^{-\langle J(z), \beta \rangle}}{\int_D e^{-\langle J(z), \beta \rangle} d\lambda(z)} \text{ with } d\lambda(z) = 2ir \frac{dz \wedge dz^*}{(1 - |z|^2)^2}$$
$$-\left\langle r \begin{pmatrix} \frac{1+|z|^2}{(1-|z|^2)} & \frac{-2z^*}{(1-|z|^2)} \\ \frac{2z}{(1-|z|^2)} & \frac{1+|z|^2}{(1-|z|^2)} \end{pmatrix}, \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix} \right\rangle$$
$$p_{Gibbs}(z) = \frac{e^{-\langle r(2\Im bb^+ - Tr(\Im bb^+)I), \beta \rangle}}{\int_D e^{-\langle J(z), \beta \rangle} d\lambda(z)} = \frac{e^{-\langle r(2\Im bb^+ - Tr(\Im bb^+)I), \beta \rangle}}{\int_D e^{-\langle J(z), \beta \rangle} d\lambda(z)}$$

$$J(z) = r(2Mbb^+ - Tr(Mbb^+)I) \text{ with } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } b = \frac{1}{1 - |z|^2} \begin{bmatrix} 1 \\ -z \end{bmatrix}$$

Souriau Gibbs density

Covariant Gibbs Density

$$p_{Gibbs}(z) = \frac{e^{-\langle J(z), \beta \rangle}}{\int_D e^{-\langle J(z), \beta \rangle} d\lambda(z)}$$
$$\rho \begin{pmatrix} \frac{1+|z|^2}{(1-|z|^2)} & \frac{-2z^*}{(1-|z|^2)} \\ \frac{2z}{(1-|z|^2)} & -\frac{1+|z|^2}{(1-|z|^2)} \end{pmatrix}, \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix}$$

- To write the Gibbs density with respect to its statistical moments, we have to express the density with respect to $Q = E[J(z)]$
- Then, we have to invert the relation between Q and β , to replace $\beta = \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix} \in \Lambda_\beta$ by $\beta = \Theta^{-1}(Q) \in \mathfrak{g}$ where $Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \Theta(\beta) \in \mathfrak{g}^*$ with $\Phi(\beta) = -\log \int_D e^{-\langle J(z), \beta \rangle} d\lambda(z)$ deduce from Legendre transform. The mean moment map is given by:

$$Q = E[J(z)] = E \left[\rho \begin{pmatrix} \frac{1+|w|^2}{(1-|w|^2)} & \frac{-2w^*}{(1-|w|^2)} \\ \frac{2w}{(1-|w|^2)} & -\frac{1+|w|^2}{(1-|w|^2)} \end{pmatrix} \right]$$

where $w \in D$

Gauss Density on Siegel Unit Disk

Moment Map of $SU(n,n)/S(U(n)\times U(n))$

> The moment map for $SU(n,n)/S(U(n)\times U(n))$ is then given by:

$$J(Z) = \rho n \begin{pmatrix} (I_n - ZZ^+)^{-1} (I_n + ZZ^+) & -2Z^+ (I_n - ZZ^+)^{-1} \\ 2(I_n - ZZ^+)^{-1} Z & (I_n + ZZ^+) (I_n - ZZ^+)^{-1} \end{pmatrix} \in \mathfrak{g}^*$$

> The Souriau Gibbs density is then given with $\beta, M \in \mathfrak{g}$ and $Z \in SD_n$ by:

$$p_{Gibbs}(Z) = \frac{e^{-\left\langle \rho n \begin{pmatrix} (I_n - ZZ^+)^{-1} (I_n + ZZ^+) & -2Z^+ (I_n - ZZ^+)^{-1} \\ 2(I_n - ZZ^+)^{-1} Z & (I_n + ZZ^+) (I_n - ZZ^+)^{-1} \end{pmatrix}, \beta \right\rangle}}}{\int_{SD_n} e^{-\langle J(Z), \beta \rangle} d\lambda(Z)}$$

$$\beta = \Theta^{-1}(Q) \in \mathfrak{g}$$

$$Q = E[J(Z)]$$

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \Theta(\beta) \in \mathfrak{g}^*$$

> Gauss density of SPD matrix is given by Cayley Transform with:

$$Z = (Y - I)(Y + I)^{-1}_{\text{OPEN}}, Y \in \text{Sym}(n)^+$$

Lars Onsager & Baptiste Coquinet: *Symplectic Foliation Model of METRIPLECTIC FLOW*



L. ONSAGER



B. COQUINET



METRIPLECTIC & GENERIC Models

- Systems that preserve energy throughout the phase are characterized by an Hamiltonian formulation of dynamics.
- Classical Hamiltonian systems cannot take into account dissipative effects, as irreversible changes from a thermodynamic standpoint (dissipative dynamics that do not preserve energy).
- A. N. Kaufman and P.J. Morrison have introduced in 1983, the **metrilectic** bracket by introducing a bracket formalism that ensures both conservation of energy and non-decrease of entropy, and that reduces to the standard Poisson bracket formalism in the limit of no dissipation.
- This model has been axiomatized in parallel by Grmela and his collaborators (**GENERIC method: General Equation for Non-Equilibrium Reversible Irreversible Coupling**).

METRIPLECTIC FRAMEWORK FOR DISSIPATIVE HEAT EQUATION

| Metriplectic Model

- Dissipation could take two forms:
 - viscosity removes energy from the system (e.g. Navier-Stokes equation)
 - thermal diffusion with conservation of energy and entropy production by heat transfer (e.g. Boltzmann operator, Transport equations with collision operators,...).
- Metriplectic dynamics includes these kind of systems compliant both with the first and second thermodynamics principles.
- In the metriplectic formalism, evolution equation is given by a new bracket: $\{\{.,.\}\}$

$$\frac{df}{dt} = \{\{f, F\}\} = \{f, F\} + (f, F)$$

Non-Dissipative Bracket
(Poisson Bracket)

Dissipative Bracket
(Metric Flow Bracket)

METRIPLECTIC FRAMEWORK FOR DISSIPATIVE HEAT EQUATION

| Metriplectic Model

$$\frac{df}{dt} = \left\{ \{f, F\} \right\} = \{f, F\} + (f, F)$$

- > Hamiltonian components is introduced by requiring:

$$F = H + S$$

- > The 2nd bracket has 2 constraints :

$$(f, F) = (F, f) \text{ and } (f, f) \geq 0$$

- > with **the entropy S selected from the set of Casimir invariants of the non-canonical Poisson bracket**, playing the role of a Lyapunov functional.

- > A metriplectic vector field induced by F is given by the dynamics:

$$\frac{dz_i}{dt} = J_{ij} \frac{\partial F}{\partial z_j} + M_{ij} \frac{\partial F}{\partial z_j}$$

OPEN

Metriplectic Model Compliance to Thermodynamics Principles

Metriplectic Model compliance with two first principles of thermodynamics

- First principle: Energy conservation

$$F = H + S$$

$$\frac{dH}{dt} = \{H, F\} + (H, F) = \{H, H\} + \{H, S\} + (H, H) + (H, S) = 0$$

because $\begin{cases} \{H, H\} = 0 \text{ by symmetry} \\ \{f, S\} = 0, \quad \forall f \\ (H, f) = 0, \quad \forall f \end{cases}$

Metriplectic Model Compliance to Thermodynamics Principles

| Metriplectic Model compliance with two first principles of thermodynamics

- Second principle: **Entropy production**

$$\frac{dS}{dt} = \{S, F\} + (S, F) = 0 + (S, H) + (S, S) = (S, S) \geq 0$$

because $\begin{cases} \{S, f\} = 0, \quad \forall f & \text{(Casimir property)} \\ (f, H) = 0, \quad \forall f \\ \text{positive semi-definite} \Rightarrow (S, S) \geq 0 \end{cases}$

- The choice of thermal equilibrium is induced by selecting **Entropy as Casimir invariant function.**

| Metriplectic Model

- Finally in metriplectic systems, the geometry is determined by two compatible brackets, a Poisson bracket and a symmetric bracket:

$$\frac{df}{dt} = \left\{ \{f, F\} \right\} = \{f, H\} + (f, S)$$

- The energy H is a Casimir invariant of the dissipative bracket, and the entropy S is a Casimir invariant of the Poisson bracket:

$$\{S, H\} = 0 \quad \forall H$$

$$(H, S) = 0 \quad \forall S$$

Foliation Structures of Thermodynamics-Informed Neural Networks

Thermodynamics-Informed Neural Networks and Symplectic Foliation

- TINN are based on metriplectic flow (also called GENERIC flow) modeling:
 - **non-dissipative dynamics (1st thermodynamic principle of energy preservation)**
 - **dissipative dynamics (2nd thermodynamic principle of entropy production).**

Metriplectic

$$\frac{dz}{dt} = \{z, E\} + (z, S) \quad \xrightarrow{\text{Flow}} \quad \frac{dz}{dt} = L(z) \frac{\partial E}{\partial z} + M(z) \frac{\partial S}{\partial z}$$

**Energy
Preservation** **Entropy
Production**

$$\frac{dE}{dt}(z) = 0 \quad \frac{dS}{dt}(z) = \frac{\partial S}{\partial z} M(z) \frac{\partial S}{\partial z} \geq 0$$

- Souriau's Lie Groups Thermodynamics allows to characterize geometrically metriplectic flow by:
 - **Symplectic foliation (non-dissipative part)**
 - **Riemannian Foliations (dissipative part), transverse to symplectic leaves**
- From symmetries of the problem, we can generate **Lie group coadjoint orbits that are symplectic leaves defined as level sets of Entropy**, where Entropy is an invariant Casimir function in coadjoint representation (invariant function on these symplectic leaves).
 - **Dynamics along these symplectic leaves**, given by Poisson bracket, characterize non-dissipative dynamics with Entropy preservation.
 - **Dissipative dynamics are then given by transverse Poisson structure and metric flow bracket**, with evolution from leaf to leaf constrained by entropy production and Energy preservation. Foliation transverse structure are linked with Riemannian foliations given by Souriau-Fisher metric.

Metriplectic Flow on Symplectic Foliation & Transverse Metric Foliation

In whole or in part
reserved.

Foliation Leaves =
Level Sets of Energy

METRIC FOLIATION

Foliation Leaves =
Level Sets of Entropy

SYMPLECTIC FOLIATION

1st Principle of
Thermodynamics
Preservation of Energy

2nd Principle of
Thermodynamics
Entropy Production

METRIPLECTIC FLOW

$$\frac{dF}{dt} = \{F, H\} + (F, S)$$

Non-dissipative
Entropy = Constant

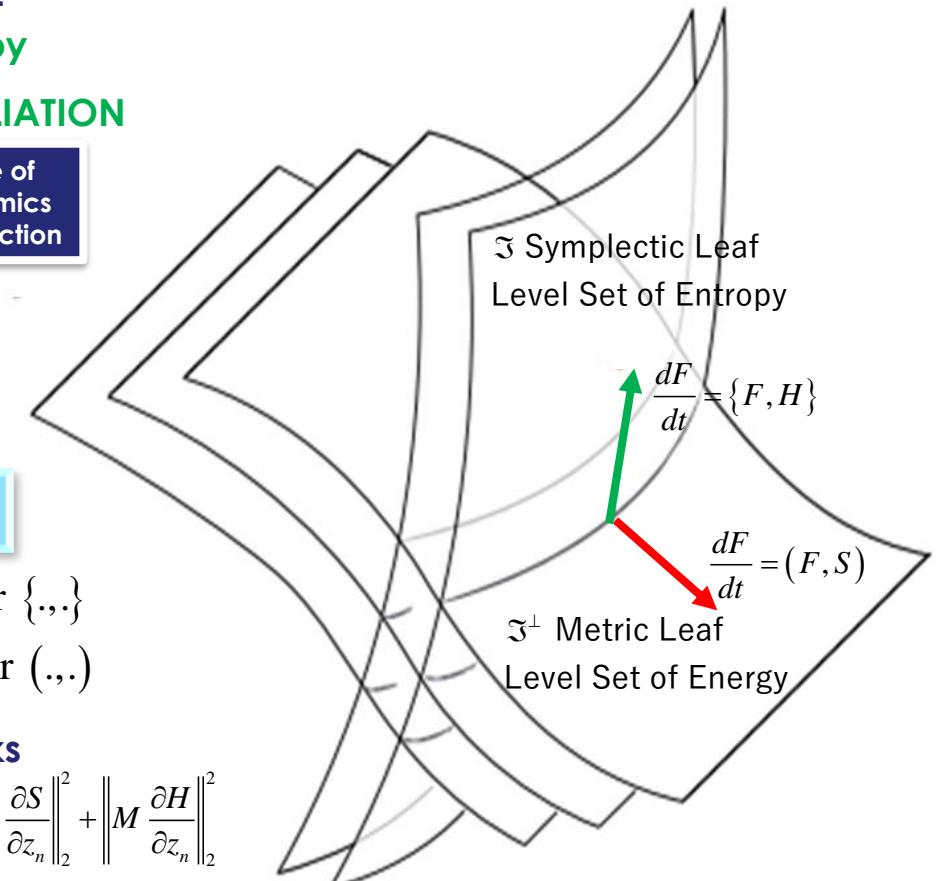
Dissipative
Energy = Constant

$\{F, S\} = 0 \quad \forall F : S$ Entropy Casimir Function for $\{.,.\}$

$(F, H) = 0 \quad \forall F : H$ Energy Casimir Function for $(.,.)$

Thermodynamics-Informed Neural Networks

$$\Lambda = \frac{1}{N_{batch}} \sum_{n=0}^{N_{batch}} (\lambda \Lambda_n^{data} + \Lambda_n^{deg}) \quad \Lambda_n^{data} = \left\| \frac{dz^{GT}}{dt} - \frac{dz^{net}}{dt} \right\|_2^2 \quad \Lambda_n^{deg} = \left\| L \frac{\partial S}{\partial z_n} \right\|_2^2 + \left\| M \frac{\partial H}{\partial z_n} \right\|_2^2$$



SINN & MINN: Symplectic-Informed NN & Metriplectic-Informed NN

Symplectic Integrator

- Capture of symmetries (Geometry-Informed)
- Capture Noether Invariants via Souriau's Moment Map as Energy, Angular Momentum (Physics-Informed)

$$\frac{dF}{dt} = \{F, H\}$$

non-dissipative

(*) The time evolution of Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}$$

is a symplectomorphism, meaning that it conserves the symplectic 2-form $dp \wedge dq$. A numerical scheme is a symplectic integrator if it also conserves this 2-form and equation are reduced to:

$$z = (p, q) \Rightarrow \dot{z} = \{z, H(z)\}$$

Metriplectic Integrator

- Capture 1st principle (Energy preservation, Moment map)
- Capture 2nd principle (Entropy Production)

$$\frac{dF}{dt} = \{F, H\} + (F, S)$$

dissipative

$$F = H + S$$

(*) 1st principle: Preservation of Energy

$$\frac{dH}{dt} = 0$$

2nd Principle Entropy production

$$\frac{dS}{dt} \geq 0$$

Metriplectic Flow and Symplectic Complete Integrable System

- They form what is well known in **symplectic geometry** under a variety of names: **dual pair, symplectically complete foliation, bifoliation, bifibration**.
- **These foliations are fibrations**, then the space of the leaves of the polar foliation is an affine manifold which has the same dimension of the invariant tori and plays the role of ‘action manifold’ of the system. Furthermore, the base of the fibration by the invariant tori is a Poisson manifold and the space of **its symplectic leaves can be identified with the action manifold**.
- A submanifold of M is said to be isotropic (resp. coisotropic) if its tangent spaces are contained in (resp. contain) their own symplectic complements. **Lagrangian submanifolds are both isotropic and coisotropic**. Let F be a foliation of M . The polar of \mathfrak{I} , if it exists, is the unique foliation \mathfrak{I}^\perp of M with the property that the tangent spaces of its leaves are the **symplectic orthogonals** of the tangent spaces of the leaves of \mathfrak{I} . Furthermore, $(\mathfrak{I}^\perp)^\perp = \mathfrak{I}$. If \mathfrak{I} is isotropic then \mathfrak{I}^\perp is coisotropic.
- A foliation \mathfrak{I} which has a polar is called **symplectically complete**. If \mathfrak{I} is symplectically complete, then the pair $(\mathfrak{I}, \mathfrak{I}^\perp)$ is called a **bifoliation**. If, moreover, both \mathfrak{I} and \mathfrak{I}^\perp are fibrations, then $(\mathfrak{I}, \mathfrak{I}^\perp)$ is also called a **bifibration**. A bifibration is a particular case of **dual pair**.

Symplectic potentials and Guillemin Metric

- We consider **Action-Angle coordinates**: $\omega = dx^i \wedge d\theta_i$
 - We also consider Moment Map $\mu: M \rightarrow \mathfrak{g}^*$ where (x^1, \dots, x^n) are coordinates on \mathfrak{g}^* given by $x^i = \langle X_i, . \rangle$ where (X_1, X_2, \dots, X_n) is a base of vectors field of group action: $dx^i = -i_{X_i} \omega$. We can select angular coordinates such that $X_i = \frac{\partial}{\partial \theta_i}$
 - Symplectic coordinates and complex structure: Consider a complexe structure J $Jdx^i = G^{ij}d\theta_j$ and $Jd\theta_i = -G_{ij}dx^j$ where $G_{ij} = (G^{ij})^{-1}$
 - We can then deduce the metric: $g = G_{ij}dx^i dx^j + G^{ij}d\theta_i d\theta_j$ where (G_{ij}) is symmetric positive definite
 - We can make appear a symplectic potential:
- $dJd\theta_i = -\frac{\partial G_{ij}}{\partial x^k} dx^k \wedge dx^j$ of type $(1,1)$
- $dJd\theta_i = 0 \Rightarrow \frac{\partial}{\partial x^k} \frac{\partial G_{ij}}{\partial x^j} = \frac{\partial G_{ik}}{\partial x^j}, \exists u \text{ convex, } G_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j} = u_{ij}$
- $g = u_{ij}dx^i dx^j + u^{ij}d\theta_i d\theta_j$ where $(u^{ij}) = (u_{ij})^{-1}$
-

GODBILLON–VEY CLASSES OF SYMPLECTIC FOLIATIONS by Kentaro Mikami

- Each transversally oriented foliation has the Godbillon–Vey characteristic class, and regular Poisson structures define symplectic foliations.
- K. Mikami has given a new interpretation and the explicit formula for a representative of the Godbillon–Vey characteristic classes of symplectic foliations in the context of Poisson geometry.
- For each transversely oriented foliation, we have the famous Godbillon–Vey characteristic class. When the symplectic foliations of regular Poisson structures are transversely oriented, they have the Godbillon–Vey characteristic classes. K. Mikami has given a formula defining their Godbillon–Vey classes in terms of Poisson structure.
- Transversely oriented foliations have secondary characteristic classes which are called the Godbillon–Vey class.
 - Reference: J.-L. Koszul, **Crochet de Schouten-Nijenhuis et cohomologie**, in ‘Elie Cartan et les Mathématiques d’aujourd’hui’, 257–271; Société Math. de France, Astérisque, hors série, 1985.

| Non-dissipative part of Metriplectic Model

- The 1st part of a metriplectic vector field relative to non-dissipative part is given by Poisson bracket. Considering the Hamiltonian function $H(q, p)$ depending on the canonical coordinates q and momenta p , with $z = (p, q)$, Hamiltonian equations:

$$\frac{dz_i}{dt} = J_{ij} \frac{\partial H}{\partial z_j} = \{z_i, H\}, \quad i, j = 1, \dots, 2N$$

$$J = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \text{ and } \{f, g\} = \frac{\partial f}{\partial z_j} J_{ij} \frac{\partial g}{\partial z_j}$$

- We note the symplectic 2-form $\omega = \sum dp^i \wedge dq_i$ such that $\omega^{ik} J_{kj} = \delta_{ij}$

Metric Flow Structures and Symmetric Bracket

Dissipative part of Metriplectic Model

- The 2nd part of a metriplectic vector field relative to dissipative part, is a flow in a metric space. A metric flow on a finite dimensional phase space manifold, has the following form in coordinates:

$$\frac{dz_i}{dt} = M_{ij} \frac{\partial S}{\partial z_j} = (z_i, S), \quad i = 1, \dots, M$$

$$(f, g) = \frac{\partial f}{\partial z_j} M_{ij}(z) \frac{\partial g}{\partial z_j}, \quad i, j = 1, \dots, M \quad \text{with} \quad (f, g) = (g, f)$$

- where S is an Entropy function. We should have the properties for the metric:

$$M \text{ positive semi-definite} \Rightarrow \frac{dS}{dt} = (S, S) \geq 0$$

Energy Conservation H (Hamiltonian) $\Rightarrow (H, f) = 0, \quad \forall f$

METRIPLECTIC MODEL

| Metriplectic Model

- > The symmetry condition is a generalization of the Onsager symmetry of linear irreversible thermodynamics to non-linear problems, but classically in Metriplectic model, they don't consider the possibility of Casimir symmetry.
- > In the Lie-Poisson framework, different dissipation Bracket $(.,.)$ that can be defined by the Lie algebra have been proposed in the literature for Metriplectic system: the double bracket, Cartan-Killing bracket, and Casimir dissipation bracket.
- > For Lie-Poisson systems, Lie-Poisson bracket for two functions f, h is given by:

$$\{f, h\}(z) = \left\langle z, \left[\frac{\partial f}{\partial z}, \frac{\partial h}{\partial z} \right] \right\rangle = \left\langle z, -ad_{\frac{\partial h}{\partial z}} \frac{\partial f}{\partial z} \right\rangle = \left\langle ad_{\frac{\partial h}{\partial z}}^* z, \frac{\partial f}{\partial z} \right\rangle$$

- > Hamiltonian dynamics is given by :

$$\frac{df}{dt} = \{f, h\}(z) = \left\langle ad_{\frac{\partial h}{\partial z}}^* z, \frac{\partial f}{\partial z} \right\rangle \Rightarrow \frac{dz}{dt} = ad_{\frac{\partial h}{\partial z}}^* z$$

| Metriplectic Model

- In coordinate realization, with a coordinate chart (z_i) , the Poisson bivector is represented by a set of coefficient function determining the Poisson bracket:

$$\{f, h\} = J_{ij} \frac{\partial f}{\partial z_i} \frac{\partial h}{\partial z_j} \Rightarrow \frac{dz_i}{dt} = J_{ij} \frac{\partial h}{\partial z_j}$$

- For Lie-Poisson structure defined on the dual of a finite dimensional Lie algebra,

we can introduce structure constants with an N dimensional Lie algebra

admitting a basis $\{e_1, \dots, e_N\}$: $[e_i, e_j] = c_{ij}^k e_k$ (with summation convention over the repeated indices).

- The Lie-Poisson dynamics is given by:

$$J_{ij} = c_{ij}^k z_k \Rightarrow \frac{dz_j}{dt} = c_{ij}^k z_k \frac{\partial h}{\partial z_i}$$

Dissipation bracket as double bracket

- The double bracket is given by:

$$(f, h) = \sum_j J_{ij} J_{lj} \frac{\partial f}{\partial z_i} \frac{\partial h}{\partial z_l} = \sum_j c_{ij}^k c_{lj}^r z_k z_r \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial z_l}$$

- with the metriplectic dynamics :

$$\frac{dz_j}{dt} = c_{ij}^k z_k \frac{\partial h}{\partial z_i} + \sum_i c_{ji}^k c_{li}^r z_r z_n \frac{\partial h}{\partial z_l}$$

METRIPLECTIC MODEL

| The symmetric dissipative metriplectic bracket as Euclidean metric tensor on the symplectic leaves foliated by the Casimir invariants

- > *Sato observed that the canonical form of the symmetric dissipative part of the metriplectic bracket is identified in terms of a ‘canonical metric tensor’ corresponding to an Euclidean metric tensor on the symplectic leaves foliated by the Casimir invariants.*
- > It makes the link with Symplectic model of Lie groups Thermodynamics. A single generating function $\Phi = \langle \beta, Q \rangle - S$ **is sufficient to generate the dynamics by the action of the metriplectic bracket:**

$$\frac{dF}{dt} = \left\{ \{ F, \Phi \} \right\} = -\beta^{-1} \{ F, \Phi \} + (F, \Phi)$$

Variational Principles of Thermodynamics

| Variational Principles

- **D'Alembert's variational principles** and Gauss' principle of the least constraint, which are differential principles
- **Maupertuis' principle of the least action** and the Hamilton principle, which are integral principles

| Thermodynamics can be embraced with variational principles

- **ONSAGER's principle of least dissipation of energy** restricted to the particular case of heat conduction in anisotropic continua
- **ONSAGER & MACHLUP's principle and TISZA & MANNING's principle of least dissipation of energy** extended for the case of adiabatically isolated, non-continuous systems.
- **PRIGOGINE's principle of minimum production of entropy** with ONO clarification of the relation with least dissipation of energy.
- **GLANSDORFF and PRIGOGINE's integrale principle** when the conductivity coefficients are not constant

COQUINOT NON-EQUILIBRIUM THERMODYNAMIC THEORY OF DISSIPATIVE BRACKETS

- Coquinot deduced that the **dynamics of out-of-equilibrium thermodynamics** on the phase space can be expressed **with a symmetric bracket** for any two functionals f and g :

$$(f, g) = \frac{1}{\Upsilon} \int_{\Omega} \nabla \left(\frac{\partial f}{\partial \zeta_a(y)} \right) L_{\alpha\beta} \nabla \left(\frac{\partial g}{\partial \zeta_\beta(y)} \right) d^3y$$

- Coquinot has observed that this equation is a **pure geometric object**, **independent of the basis** $\{\zeta_\alpha\}$, where the functional derivatives can be seen as functional gradients, and both functional gradients are contracted thanks to the pseudometric $[L_{\alpha\beta}]$ and where **the bracket is symmetric thanks to the Onsager-Casimir relations**.
- As previously, has been demonstrated that:

$$\frac{\partial \sigma(x, t)}{\partial t} = \int_{\Omega} \left[\nabla \left(\frac{\partial \sigma(x, t)}{\partial \zeta_\alpha(y, t)} \right) L_{\alpha\beta}(y, t) \nabla \left(\frac{\partial S(t)}{\partial \zeta_\beta(y, t)} \right) \right] d^3y = \Upsilon(\sigma, S)$$

COQUINOT NON-EQUILIBRIUM THERMODYNAMIC THEORY OF DISSIPATIVE BRACKETS

- Baptiste Coquinot proves a **formal equivalence between the classical out-of-equilibrium thermodynamics and a subclass of metriplectic dynamical systems**, showing that the pseudometric nature of the dissipative bracket, usually an ad hoc hypothesis, is the **exact transcription of the well-known second law of thermodynamics and Onsager's relations** through this equivalence.
- Coquinot's construction shows that the dissipative brackets are completely natural for non-equilibrium thermodynamics, just as Poisson brackets are natural for Hamiltonian dynamics, **deriving a general dissipative bracket, for the first time, from basic thermodynamic first principles**.
- Baptiste Coquinot has considered **the role of entropy, a Casimir invariant, as counterpart to the role of the Hamiltonian in analytical mechanics**, the non-negativity of the pseudometric ensures the entropy growth, as given by the second law of thermodynamics.

Baptiste Coquinot, Philip J. Morrison, A General Metriplectic Framework With Application To Dissipative Extended Magnetohydrodynamics, Journal of Plasma Physics (2020)



L. ONSAGER



B. COQUINOT



Transverse Symplectic Foliation Structure



Foliation

Foliation Theory Inventors: Ehresmann & Reeb

- > André Haefliger passed away 7th March 2023 : **Haefliger, A.: Naissance des feuilletages d'Ehresmann-Reeb à Novikov. Journal 2(5), 99–110 (2016)**
- > G. REEB, **Sur certaines propriétés topologiques des variétés feuilletées (Thèse)**, Hermann 1952. supervised by Charles Ehresmann
- > "**Sur une durée de quarante années l'immeuble s'est édifié; des centaines d'ouvriers ont œuvré. L'édifice n'est pas achevé, mais on peut visiter. Oui, visiter est le mot**" - Georges REEB



Charles Ehresmann



Georges Henri Reeb

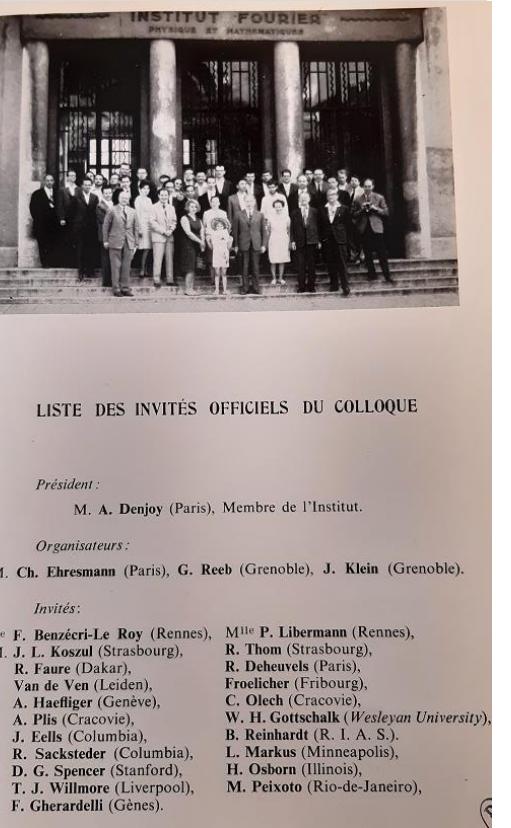
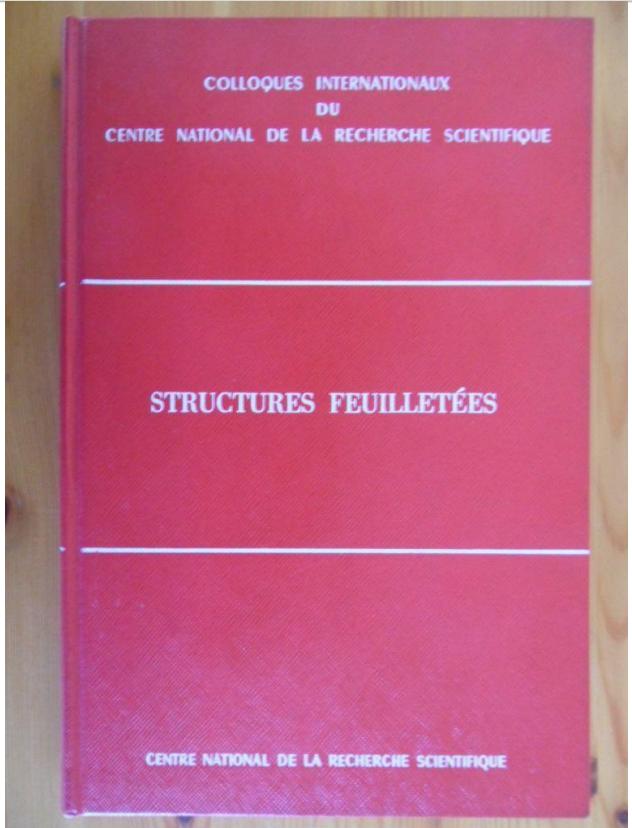


Sergei Petrovich Novikov



André Haefliger

1963, naissance des « Structures feuilletées »



Structures feuilletées - Grenoble, 25-30 juillet 1963 - Georges Reeb

Auteur(s) : Georges Reeb - Charles Ehresmann - René Thom - Paulette Libermann

Editeur : Centre National De La Recherche Scientifique Collection : Colloques Int. Du Cnrs, 1964

C. EHRESMANN, *Structures feuilletées*,
Proceedings of the Fifth Canadian
Mathematical Congress, 109-172., 1963

STRUCTURES FEUILLETÉES

CHARLES EHRESMANN, *Institut Henri Poincaré*

Introduction

Cet article a pour but la définition précise et l'étude des structures feuilletées dans le cadre de la théorie des structures locales telle qu'elle est exposée dans (3; 5; 6). Les résultats connus dans le cas des variétés feuilletées sont précisés et généralisés au cas d'un feuilletage topologique localement simple. Les notions d'holonomie, de déroulements et de tubes analysées ici permettent d'étudier les questions de stabilité. Seuls des problèmes généraux sont abordés, les applications étant réservées pour une publication ultérieure.

La plupart des idées contenues dans ce travail ont été exposées dans mes cours (en particulier, Paris 1955-56, 1958, 1961) et dans des conférences (par exemple Princeton 1953, Buenos-Aires 1959-60, Montréal 1961). Rappelons que la notion de variété feuilletée a été introduite dans une Note en collaboration avec Reeb (1), puis étudiée d'une façon approfondie par Reeb (13; 14) dans différentes publications. Les structures feuilletées d'espèce $\mathfrak{B} \# \mathfrak{F}$ élargie et de seconde espèce ont été définies dans (3). Les Γ -structures étudiées par Haefliger (16), qui sont étroitement liées aux feuilletages de seconde espèce, ne seront pas considérées ici. Les feuilletages localement simples ont été introduits dans une Note en collaboration avec Shih Weishu (2).

I. Définitions de diverses espèces de structures feuilletées

1. Feuilletages topologiques

Soit E un ensemble muni de deux topologies T et T' . On dira que (T, T') définit sur E un *feuilletage topologique* ou une structure d'espace feuilleté topologique si la condition suivante est vérifiée : Pour tout $x \in E$, il existe un voisinage ouvert U' de x relativement à T' sur lequel T et T' induisent la même topologie.

Si (T, T') est un feuilletage topologique sur E , alors T' est une topologie plus fine que T . Nous supposerons désormais que T' est

Transverse Structure of Foliations

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ASTÉRISQUE

1984

STRUCTURE TRANSVERSE DES FEUILLETAGES

Toulouse, 17-19 février 1982

Astérisque

no. 116

Structure transverse des feuilletages

Collectif

Astérisque, no. 116 (1984), 302 p.



SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

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- N.A'CAMPO - Une mesure $SL(n, \mathbb{R})$ -invariante sur l'espace des couples de drapeaux sur \mathbb{R}^n .
- R.BARRE - Théorie des Q-variétés et structures de Hodge mixtes.
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Georges Reeb & Analytical Mechanics

Symplectic Model of Analytical Mechanics

- Symplectic Geometry in conjunction with Analytical Mechanics has grown considerably over the past decades; inspired by the work of S. Lie and E. Cartan, A. Lichnerowicz, G. Reeb, J.M. Souriau, as well as F. Gallissot, who were the initiators of this revival of Analytical Mechanics.

Foliation in Lie Groups Actions and Thermodynamics Integrable Pfaff forms

- In paper "Structures feuilletées", G. Reeb asked the following questions about foliation structures "Why have they been studied. How were they studied? Is it "profitable" to continue these investigations?" and proposed motivations for studying foliations. Among motivations, G. Reeb identified two key use-cases, Lie Groups action and integrable Pfaff forms of Thermodynamics:
 - "**Lie groups action theory** (theory much older than that of foliations) often leads to consider generated foliations. Similarly, the "moving frame" theory (Cartan) ("dual" in a rather vague sense of the previous one) suggests classes of foliations with a remarkable transverse structure."
 - "**Thermodynamics** has long accustomed mathematical physics [cf. Duhem Pierre] to the consideration of completely integrable Pfaff forms: the elementary heat dQ [notation of thermodynamicists] representing the elementary heat yielded in an infinitesimal reversible modification is such a completely integrable form. This point does not seem to have been explored since then.".
- Foliation and Cartan's moving frame was developed by Reeb PhD student, Edmond Férida.

Structures feuilletées – Georges Henri REEB

Motivations for Foliations Studies

G. Reeb, Structures feuilletées, Differential Topology, Foliations and Gelfand-Fuks cohomology, Rio de Janeiro, 1976, Springer Lecture Notes in Math. 652 (1978), 104-113.

STRUCTURES FEUILLETÉES

Pourquoi les a-t-on étudiées. Comment les a-t-on étudiées. Est-il "rentable" de continuer ces investigations ?

Orbites coadjointes (action d'un groupe sur le dual de l'algèbre de Lie)

M₃ : La théorie des actions de groupes de Lie (théorie bien plus ancienne que celle des feuilletages) conduit souvent à considérer des feuilletages engendrés. De même la théorie du "repère mobile" (CARTAN) ("duale" en un sens assez vague de la précédente) suggère des classes de feuilletages à structure transversale remarquable.

Thermodynamiques et formes de Pfaff

M₄ : La thermodynamique a habitué de longue date la physique mathématique [cf. DUHEM P.] à la considération de formes de Pfaff complètement intégrables : la chaleur élémentaire dQ [notation des thermodynamiciens] représentant la chaleur élémentaire cédée dans une modification infinitésimale réversible est une telle forme complètement intégrable. Ce point ne semble guère avoir été creusé depuis lors.



"Sur une durée de quarante années l'immeuble s'est édifié; des centaines d'ouvriers ont œuvré. L'édifice n'est pas achevé, mais on peut visiter. Oui, visiter est le mot" - Georges REEB

Georges REEB contribution to Analytical Mechanics (in « Œuvres complètes de Charles Ehresmann »)

La Géométrie Symplectique en liaison avec la Mécanique Analytique, a pris un extension considérable ces trente dernières années ; inspirés par les travaux de S. Lie et E. Cartan, A. Lichnerowicz [27] , G. Reeb [44, 45], J.M. Souriau [48], ainsi que F. Gallissot [8], ont été les initiateurs de ce renouveau de la Mécanique Analytique.

References:

- > G. REEB, Propriétés topologiques des trajectoires des systèmes dynamiques, Mém. Acad. Se. Bruxelles 27 (1952).
- > G. REEB, Variétés symplectiques, variétés presque-complexes et systèmes dynamiques, Comptes Rendus de l'Académie de Sciences, 235, (1952) 776–778
- > G. REEB, Espaces de Finsler et espaces de Cartan, Coll. Int. CNRS Géom. Diff. Strasbourg (1953), 35-40.
- > G. REEB, Problèmes relatifs aux variétés presque symplectiques et systèmes dynamiques, Convegno Int. Geom. Diff. Venise-Bologne-Pise (1953) 104-113



Symplectic Leaves & transverse Metric Leaves

PHYSICAL REVIEW E 91, 042138 (2015)

Essential equivalence of the general equation for the nonequilibrium reversible-irreversible coupling (GENERIC) and steepest-entropy-ascent models of dissipation for nonequilibrium thermodynamics

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(Received 19 November 2014; published 28 April 2015)

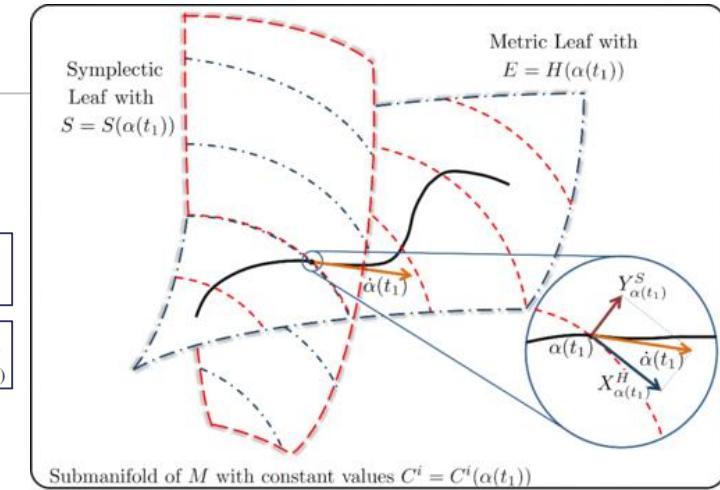
$$\{S, A\} = P(dS, dA) = dA[P^\sharp(dS)] = 0 \quad \forall A \in C^\infty(\mathcal{M}),$$

or, equivalently, $P_{\alpha(t)}^\sharp(dS_{\alpha(t)}) = 0.$ (60)

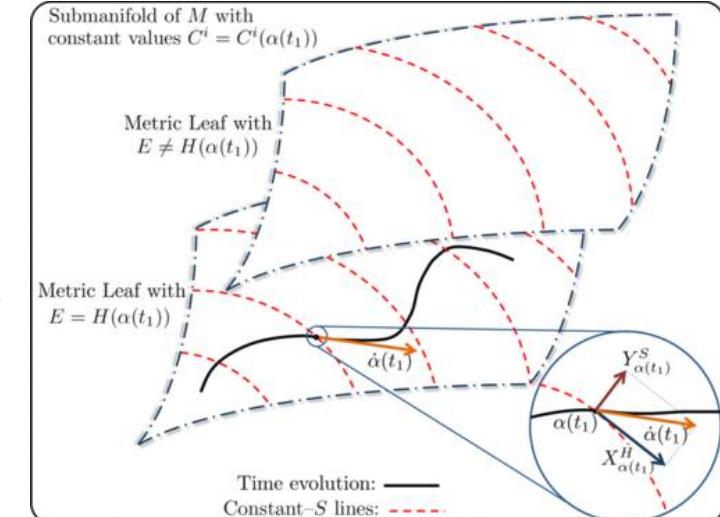
$$[H, A] = D(dH, dA) = dA[D^\sharp(dH)] = 0 \quad \forall A \in C^\infty(\mathcal{M}),$$

or, equivalently, $D_{\alpha(t)}^\sharp(dH_{\alpha(t)}) = 0.$ (61)

- In the context of GENERIC dynamics, the degeneracy condition (60) implies that **symplectic leaves are at constant entropy** (and the other distinguished functionals of the Poisson bracket) while the degeneracy condition (61) implies that **metric leaves are at constant energy** (and the other distinguished functionals of the dissipative bracket).
- Each trajectory is effectively constrained on a single metric leaf.
- The relationship between metric leaves, where GENERIC dynamics (of an overall closed and isolated thermodynamic system) takes place, and symplectic leaves, where purely Hamiltonian dynamics takes place.
- **Metric leaves are surfaces with constant energy, while the symplectic leaves are surfaces with constant entropy** (because Hamiltonian dynamics is reversible).
- **The intersection of symplectic leaves on a metric leaf produces isentropic contours** and the GENERIC non dissipative vector (for an overall closed and isolated thermodynamic system) is always contained in such an intersection.



Submanifold of M with constant values $C^i = C^i(\alpha(t_1))$
Time evolution: — Constant- S lines: - - - Constant- E lines: - - -



Transverse Structure of Foliations

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Transverse Structure of Foliations

P R É F A C E

Issue de la rencontre de W.T. Van Est et de P. Molino, la première "Journée Transverse", consacrée à la structure de l'espace des feuilles d'un feuilletage, fut organisée en 1979 à Paris VII par P. Libermann.

L'intérêt d'une rencontre élargie, suggérée par P. Molino, apparut rapidement, renforcé par la convergence des points de vue de W.T. Van Est et de A. Haefliger.

L'impulsion décisive devait venir de P. Cartier, à qui sa culture universelle fit apercevoir la confluence des méthodes des géomètres différentiels avec celles introduites par A. Connes, en liaison avec les travaux de Ruelle-Sullivan, et reconnaître la convergence de courants mathématiques lointains issus de domaines variés (Géométrie Algébrique, topos de Grothendieck, sous-groupes virtuels de Mackey, etc...).

Son appui assura le succès de ces Journées, dépassant les prévisions, attesté par la quantité et la qualité des conférenciers et des participants. Un horaire très chargé, rançon de ce succès, ne fit pas faiblir le nombre et l'attention des auditeurs.

L'Université Paul Sabatier de Toulouse a accueilli le Colloque sur son campus et assuré la base du financement, notamment grâce à des crédits sur programme accordés à l'Equipe de Topologie Algébrique et Difféentielle.

La Société Mathématique de France, avec le concours du C.N.R.S., a accordé son soutien moral et financier, complété par une subvention de la D.C.R.I.

On trouvera ci-après les textes rédigés des 2/3 des conférences prononcées à ce Colloque, la plupart avec des développements et améliorations considérables, fruits des discussions tenues au cours de ces Journées.

STRUCTURE TRANSVERSE DES FEUILLETAGES

Il était prévu d'ouvrir et de fermer ce recueil par le texte de l'Introduction et de la Conclusion prononcées par P. Cartier, qui devait être rédigé en collaboration avec A. Connes. Il a fallu malheureusement y renoncer, sous peine de retarder exagérément la publication.

Cette lacune est partiellement comblée par les exposés de W.T. Van Est et de A. Haefliger, qui fondent la théorie, tout en l'illustrant d'applications concrètes. La structure de l'espace des feuilles y apparaît comme une classe d'équivalence, en un sens naturel, de pseudogroupes, ou plus généralement de groupoïdes (dits d'holonomie), munis d'une structure (topologique, différentielle, etc...) invariante.

Il résulte d'une construction de G. Hector que ce point de vue équivaut exactement à celui de P. Molino, très brièvement rappelé dans son article, qui consiste à considérer une certaine classe d'équivalence de feuilletages, qu'il appelle F -variété. Notons que les QF -variétés sont définies par une relation d'équivalence plus large, mais conduisent aux mêmes invariants continus que les F -variétés, du fait que le groupoïde d'holonomie est dense dans le groupoïde d'holonomie transverse utilisé par C. Godbillon.

Les autres conférences illustrent la variété et la richesse des méthodes d'étude de cette structure transverse et des applications géométriques que l'on en tire, tant pour les propriétés globales du feuilletage que pour celles des feuilles. On y trouvera une moisson de résultats inédits.

Cet échantillonnage ne saurait cependant prétendre être exhaustif, ni statistiquement représentatif des applications des propriétés transverses. C'est par un concours de circonstances, et non par suite d'un choix prémedité, que les thèmes le plus souvent abordés sont les feuilletages riemanniens et la cohomologie basique, alors que les propriétés de croissance des feuilles et d'ergodicité le sont beaucoup plus rarement.

Il semble intéressant de souligner le fait que certaines notions de variétés singulières, apparues dans des contextes éloignés

PRÉFACE

des feuilletages, se sont trouvées être des cas particuliers importants d'espaces de feuilles : les Q -variétés de Barre, et les V -variétés de Satake, redécouvertes comme orbifolds de Thurston, que l'on voit fréquemment apparaître dans ce qui suit.

Je ne saurais terminer sans rendre hommage à la mémoire du grand géomètre Ch. Ehresmann, disparu peu d'années avant la tenue de ce Colloque, fondateur avec G. Reeb, de la théorie des feuilletages. Beaucoup des notions fondamentales, qui sont à la base des travaux que l'on va lire, lui sont dues, notamment notamment les diverses variantes du groupoïde d'holonomie et la notion de groupoïde différentiable (et structuré), qui revient en surface après un long cheminement souterrain. Ce colloque, auquel participaient nombreux de ses disciples, illustre l'actualité de sa pensée.

J. Pradines

Transverse Structure of Foliations

LISTE DES PARTICIPANTS

(Journées SMF, Toulouse 17-18-19 février 1982)

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BOYOM N'Guiffo (Montpellier)
BRYLINSKI Jean-Luc (Ecole Polytechnique, Paris)
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CASCON Ana (Dijon)
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CRUMEYROLLE Albert (Toulouse)
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LEGRAND Claude (Toulouse)
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ROCHE Claude (Grenoble)
ROGER Claude (Metz)
ROSENBERG Harold (Paris 7)
SACKSTEDER Richard (New-York)

ons"

Transverse Foliation Structures

- Dazord, P., Molino, P. : Γ -Structures poissonniennes et feuilletages de Libermann, Publications du Département de Mathématiques de Lyon, fascicule 1B, « Séminaire Sud-Rhôdanien 1ère partie », chapitre II , p. 69-89 (1988)

Liebermann
Foliation
and
Haefliger
 Γ -Structure

Hervé
Sabourin
Transverse
Poisson
Structure

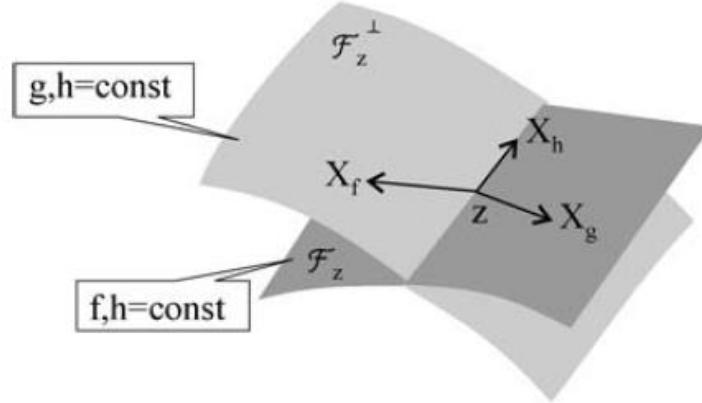
Foliation
based on
Elie Cartan
Moving
Frame

Polar
Foliation,
Bifoliation
and
Bifibration

- Fedida, E. : Sur la théorie des feuilletages associée au repère mobile : cas des feuilletages de Lie. Lecture Notes in Mathematics, vol 652. Springer (1978)

- Sabourin, H. : Quelques aspects de la méthode des orbites en théorie de Lie, HDR spécialité Mathématique, Université de Poitiers, 15 Décembre 2005

OPEN

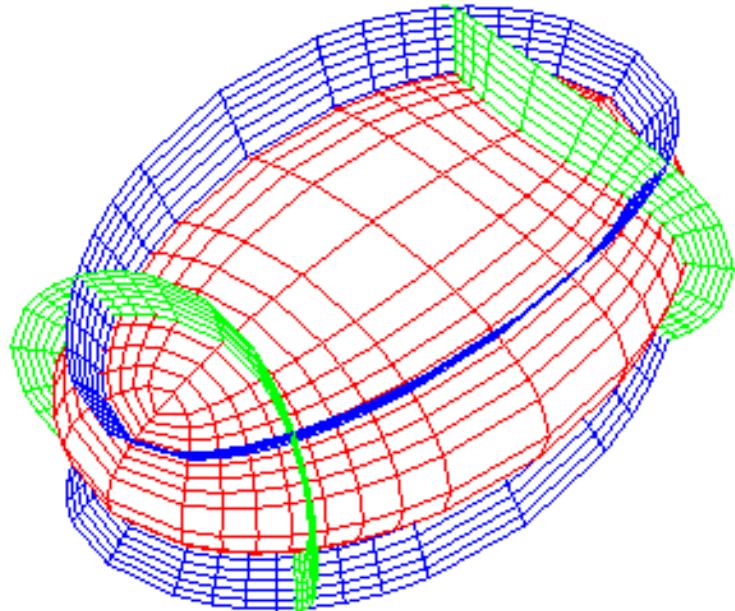
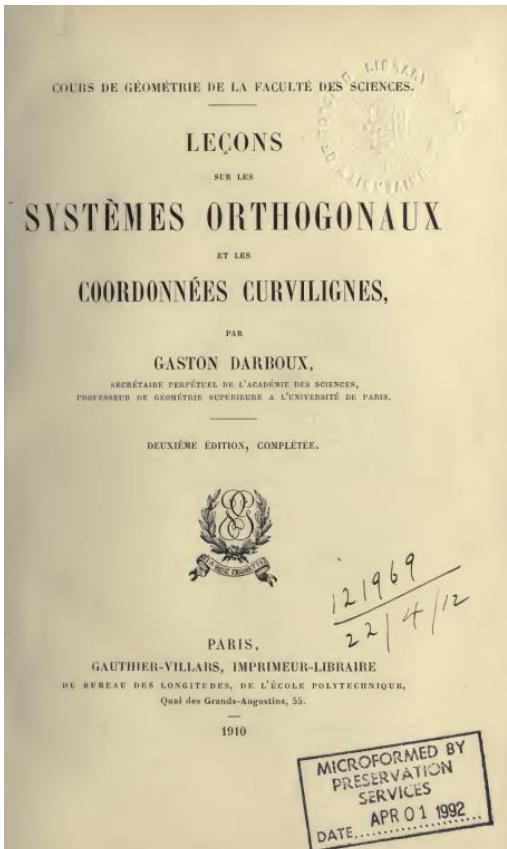
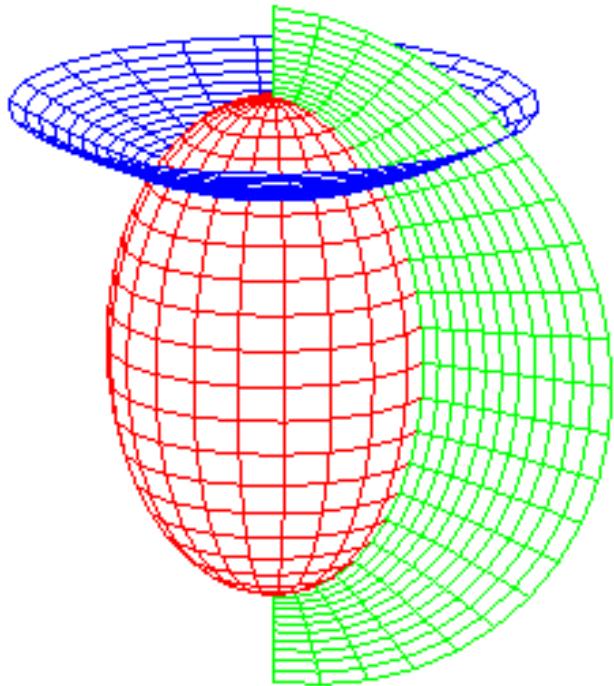


Cours 1982-1983 – Étude systématique des rapports entre feuilletages et variétés de Poisson | Étude de cohomologies d'algèbres de Lie attachées à une variété de contact

- https://www.college-de-france.fr/sites/default/files/media/document/2023-05/1982-1983_lichnerowicz.pdf

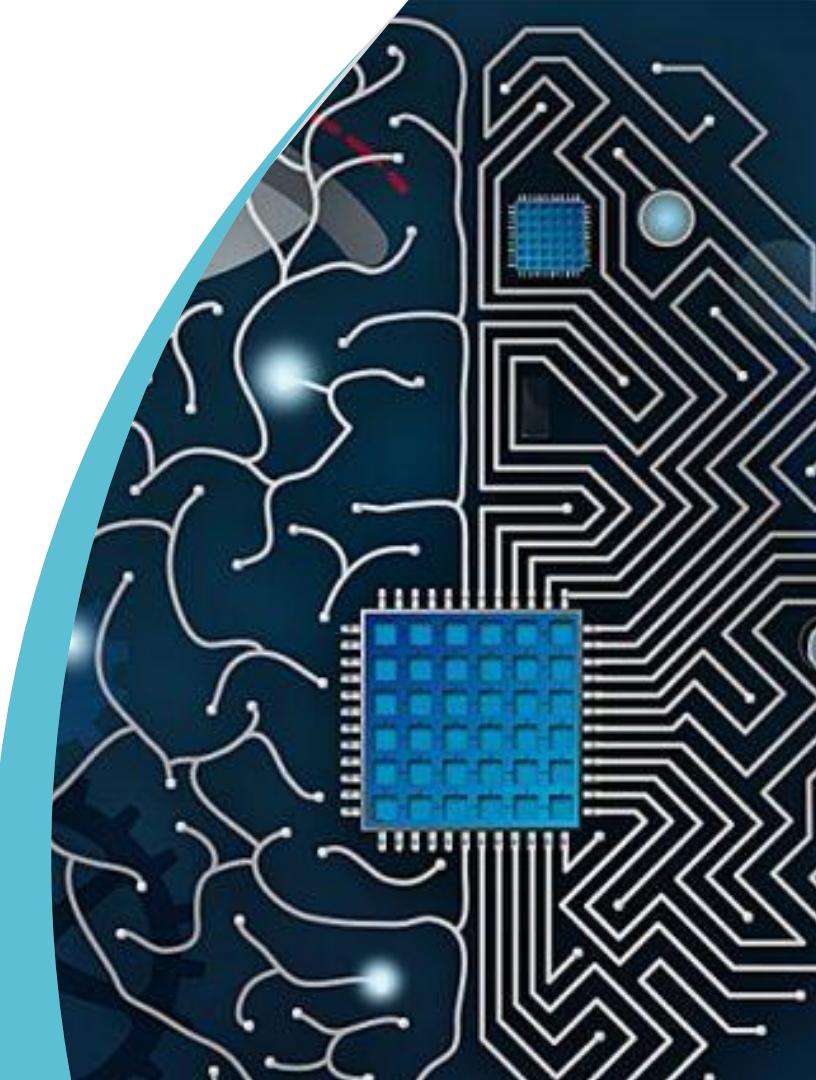
Soit (M, F) une variété symplectique munie d'un *feuilletage lagrangien* \mathcal{F} . On a montré qu'il existe toujours sur M une connexion adaptée au feuilletage qui induit sur chaque feuille une *connexion plate* sans torsion. Si la variété admet une métrique riemannienne de type fibré pour \mathcal{F} , elle admet une métrique riemannienne qui induit sur chaque feuille une métrique plate. On a ainsi précisé et généralisé des résultats récents d'A. Weinstein et P. Dazord. Les mêmes résultats sont valables si, au lieu d'un feuilletage lagrangien, on considère un feuilletage isotrope de (M, F) tel que le champ des plans orthogonaux symplectiques soit un feuilletage coisotrope.

Triple Orthogonal Systems : Darboux and Lamé

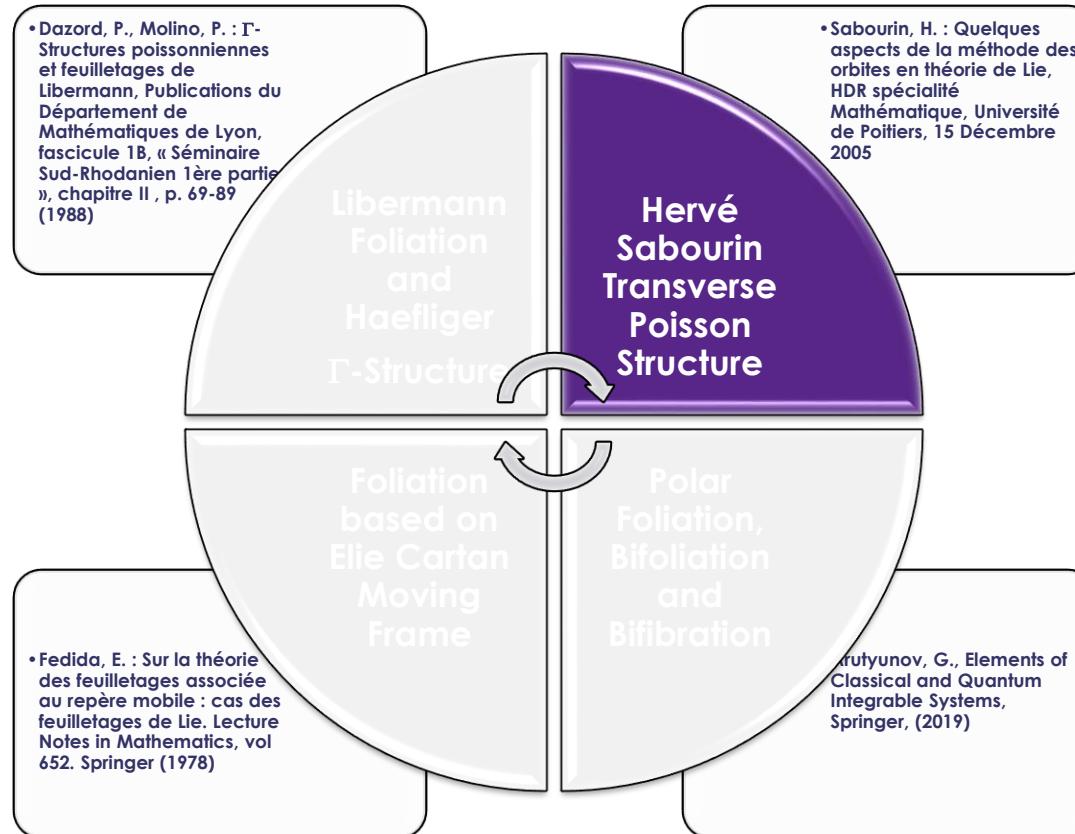




HERVE SABOURIN TRANSVERSE POISSON STRUCTURE



Transverse Foliation Structures



Hervé SABOURIN HDR 2005, Université de Poitiers

DIPLOME D'HABILITATION

A

DIRIGER DES RECHERCHES

présenté à l'Université de Poitiers

par

Hervé Sabourin,

Le 15 Décembre 2005.

Spécialité : Mathématiques.

Quelques aspects de la méthode des orbites en théorie de Lie.

JURY

Michel DUFOLO, Professeur Emerite, Université de Paris 7

Examinateur.

Thierry LEVASSEUR, Professeur, Université de Brest,

Rapporteur.

Bent ORSTED, Professeur, Université de Aarhus, Danemark,

Rapporteur.

Hubert RUBENTHALER, Professeur, Université de Strasbourg,

Rapporteur.

Pierre TORASSO, Professeur, Université de Poitiers,

Examinateur.

Pol VANHAECKE, Professeur, Université de Poitiers,

Examinateur.

0. Introduction

L'essentiel de mon travail de recherche a pour cadre général l'espace des orbites nilpotentes co-adjointes d'une algèbre de Lie semi-simple réelle ou complexe.

Considérons, plus précisément, un groupe de Lie réel ou complexe G , d'algèbre de Lie \mathfrak{g} , qui opère sur \mathfrak{g} par la représentation adjointe, puis sur le dual \mathfrak{g}^* de \mathfrak{g} suivant l'action "duale". On obtient ainsi un ensemble d'orbites "coadjointes", noté \mathcal{N}^*/G . Lorsque G est semi-simple, on peut identifier \mathfrak{g} à son dual, via la forme de Killing \mathcal{K} , et donc l'action co-adjointe à l'action adjointe. On considère alors l'ensemble \mathcal{N}' des éléments nilpotents de \mathfrak{g} , puis l'ensemble \mathcal{N}'/G des orbites nilpotentes adjointes. Cet ensemble joue en fait un rôle essentiel en théorie de Lie.

• L'une des questions importantes à envisager concerne l'étude et la classification éventuelle de l'ensemble des représentations d'un groupe de Lie réel et, plus précisément, de l'ensemble des représentations unitaires irréductibles, appelé dual unitaire. Historiquement cette étude a été initiée dans le cadre des groupes nilpotents; une réponse complète a été apportée dans ce cas par A.A.Kirillov ([30]), qui a établi une bijection naturelle entre le dual unitaire et l'ensemble des orbites coadjointes introduisant de ce fait, pour la première fois, la notion de *méthode des orbites*.

Cette méthode a été ensuite généralisée aux groupes exponentiels par les travaux de P.Bernat ([12]), aux groupes résolubles de type I par L.Auslander et B.Kostant ([11]) et enfin aux groupes résolubles quelconques par L.Pukansky ([40]). L'ensemble des orbites coadjointes sert encore à donner dans ce cas une paramétrisation du dual unitaire mais, cette fois, il ne suffit pas à tout décrire.

Il est naturel donc de s'intéresser au cas d'un groupe réductif et de tenter, entre autres, d'y développer une méthode des orbites. Un tel travail est actuellement en cours et de nombreux auteurs ont apporté une contribution considérable à cette question difficile. En voici une liste non exhaustive : D.Vogan ([53], [54], [55]), R.Brylinski et B.Kostant ([16]) T.Kobayashi et B.Orsted ([31], [32], [33]), M.Duflo ([23]) et P.Torasso ([51], [52]). Comme on le décrira de manière plus précise ultérieurement, la question consiste souvent à essayer de relier de manière la plus naturelle possible le dual unitaire et l'ensemble des orbites nilpotentes coadjointes.

• L'espace \mathcal{N}'/G peut aussi servir à l'étude de certains invariants algébriques d'une algèbre de Lie complexe \mathfrak{g} . Soit $\mathcal{C}(\mathfrak{g}) = \{(X, Y) \in \mathfrak{g} \times \mathfrak{g} \mid [X, Y] = 0\}$ la variété commutante de \mathfrak{g} . La question se pose de déterminer les composantes irréductibles de cette variété algébrique. Dans [42], W.Richardson montre que, lorsque \mathfrak{g} est semi-simple, $\mathcal{C}(\mathfrak{g})$ est irréductible; le résultat s'obtient par récurrence sur la dimension de \mathfrak{g} en utilisant en particulier les propriétés de certaines orbites nilpotentes dites *orbites distinguées*.

On peut alors s'intéresser à la situation suivante : l'algèbre \mathfrak{g} se décompose, sous l'action d'une involution θ , en sous-espaces propres suivant les valeurs ± 1 , soit $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

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L'espace propre \mathfrak{k} associé à la valeur propre 1 est une algèbre de Lie et l'on désigne par K le sous-groupe de G correspondant, qui agit sur l'espace \mathfrak{p} . On peut considérer la variété commutante de \mathfrak{p} et se demander si celle-ci est aussi irréductible; une fois encore, la structure des K -orbites nilpotentes de \mathfrak{p} joue un rôle prépondérant et permet de conclure dans certains cas.

• On sait enfin que le dual \mathfrak{g}^* d'une algèbre de Lie complexe peut-être muni d'une structure de Poisson canonique, appelée *structure de Lie-Poisson*. Selon des résultats de A.Weinstein ([56]), on peut associer, canoniquement, à chaque feuille symplectique d'une variété de Poisson quelconque, une sous-variété transverse munie d'une structure de Poisson, appelée *structure de Poisson transverse à la feuille symplectique*. Or, les feuilles symplectiques de \mathfrak{g}^* sont les orbites coadjointes, dont il semble donc naturel d'étudier la structure de Poisson transverse. Un cas s'avérera particulièrement intéressant, celui des orbites nilpotentes d'une algèbre de Lie semi-simple.

Voici donc les trois thèmes sur lesquels portent mes travaux et mes résultats que je vais maintenant détailler. Je ne peux terminer cette introduction sans exprimer toute ma reconnaissance aux nombreux collègues avec qui j'ai pu partager mes questionnements mathématiques et de qui j'ai reçu tant de réponses utiles. Je commencerai par Michel Duflo et Pierre Torasso dont les travaux ont été si essentiels pour moi et dont la disponibilité a toujours été sans faille tout au long de ces dernières années, David Vogan ensuite pour ses suggestions si souvent déterminantes. Je pense aussi à Toshi Kobayashi, Thierry Levasseur et Mustapha Rais pour les nombreuses et fructueuses discussions que nous avons pu avoir sur ces différents sujets. Je n'oublierai pas mes collègues de Poitiers, Abderrazak Bouaziz, Patrice Tauvel, Pol Vanhaecke et Rupert Yu, leur présence, nos échanges réguliers et nos collaborations.

Transverse Poisson Structure

- On sait enfin que le dual \mathfrak{g}^* d'une algèbre de Lie complexe peut-être muni d'une structure de Poisson canonique, appelée *structure de Lie-Poisson*. Selon des résultats de A. Weinstein ([56]), on peut associer, canoniquement, à chaque feuille symplectique d'une variété de Poisson quelconque, une sous-variété transverse munie d'une structure de Poisson, appelée *structure de Poisson transverse à la feuille symplectique*. Or, les feuilles symplectiques de \mathfrak{g}^* sont les orbites coadjointes, dont il semble donc naturel d'étudier la structure de Poisson transverse. Un cas s'avérera particulièrement intéressant, celui des orbites nilpotentes d'une algèbre de Lie semi-simple.

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

- Hervé Sabourin has studied more deeply the transverse Poisson structure to coadjoint orbits **in a complex semisimple Lie algebra, by reducing to the case of nilpotent orbits.**
- For subregular nilpotent orbits, Hervé Sabourin showed that **the transverse Poisson structure could be described by a determinantal formula** based on the Chevalley's restriction of the invariants on the slice.
- Based on Slodowy slice model, the transverse Poisson structure is reduced to a three dimensional Poisson bracket.

Notes: Nilpotent means $[ad_g(X)]^n = 0$ for large enough n

Equivalently, X is nilpotent if its characteristic polynomial $P_{ad_X}(t)$ is equal to $t^{\dim g}$.

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

$$\mathfrak{g} = \mathfrak{g}(x) \oplus n_e \oplus n_s \quad \mathfrak{g}(x) = \{y \in \mathfrak{g} / [x, y] = 0\} \quad n = n_e \oplus n_s \quad N_x = x + n^\perp$$

> Considering the basis vectors X_1, \dots, X_{2r} of n such that $X_1, \dots, X_{2p} \in n_e$ and $X_{2p+1}, \dots, X_{2r} \in n_s$ and as $\langle n, [\mathfrak{g}(x), n_s] \rangle = \{0\}$ and $\langle n, [n_e, n_s] \rangle = \{0\}$, then the Poisson Matrix takes at $n \in N_x$ the form :

$$\Lambda(n) = \begin{pmatrix} A(n) & B_e(n) & 0 \\ -B_e(n)^T & C_e(n) & 0 \\ 0 & 0 & C_s(n) \end{pmatrix} \text{ where } \begin{cases} A_{i,j}(n) = \langle n, [Z_i, Z_j] \rangle, \text{ for } 1 \leq i, j \leq k \\ B_{e;i,m}(n) = \langle n, [Z_i, X_m] \rangle, \text{ for } 1 \leq i \leq k, 1 \leq m \leq 2p \\ C_{e;l,m}(y) = \langle n, [X_i, X_m] \rangle, \text{ for } 1 \leq l, m \leq 2p \\ C_{s;l,m}(y) = \langle n, [X_i, X_m] \rangle, \text{ for } 2p+1 \leq l, m \leq 2r \end{cases}$$

> Sabourin has deduced from it that the Poisson matrix of the transverse Poisson structure on N_x is given by: $\Lambda_{N_x}(n) = A(n) + B_e(c)C_e(n)^{-1}B_e(n)^T$

> Sabourin has also proved that:

$$\Lambda_N = A(n) + B_e(c)C_e(n)^{-1}B_e(n)^T \text{ where } n \in N$$

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

- M. Saint-Germain has proved in his PhD that, **in the Lie–Poisson case, the transverse Poisson structure is always rational**. This result has been completed by P. Damianou for coadjoint orbits in a semisimple Lie algebra, and by Hervé Sabourin in 2005, with a more general class of complements having a polynomial transverse structure.
- Hervé Sabourin has studied the **transverse Poisson structure** to any adjoint orbit $G.x$ and has proved that it can be reduced, via the Jordan–Chevalley decomposition of $x \in \mathfrak{g}$ to the case of an adjoint nilpotent orbit.
- As the transverse structure to the regular nilpotent orbit O_{reg} of \mathfrak{g} is always trivial, Hervé Sabourin has considered the case of the **subregular nilpotent orbit** O_{sr} of \mathfrak{g} with dimension of O_{sr} two less than the dimension of the regular orbit, so that the transverse Poisson structure has rank 2. **Sabourin replaced, for the transverse Poisson structure, the complicated Dirac constraints, by a simple determinantal formula:**

$$\{f, g\}_{det} = \det(\nabla f, \nabla g, \nabla \chi_1, \dots, \nabla \chi_l) = \frac{df \wedge dg \wedge d\chi_1 \wedge \dots \wedge d\chi_l}{dq_1 \wedge dq_2 \wedge \dots \wedge dq_{l+2}}$$

where χ_1, \dots, χ_l are **independent polynomial Casimir functions** with χ_i the **restriction of the i-th Chevalley invariant** G_i to the slice N , and q_1, \dots, q_{l+2} linear coordinates on N .

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

> **Sabourin Theorem:** Let O_{sr} be the subregular nilpotent adjoint orbit of a complex semisimple Lie algebra \mathfrak{g} , and let (h, e, f) be the canonical triple associated to O_{sr} . Let $N = e + n^\perp$ be a slice transverse to O_{sr} , where n is an ad_h -invariant complementary subspace to $\mathfrak{g}(e)$. Let $\{\cdot, \cdot\}_N$ and $\{\cdot, \cdot\}_{\det}$ denote respectively **the adjoint transverse Poisson structure and the determinantal structure on N** . Then:

$$\{\cdot, \cdot\}_N = c \{\cdot, \cdot\}_{\det}$$

for some $c \in \mathbb{C}^*$.

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

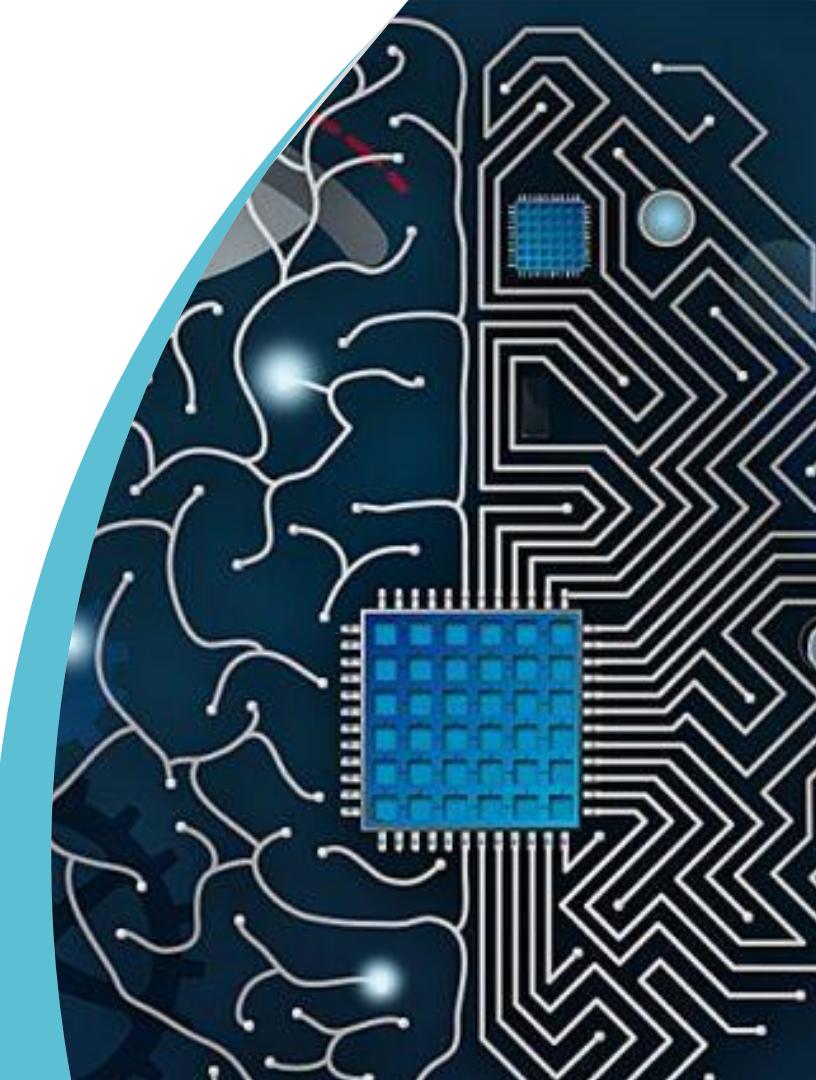
- From this determinantal formula, Sabourin has deduced that the Poisson matrix of the transverse Poisson on N takes, in suitable coordinates, the block form:

$$\tilde{\Lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix} \quad \text{where} \quad \Omega = c' \begin{pmatrix} 0 & \frac{\partial F}{\partial q_{l+2}} & -\frac{\partial F}{\partial q_{l+1}} \\ -\frac{\partial F}{\partial q_{l+2}} & 0 & \frac{\partial F}{\partial q_l} \\ \frac{\partial F}{\partial q_{l+1}} & -\frac{\partial F}{\partial q_l} & 0 \end{pmatrix}$$

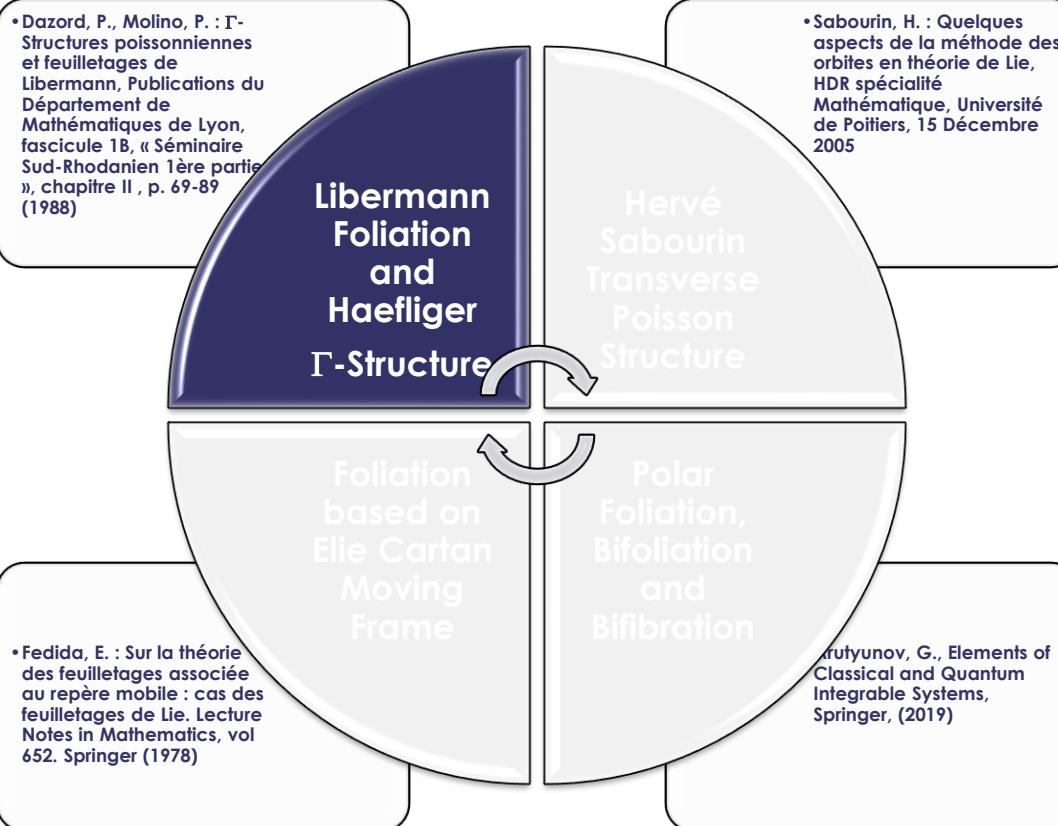
- where F is the polynomial $F(u_1, \dots, u_{l-1}, q_l, q_{l+1}, q_{l+2})$ with u_1, \dots, u_{l-1} the deformation parameters, given by **Casimir for the Poisson structure on N** , of simple singularity of the singular surface $N \cap \mathfrak{N}$, where \mathfrak{N} is the nilpotent cone of \mathfrak{g} .



LIBERMANN FOLIATION & HAEFLIGER Γ -STRUCTURE



Transverse Foliation Structures



LIBERMANN FOLIATION & HAEFLIGER Γ -STRUCTURE

- > In the framework of Stefan foliations, Libermann's foliations are the generalization of foliations studied in regular case by P. Libermann under the name of symplectically complete foliations.
- > The study of these foliations has been clarified by the introduction of the notion of Poisson Γ -structure, which is the maximum extension of the notion of moment map in the sense of J.M. Souriau. This notion of foliation then appears as dual to that of Poisson Γ -structure.
- > Generalized Moment has been introduced by P. Molino, M. Condevaux et P. Dazord in papers of "Séminaire Sud-Rhodanien de Géométrie", with the translation in terms of symplectic duality between Haefliger Γ -structures and Libermann singular foliations of the notion of Souriau's moment of a Hamiltonian action.

LIBERMANN FOLIATION & HAEFLIGER Γ -STRUCTURE

- We define a map J from M to \mathfrak{g}^* , moment of Hamiltonian action ϕ such that:

$$\left\langle J(x), \sum_{i=1}^p \lambda^i \xi_i \right\rangle = \sum_{i=1}^p \lambda^i f_{\xi_i}(x) , \quad \forall x \in M \text{ and } \lambda^1, \dots, \lambda^p \in \mathbb{R}$$

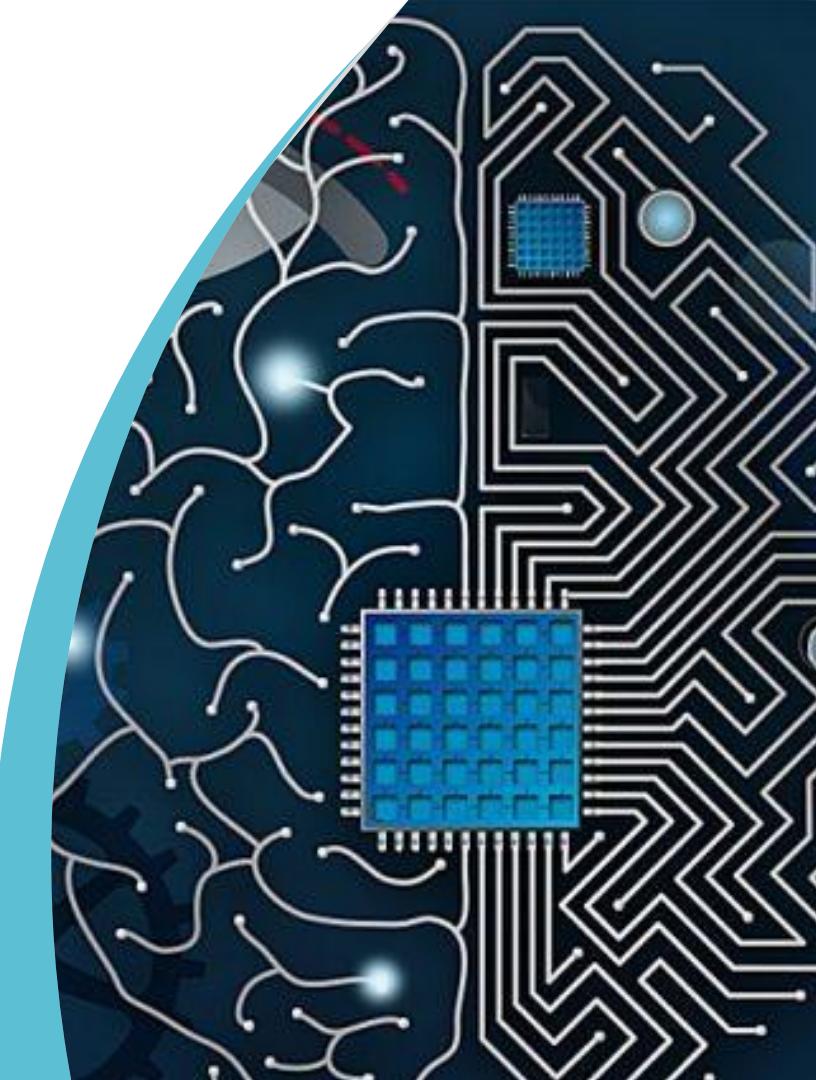
- In the particular case where the orbits of ϕ define a foliation \mathfrak{J} of dimension r on M , the moment map J has constant rank r .
- The connected components of the level submanifolds of J then form on M a new foliation \mathfrak{J}^\perp , of codimension r .
- The foliations \mathfrak{J} and \mathfrak{J}^\perp are orthogonal in the symplectic sense, and the local first integrals of each of them define Hamiltonian fields tangent to the other.
- Based on work of Paulette Libermann, a Libermann Foliation is a foliation \mathfrak{J} on the symplectic manifold (M, ω) , with the following properties that are equivalent:
- **(P1) The field of contact elements orthogonal to \mathfrak{J} is completely integrable.**
 - **(P2) \mathfrak{J} is locally generated by Hamiltonian fields.**
 - **(P3) The Poisson bracket of 2 local first integrals of \mathfrak{J} is again a local first integral.**

LIBERMANN FOLIATION & HAEFLIGER Γ -STRUCTURE

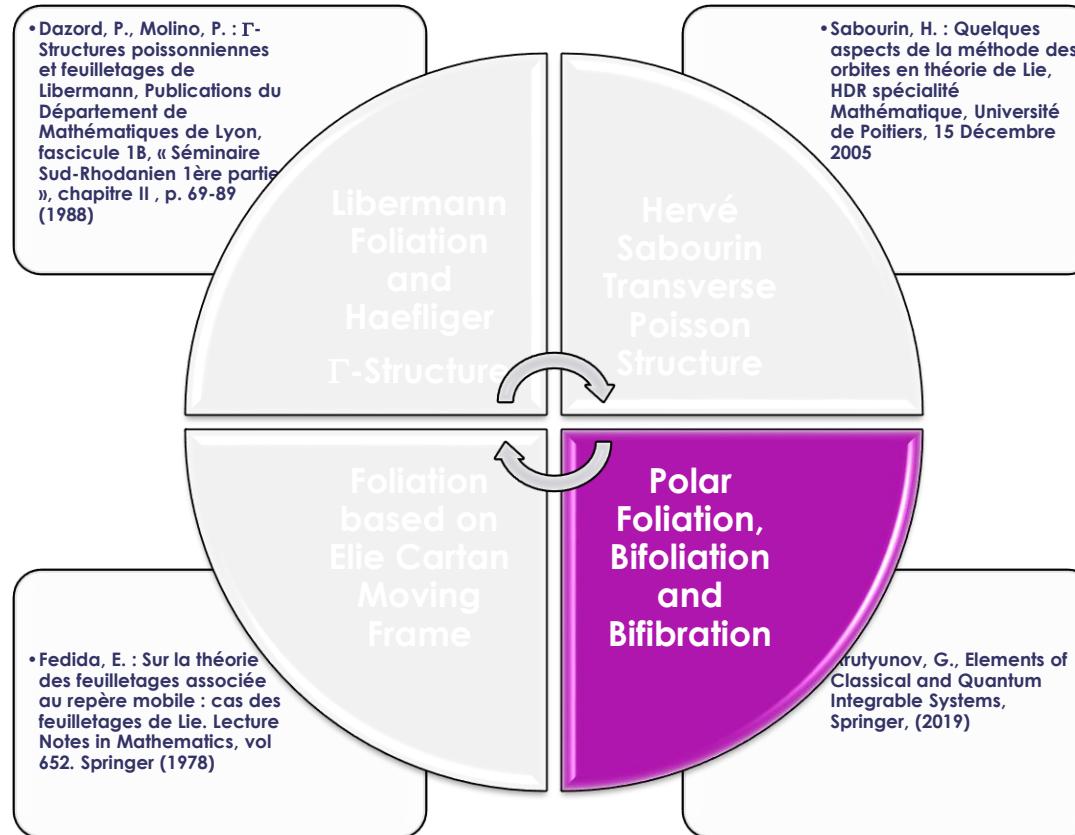
- If \mathfrak{I} is a Libermann foliation on (M, ω) , the fact that the Poisson bracket of two local first integrals is still a local first integral makes it possible to define a natural structure of Poisson manifold on the transversals to the foliation.
- The transverse structure of a Libermann foliation is a Poisson manifold structure, and the local projection onto a transverse is a Poisson morphism.
- An important Poisson structure is the Lie-Poisson structure $(\mathfrak{g}^*, \Lambda_0)$ on the dual of the Lie algebra \mathfrak{g} of the Lie group G by setting an affine Poisson structure $\{\eta, \xi\}_{\Lambda_0}(v) = \{\eta, \xi\}(v) + \tilde{\Theta}$ where $\tilde{\Theta}$ is a 2-cocycle on the Lie algebra \mathfrak{g} .
- We can observe that the Souriau moment of a Hamiltonian action is a Poisson morphism endowed with such an affine structure. If we consider the Hamiltonian action of a Lie group on (M, ω) , its orbits in general no longer form a foliation, but a singular foliation. Similarly, the Souriau moment no longer determines a foliation, but simply a Poisson morphism on a transverse Poisson manifold.



POLAR FOLIATION, BIFOLIATION & BIFIBRATION



Transverse Foliation Structures



POLAR FOLIATION, BIFOLIATION & BIFIBRATION

- We study bifoliation as described in:
 - Arutyunov, G., **Elements of Classical and Quantum Integrable Systems**, Springer, (2019)
- Consider P as a symplectic manifold equipped with a form ω and consider Ψ be foliation of P such that the quotient space N_Ψ of P is a manifold over an equivalence relation set up by Ψ . Let \mathfrak{I}_Ψ the space of functions on P which are constant along the leaves of Ψ .
- Consider $T\Psi$ be a bundle of vectors tangent to the leaves of Ψ , and consider $T\Psi^\perp$ be its orthogonal complement with respect to ω . Let χ_f be the hamiltonian vector field of $f \in \mathfrak{I}_\Psi$. It follows as f is constant along Ψ , for any $v \in T\Psi$, that $\omega(v, \chi_f) = vf = 0$. Then χ_f lies in $T\Psi^\perp$. We observe that $T\Psi^\perp$ is spanned at each point by the hamiltonian vector fields of functions in \mathfrak{I}_Ψ .

Symplectic Transverse Foliation Structure

Lie-Poisson Bracket

$\{e_i\}$ basis of \mathfrak{g} and $\{e^i\}$ basis of \mathfrak{g}^*

$$[e_i, e_j] = c_{ij}^k e_k \quad \text{and} \quad \langle e^i, e_j \rangle = \delta_{ij}$$

$$l = l_i e^i$$

and l_i coordinates of l on \mathfrak{g}^*

$$\nabla f = e_i \frac{\partial f}{\partial l_i} \quad \text{so that} \quad \nabla l_i = e_i$$

$$\{l_i, l_j\} = \left\langle l, [e_i, e_j] \right\rangle = c_{ij}^k l_k$$

Foliation Ψ^\perp skew-orthogonal to Ψ at any point
Foliation Ψ by levels of a function set \mathfrak{J}_Ψ

Casimir Function

$$i_{\xi_f} \omega + df = 0 \quad \text{with}$$

$$\omega(\xi_f, \xi_h) = \{f, h\} = \xi_f h = -\xi_h f$$

Casimir function C such that

$$\{C, f\} = 0, \forall f \in \mathfrak{J}_\Psi$$

$$\Lambda_c = \{x \in P : C(x) = c\}$$

$$\Rightarrow \xi_f C = \{f, C\} = 0, \forall f \in \mathfrak{J}_\Psi$$

$$v \in T\Psi$$

$$\xi_f = \{f, \cdot\}$$

$$\omega(v, \xi_f) = vf = 0$$

f constant along Ψ

Symplectic Transverse Foliation Structure

Foliation Ψ^\perp

skew-orthogonal to Ψ
at any point

$$i_{\xi_f} \omega + df = 0 \text{ with}$$

$$\omega(\xi_f, \xi_h) = \{f, h\} = \xi_f h - \xi_h f$$

Casimir function C such that

$$\{C, f\} = 0, \forall f \in \mathfrak{I}_\Psi$$

$$\Lambda_c = \{x \in P : C(x) = c\}$$

$$\Rightarrow \xi_f C = \{f, C\} = 0, \forall f \in \mathfrak{I}_\Psi$$

Foliation Ψ
by levels of
a function set \mathfrak{I}_Ψ

$\{e_i\}$ basis of \mathfrak{g} and $\{e^i\}$ basis of \mathfrak{g}^*

$$[e_i, e_j] = c_{ij}^k e_k \text{ and } \langle e^i, e_j \rangle = \delta_{ij}$$

$$l = l_i e^i$$

and l_i coordinates of l on \mathfrak{g}^*

$$\nabla f = e_i \frac{\partial f}{\partial l_i} \text{ so that } \nabla l_i = e_i$$

$$\{l_i, l_j\} = \langle l, [e_i, e_j] \rangle = c_{ij}^k l_k$$

$$\xi_f = \{f, \cdot\}$$

$$\omega(v, \xi_f) = vf = 0$$

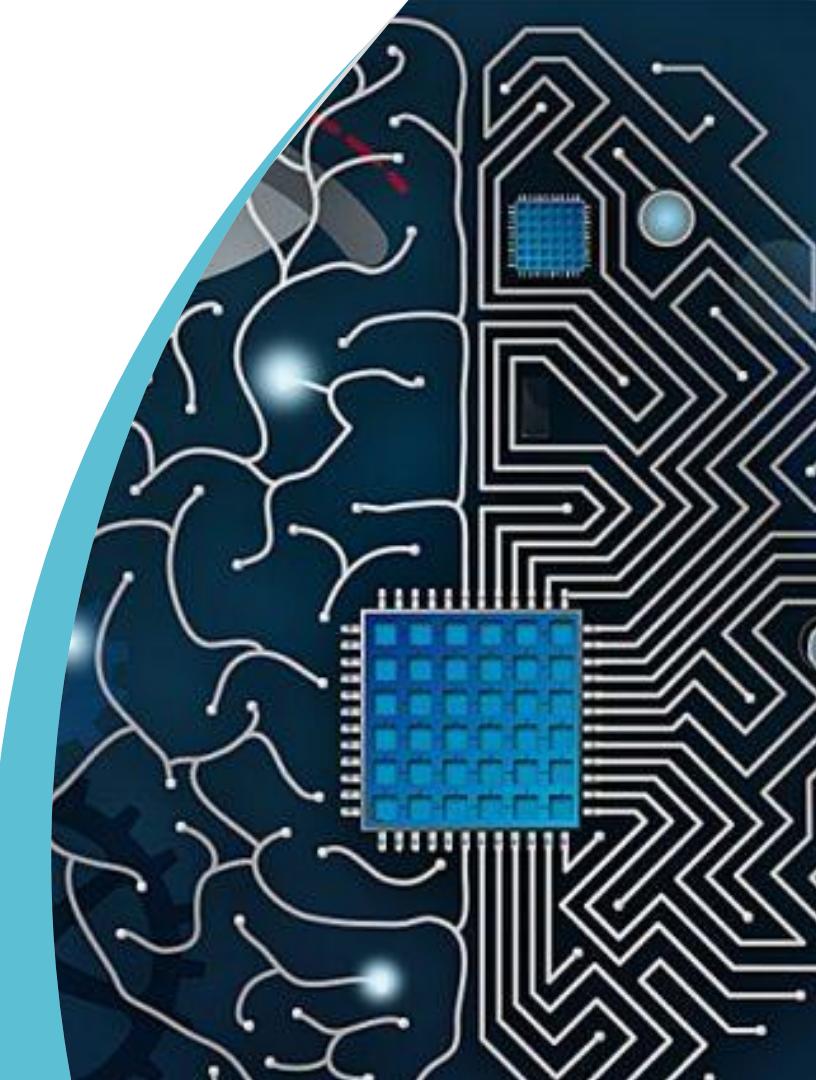
f constant along Ψ

$$\xi_f = \{f, \cdot\} = J^{ij} \partial_i f \partial_j = \xi_f^j \partial_j$$

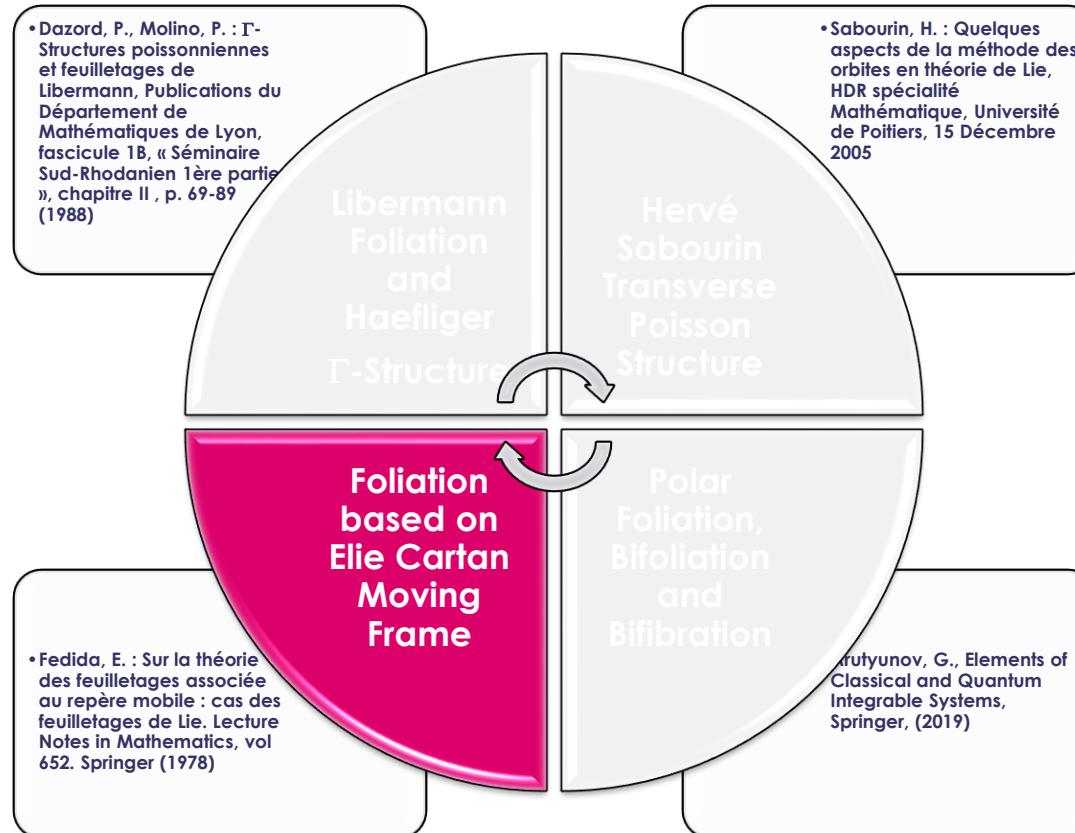
$$\xi_f^j = J^{ij} \partial_i f, \quad \partial_i f = \omega_{ij} \xi_f^i \quad \text{and} \quad \xi_{\{f,g\}} = [\xi_f, \xi_g]$$



FOLIATION BASED ON ELIE CARTAN MOVING FRAME



Transverse Foliation Structures



FOLIATION BASED ON ELIE CARTAN MOVING FRAME

- Based on the duality between the algebra of exterior differential forms and the Lie algebra of vector fields, **Edmond Fédida** has explored the theory of foliation structures, either in the language of **fully integrable Pfaff systems**, or in that of **involution vector fields**. Within the theory of foliation structures, **Lie groups transformations, which correspond to systems of vector fields in involution, associated with a Lie algebra, has a counterpart, which is the theory of foliations associated with the Cartan's moving frame.**
- For group of Lie transformation, a Manifold M of dimension n is equipped with vector fields X_i and structure coefficients given by C_{ij}^k by $[X_i, X_j] = C_{ij}^k X_k$, and **as counterpart for moving frame, we consider a system of Pfaff form ω_i with structure coefficients where $d\omega_i = C_{ij}^k \omega_j \wedge \omega_k$.**
- The ω_i form a constant rank system at all points. **The classes form a foliation M of whose transverse structure is modeled on that of a subspace of G .** On the counterpart, The X_i form a constant rank system at all points: trajectories, the trajectories then constitute a foliation of M ; leaves are homogeneous spaces. The ω_i are linearly independent at any point: **the associated foliation has a transverse structure modeled on G .** Such a foliation then deserves the name of **Lie foliation.**

FOLIATION BASED ON ELIE CARTAN MOVING FRAME

- We can always come back to this case, a structure is given on M by a 1-form ω with values in a Lie algebra of dimension q that verify **Maurer-Cartan equation**:

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

- One can "desingularize" the foliation defined by ω on M by considering on the trivial principal bundle $M \times G$, the form of **connection** Ω induced by ω ; implying that Ω **is a form of flat connection**. In particular Ω is a 1-form on $M \times G$ satisfying the **Maurer-Cartan equation**:

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$$

- $\omega: T_x(M) \rightarrow \mathfrak{g}$ is surjective for all $x \in M$. Under these conditions ω **determines a foliation of codimension q of M , and is a Lie G -foliation of M** .
- **Molino has proved that this foliation is transversely parallelizable. We can then complete in a Riemannian structure of M by an "horizontal" metric subject to the only condition that at any point $x \in M$, the space tangent to the leaf is orthogonal to the transverse space. Hence the foliation has a quasi-fibered metric.**

Cartan-Reeb-Godbillon-Vey: Maurer-Cartan Algebra for $sl(2, \mathbb{R})$

- > Maurer-Cartan form with coordinates $[X_i, X_j] = \sum_k c_{ij}^k X_k \Rightarrow d\omega_k = -\sum_{i<j} c_{ij}^k \omega_i \wedge \omega_j$
- > Maurer-Cartan Algebra bases of $sl(2, \mathbb{R})$

$$d\omega_0 = \omega_0 \wedge \omega_1 , \quad d\omega_1 = \omega_0 \wedge \omega_2 , \quad d\omega_2 = \omega_1 \wedge \omega_2$$

$$X \in SL(2, \mathbb{R}) , \quad X^{-1} dX = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

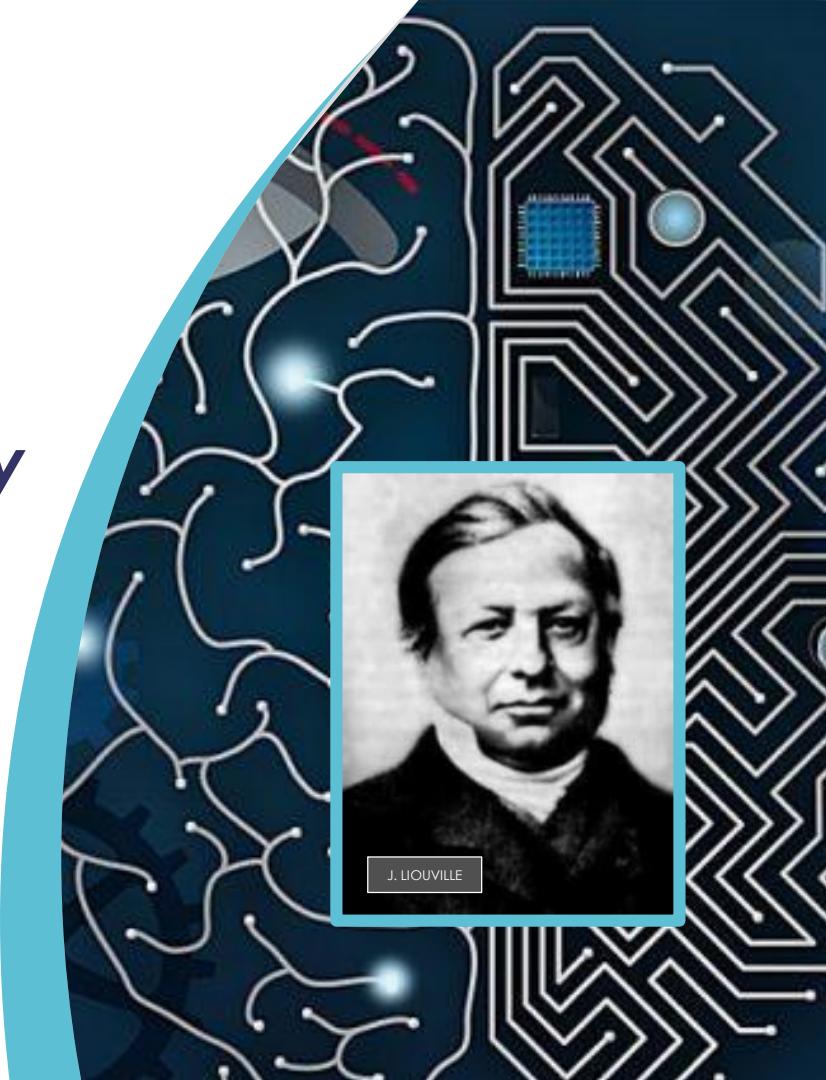
$$\omega_0 = \omega_{12} , \quad \omega_1 = 2\omega_{11} , \quad \omega_2 = -2\omega_{21}$$

$$X \in SL(2, \mathbb{R}) \text{ such that } X = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \rho & 0 \\ \sigma & \rho^{-1} \end{pmatrix} = \begin{pmatrix} \rho \cos \varphi - \sigma \sin \varphi & -\rho^{-1} \sin \varphi \\ \rho \sin \varphi + \sigma \cos \varphi & \rho^{-1} \cos \varphi \end{pmatrix} , \quad \rho > 0$$

$$\omega_0 = \omega_{12} = -\frac{d\varphi}{\rho^2}$$



Liouville Complete Integrability and Information Geometry



Integrability

- Integrability refers to the existence of invariant, regular foliations, related to the degree of integrability, depending on the dimension of the leaves of the invariant foliation.
- In the case of Hamiltonian systems, this integrability is called “complete integrability” in the sense of Liouville (Liouville-Mineur Theorem).
- In case of Liouville integrability, a regular foliation of the phase space by invariant manifolds such that the Hamiltonian vector fields associated to the invariants of the foliation span the tangent distribution. In this case, a maximal set of Poisson commuting invariants exist (functions on the phase space whose Poisson brackets with the Hamiltonian of the system, and with each other, vanish).
- If the phase space is symplectic, the leaves of the foliation are totally isotropic with respect to the symplectic form and such a maximal isotropic foliation is called Lagrangian.

Liouville Complete Integrability on statistical manifolds

In 1993, in the framework of Liouville Completely Integrable Systems on Statistical Manifolds, Y. Nakamura developed gradient systems on manifolds of various probability distributions. He proved the following results

- the gradient systems on the even-dimensional manifolds of Gaussian and multinomial distributions are completely integrable Hamiltonian systems.
- the gradient systems can always be linearized by using Information Geometry dual coordinate system of the original coordinates.
- the gradient flows on the statistical manifolds converge to equilibrium points exponentially.
- the gradient systems are completely integrable Hamiltonian systems if the manifolds have even dimensions. There is a $2m$ -dimensional Hamiltonian System which is integrable so that the Dynamics restricted to the common level sets of first integrals is a gradient.
- the gradient system associated with the Gaussian distribution can be related to an Orstein-Uhlenbeck process

Nakamura Theorem

We consider the statistical Manifolds with Fisher Metric

$p(x, \theta)$ with $\theta = (\theta_1, \dots, \theta_n)^T$ and $l(x, \theta) = \log p(x, \theta)$

$$G = [g_{i,j}] \text{ with } g_{i,j} = E \left[\frac{\partial l(x, \theta)}{\partial \theta_i} \frac{\partial l(x, \theta)}{\partial \theta_j} \right] = \frac{\partial^2 \Psi(\theta)}{\partial \theta_i \partial \theta_j} \quad (\otimes)$$

Gradient flow on Statistical Manifolds

$$(\mathfrak{X}) \quad \frac{d\theta}{dt} = -G^{-1} \frac{\partial \Psi}{\partial \theta} \quad \text{with} \quad \frac{\partial \Psi}{\partial \theta} = \left(\frac{\partial \Psi}{\partial \theta_1}, \dots, \frac{\partial \Psi}{\partial \theta_n} \right)^T$$

First Y. Nakamura Theorem

- > The gradient system (\mathfrak{X}) is always linearizable. The induced flow on an open subset of the Riemannian Statistical Manifold such that there exists a potential function satisfying (\otimes) converges to equilibrium points exponentially.

$$\eta = \frac{\partial \Psi(\theta)}{\partial \theta} \quad \text{with} \quad \eta = (\eta_1, \dots, \eta_n)^T \quad \text{and} \quad (\mathfrak{X}) \Rightarrow \frac{d\eta}{dt} = -\eta \quad , \quad \eta(t) = e^{-t} \eta(0)$$

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Liouville Completely Integrable Systems on Statistical Manifolds

| Second Y. Nakamura Theorem

- If n is even ($n = 2m$) , then the gradient system (x) is a completely integrable Hamiltonian system:

$$P_j = \eta_j^{-1} , Q_j = \eta_{m+j} \text{ and } H_j = P_j Q_j \text{ for } j = 1, \dots, m$$

$$\frac{d\eta}{dt} = -\eta \Rightarrow \frac{dH_j}{dt} = \frac{dP_j}{dt} Q_j + P_j \frac{dQ_j}{dt} = \left(-\eta_j^{-2} \frac{d\eta_j}{dt} \right) \eta_{m+j} + \eta_j^{-1} \frac{d\eta_{m+j}}{dt} = \eta_j^{-1} \eta_{m+j} - \eta_j^{-1} \eta_{m+j} = 0$$

$$\Rightarrow \frac{dH_j}{dt} = 0$$

- Define the Hamiltonian $H = \sum_{j=1}^m H_j$, then the corresponding equations of motion,

$$\frac{d\Xi_k}{dt} = \{\Xi_k, H\} \text{ with } \begin{cases} \Xi_k = P_k , k = 1, \dots, m \\ \Xi_k = Q_{k-m} , k = m+1, \dots, n \end{cases} \text{ coincides with } \frac{d\eta}{dt} = -\eta \text{ and } \begin{cases} \{H_j, H\} = 0 \\ \{H_i, H_j\} = 0 \end{cases}$$

Liouville Completely Integrable Systems on Gaussian Manifolds

| Gaussian distributions use-case $p(x, \theta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$

$$\theta = (\theta_1, \theta_2) = \left(\frac{m}{\sigma^2}, \frac{1}{2\sigma^2} \right) \text{ and } E[x] = \frac{1}{2} \theta_1 \theta_2^{-1}, E[x^2] = \frac{1}{4} \theta_1^2 \theta_2^{-2} + \frac{1}{2} \theta_2^{-1}$$

$$G = \frac{1}{2} \begin{pmatrix} \theta_2^{-1} & \theta_1 \theta_2^{-2} \\ \theta_1 \theta_2^{-2} & \theta_2^{-2} + \theta_1^2 \theta_2^{-3} \end{pmatrix} = \begin{bmatrix} \frac{\partial^2 \Psi}{\partial \theta_i \partial \theta_j} \end{bmatrix}, \quad \Psi(\theta) = \frac{\theta_1^2}{4\theta_2} - \log(\sqrt{\theta_2}) + \log \sqrt{\pi}$$

$$\frac{d\theta}{dt} = -G^{-1} \frac{\partial \Psi}{\partial \theta} \Rightarrow \begin{pmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \theta_1^3 \theta_2^{-1} \\ \theta_1^2 - 2\theta_2 \end{pmatrix} \Rightarrow \begin{cases} \theta_1(t) = \frac{\eta_1(0)}{\eta_2(0) - \eta_1(0)^2 e^{-t}} \\ \theta_2(t) = \frac{1}{2(\eta_2(0)e^{-t} - \eta_1(0)^2 e^{-2t})} \end{cases}$$

$$\frac{dH}{dt} = 0 \text{ where } H = PQ = \theta_1^{-1} + \frac{1}{2} \theta_1 \theta_2^{-1} \text{ with } P = 2\theta_2 \theta_1^{-1} = E[x]^{-1}, Q = \frac{1}{4} \theta_1^2 \theta_2^{-2} + \frac{1}{2} \theta_2^{-1} = E[x^2]$$

$$\frac{dQ}{dt} = -\frac{\partial H}{\partial P}, \quad \frac{dP}{dt} = \frac{\partial H}{\partial Q}$$

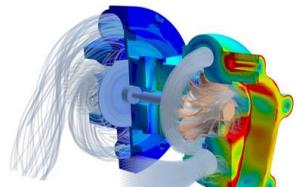
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Thermodynamics-Informed Neural Network and Symplectic Foliation Structure



PINN Physics-Informed Neural Networks

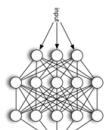
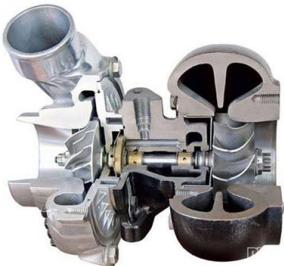
Physics-Based Model



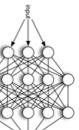
FEM simulation

Data collection

Physical asset



MACHINE
LEARNING
TECHNIQUES



MOR /
Surrogate

Ignorance
model

$$\frac{dx}{dt} = f^{\text{physics-based}}(x, \mathbf{z}) + f^{\text{data-driven}}(x, \mathbf{z})$$

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THERMODYNAMICS OF LEARNING PHYSICAL PHENOMENA

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July 27, 2022

ABSTRACT

Thermodynamics could be seen as an expression of physics at a high epistemic level. As such, its potential as an inductive bias to help machine learning procedures attain accurate and credible predictions has been recently realized in many fields. We review how thermodynamics provides helpful insights in the learning process. At the same time, we study the influence of aspects such as the scale at which a given phenomenon is to be described, the choice of relevant variables for this description or the different techniques available for the learning process.

Viscosity Simulation of water and honey

Computational sensing, understanding, and reasoning: an artificial intelligence approach to physics-informed world modeling

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Francisco Chinesta^{1,4}, Elias Cueto³

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Algorithm 1 Hybrid twin

Require: Free surface $\mathbf{x}_n \in X$, full state $\mathbf{s}_n \in S$, source model π_θ

Ensure: Next state \mathbf{s}_{n+1}

for Iterations do

 for $n = 1$ to N sequences do

 Encode $\bar{\mathbf{z}}_n \leftarrow \phi_{\text{GRU}}(\mathbf{x}_n)$;

 Compute SPNN $a_n \leftarrow \pi_\theta(\bar{\mathbf{z}}_n)$, with $a_n = [\mathbf{L}_n, \mathbf{M}_n, \mathbf{DE}_n, \mathbf{DS}_n]$;

 Determine next integration step $\hat{\mathbf{z}}_{n+1} \leftarrow \Delta t(\mathbf{L}_n \mathbf{DE}_n + \mathbf{M}_n \mathbf{DS}_n) + \bar{\mathbf{z}}_n$;

 Decode $\hat{\mathbf{s}}_{n+1} \leftarrow \psi(\hat{\mathbf{z}}_{n+1})$;

 end for

 Extract free surface of $\hat{\mathbf{s}}_{n+1}$;

 Compute loss $\mathcal{L}_{\text{correction}}$;

 Update π_θ ;

end for

return Optimized hybrid twin π_θ

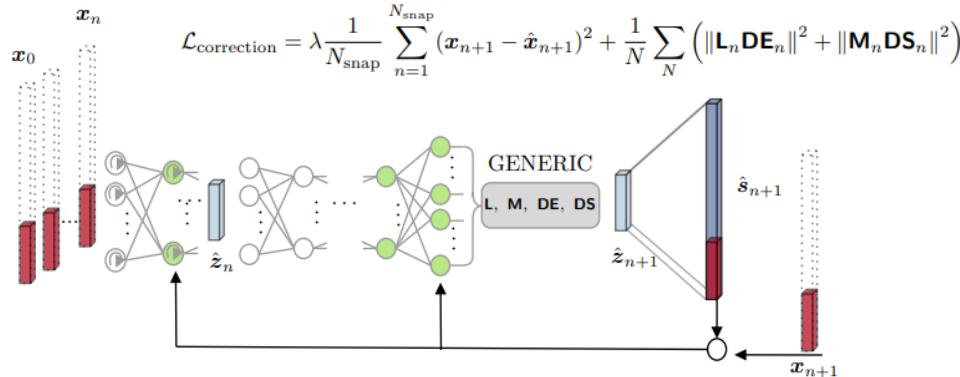
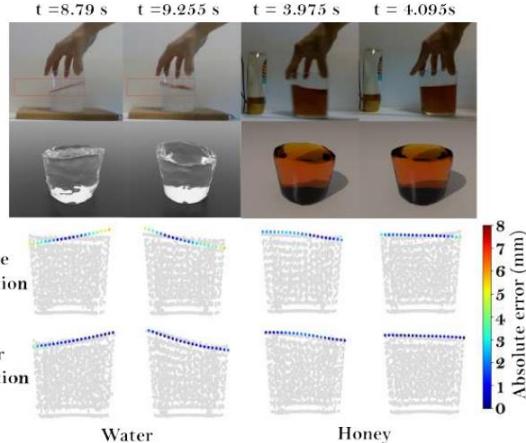


Fig. 5 Hybrid twin algorithm for the adaptation to new physics. The network as a whole adapts to the new data by fine-tuning only part of the layers, highlighted in green. The result of the integration algorithm is compared with the ground truth liquid by evaluating the reconstruction of the free surface in time. As a result the twin adapts when it detects any deviation between the prediction and the ground truth¹



Port-Metriplectic Neural-Network

Port-metriplectic neural networks: thermodynamics-informed machine learning of complex physical systems

Quercus Hernández¹ · Alberto Badiás² · Francisco Chinesta^{3,4} · Elías Cueto¹ 

Received: 3 November 2022 / Accepted: 17 February 2023 / Published online: 21 March 2023

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Abstract

We develop inductive biases for the machine learning of complex physical systems based on the port-Hamiltonian formalism. To satisfy by construction the principles of thermodynamics in the learned physics (conservation of energy, non-negative entropy production), we modify accordingly the port-Hamiltonian formalism so as to achieve a port-metriplectic one. We show that the constructed networks are able to learn the physics of complex systems by parts, thus alleviating the burden associated to the experimental characterization and posterior learning process of this kind of systems. Predictions can be done, however, at the scale of the complete system. Examples are shown on the performance of the proposed technique.

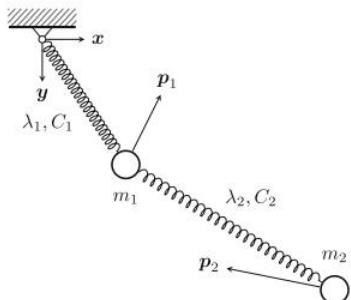
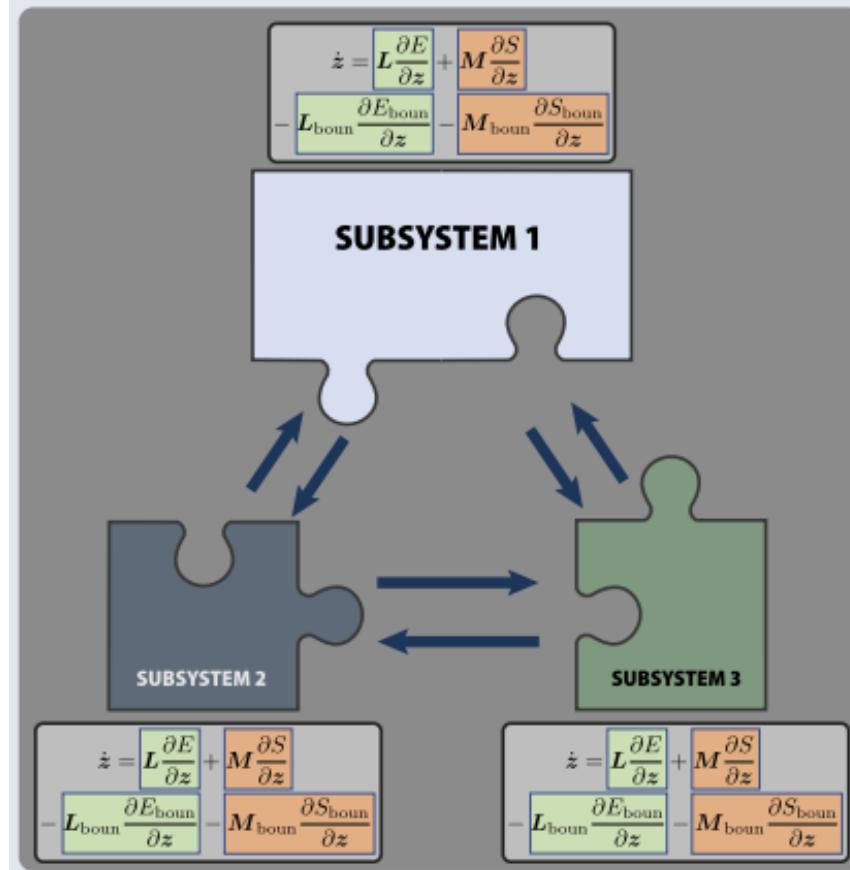


Fig. 2 Double thermoelastic pendulum. A single pendulum with external perturbations is learned, and the coupling between both systems is achieved via the port-metriplectic framework



Thermodynamics-Informed Neural Network

- The Fokker-Planck equation and, more particularly, its equivalent Itô stochastic differential equation takes the form:

$$dz = \left[L(z) \frac{\partial E}{\partial z} + M(z) \frac{\partial S}{\partial z} + k_B \nabla M(z) \right] dt + d\tilde{z}$$

- where k_B is the Boltzmann constant, $M(z)$ is a symmetric, positive semi-definite dissipation matrix, S is a second potential (the so-called Massieu potential, entropy at this level of description) and $d\tilde{z}$ is a Wiener process that satisfies

$$d\tilde{z} = B(z) dW(t)$$

- with B a non-square matrix satisfying

$$B(z) B(z)^T = 2k_B M(z)$$

- The importance of thermal fluctuations is controlled by the relative value of the Boltzmann constant k_B with respect to the average value of entropy. Given that E , S , L and M do not depend on k_B , if these effects are of low importance, we can take the limit $k_B \rightarrow 0$, resulting in

$$dz = L(z) \frac{\partial E}{\partial z} + M(z) \frac{\partial S}{\partial z}$$

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Thermodynamics-Informed Neural Network

$$\frac{dz}{dt} = L(z) \frac{\partial E}{\partial z} + M(z) \frac{\partial S}{\partial z}$$

$$\frac{dz}{dt} = \{z, E\} + (z, S) \quad \text{Metriplectic Flow}$$

- This same assumption, $k_B \rightarrow 0$, induces two additional consequences:

$$L(z) \frac{\partial S}{\partial z} = 0 \quad \text{and} \quad M(z) \frac{\partial E}{\partial z} = 0$$

- which constitute the ingredients of the celebrated General Equation for the non-Equilibrium Reversible-Irreversible Coupling, **GENERIC equations**. This type of formulations are also known as metriplectic formulations, since they combine metric and symplectic terms. However, in GENERIC equations, known as degeneracy conditions, play a fundamental role. They are key ingredients in the demonstration of the a priori satisfaction of the two laws of thermodynamics:

- **Conservation of energy** in closed systems Given the **anti-symmetry of L**:

$$\frac{dE}{dt}(z) = \frac{\partial E}{\partial z} \frac{\partial z}{\partial t} = 0$$

- **Non-negative entropy production**, given the **positive semi-definiteness of M**:

$$\frac{dS}{dt}(z) = \frac{\partial S}{\partial z} \frac{\partial z}{\partial t} = \frac{\partial S}{\partial z} M(z) \frac{\partial S}{\partial z} \geq 0$$

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Thermodynamics-Informed Neural Network

- Therefore, the GENERIC structure consistently guarantees the satisfaction of the laws of thermodynamics by construction. This makes GENERIC a very appealing choice for the construction of inductive biases in the learning of physical phenomena.
- For Thermodynamics of learning physical phenomena, we assume that D_i data sets contain labelled pairs of a single-step state vector z_t and its time evolution z_{t+1}

$$D = \{D_i\}_{i=1}^{N_{sim}}, D_i = \{(z_t, z_{t+1})\}_{t=0}^T$$

- so that a neural network can be constructed by means of two loss terms, a data loss term that takes into account the correct prediction of the state vector time evolution using the GENERIC integrator, defined as:

$$\Lambda_n^{data} = \left\| \frac{dz^{GT}}{dt} - \frac{dz^{net}}{dt} \right\|_2^2$$

Thermodynamics-Informed Neural Network

- where $\|\cdot\|_2$ denotes the L2-norm. The choice of the time derivative instead of the state vector itself is to regularize the global loss function to a uniform order of magnitude with respect to the degeneracy terms. A second loss term takes into account the fulfillment of the degeneracy equations,

$$\Lambda_n^{\text{deg}} = \left\| L \frac{\partial S}{\partial z_n} \right\|_2^2 + \left\| M \frac{\partial E}{\partial z_n} \right\|_2^2$$

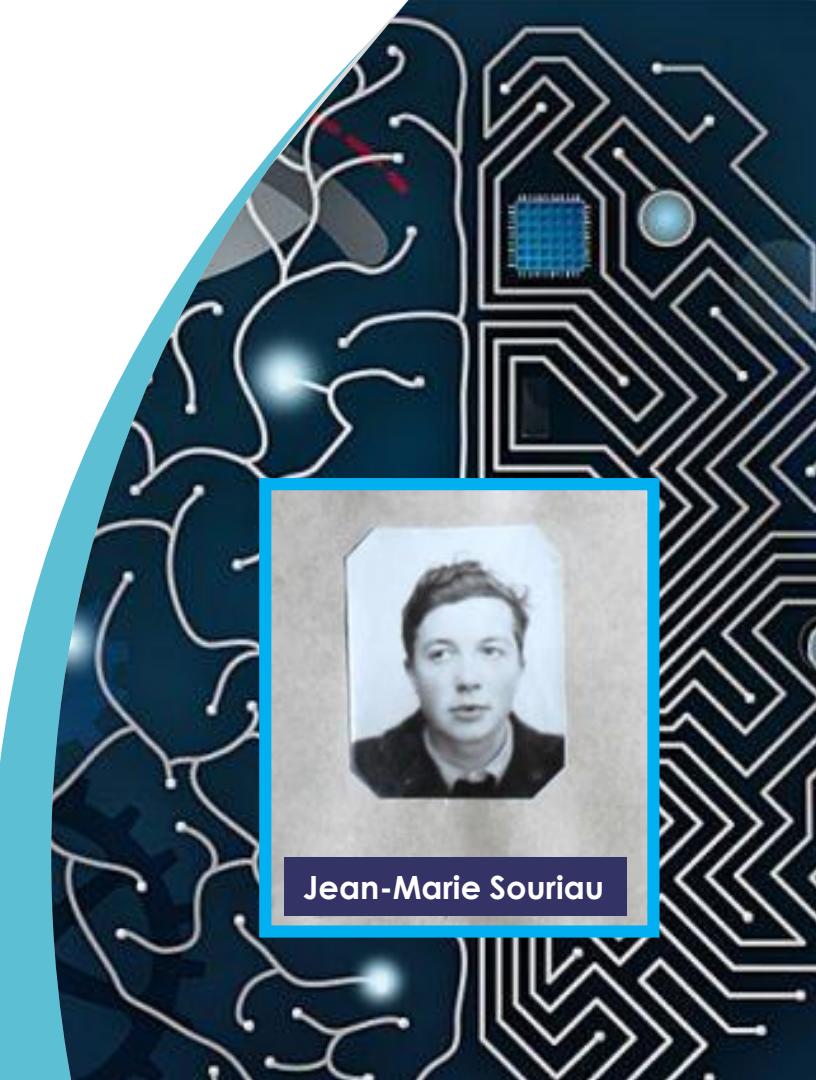
- This formulation gave rise to the so-called structure-preserving neural networks and thermodynamics-informed neural networks. These networks have been employed recently in the development of physics perception with the help of computer vision techniques.
- The global loss term is a weighted mean of the two terms over the shuffled N_{batch} batched snapshots.

$$\Lambda = \frac{1}{N_{\text{batch}}} \sum_{n=0}^{N_{\text{batch}}} (\lambda \Lambda_n^{\text{data}} + \Lambda_n^{\text{deg}})$$

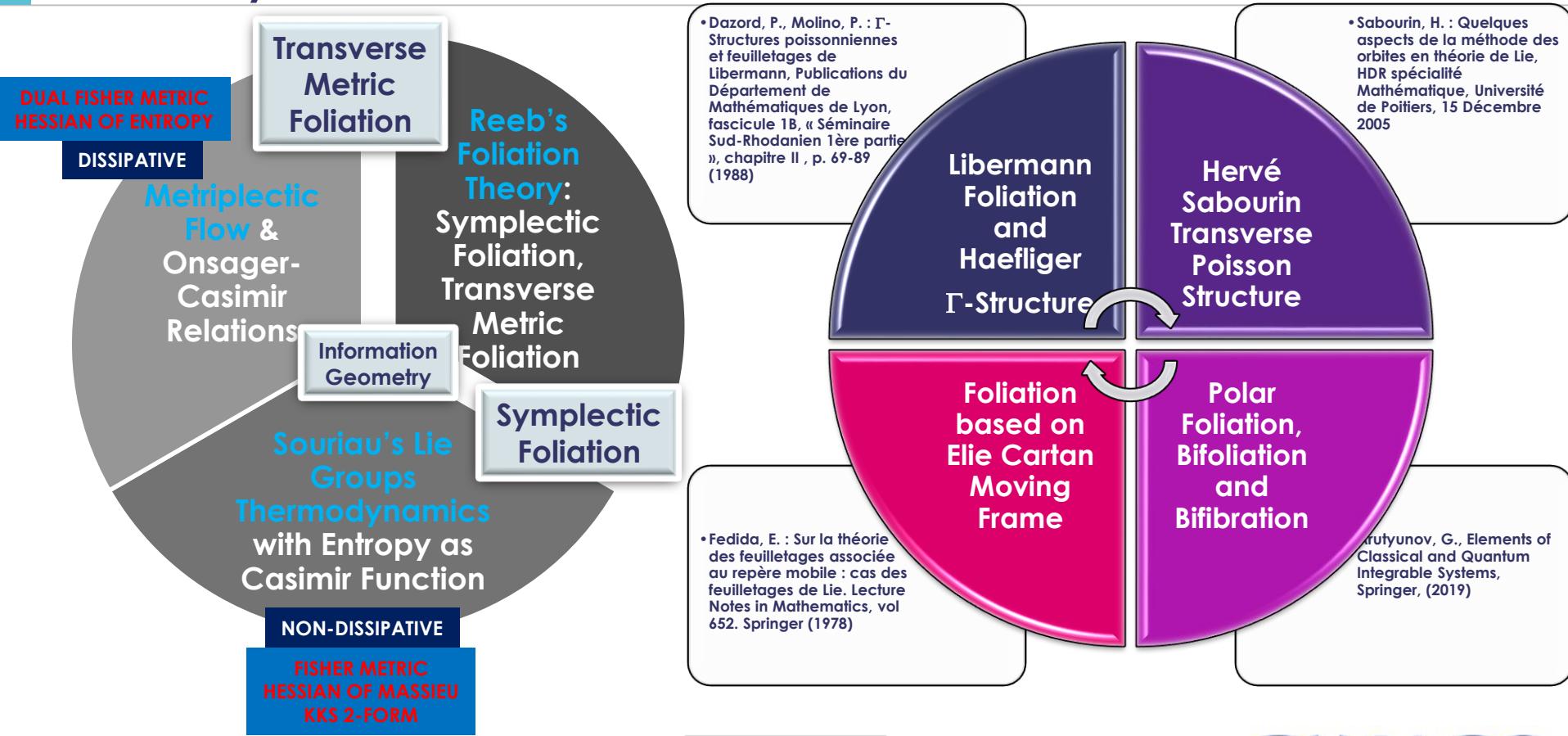


Conclusion

Jean-Marie Souriau



Symplectic Foliation Transverse Structure of Dissipative Thermodynamics



EUROPEAN ACTIONS : COST CaLISTA & MSCA CaLIGOLA

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CaLIGOLA

Cartan geometry, Lie and representation theory, Integrable Systems, quantum Groups and quantum computing towards the understanding of the geometry of deep Learning and its Applications

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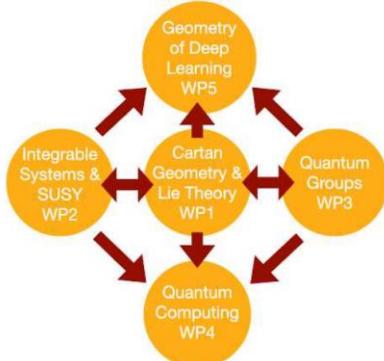
Horizon Europe Framework Programme MSCA Staff Exchanges 2021 (HORIZON-MSCA-2021-SE-01) Project ID: 101086123

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WP5 The geometry of Deep Learning

Main research themes:

- Lie groups thermodynamics
- Persistent Homology and Geneos
- The geometry of (Geometric) Deep Learning



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Cartan geometry, Lie, Integrable Systems, quantum group Theories for Applications - CaLISTA aims to advance cutting-edge research in mathematics and physics through a systematic application of the ideas and philosophy of Cartan geometry, a thorough Lie theoretic approach to differential geometry.



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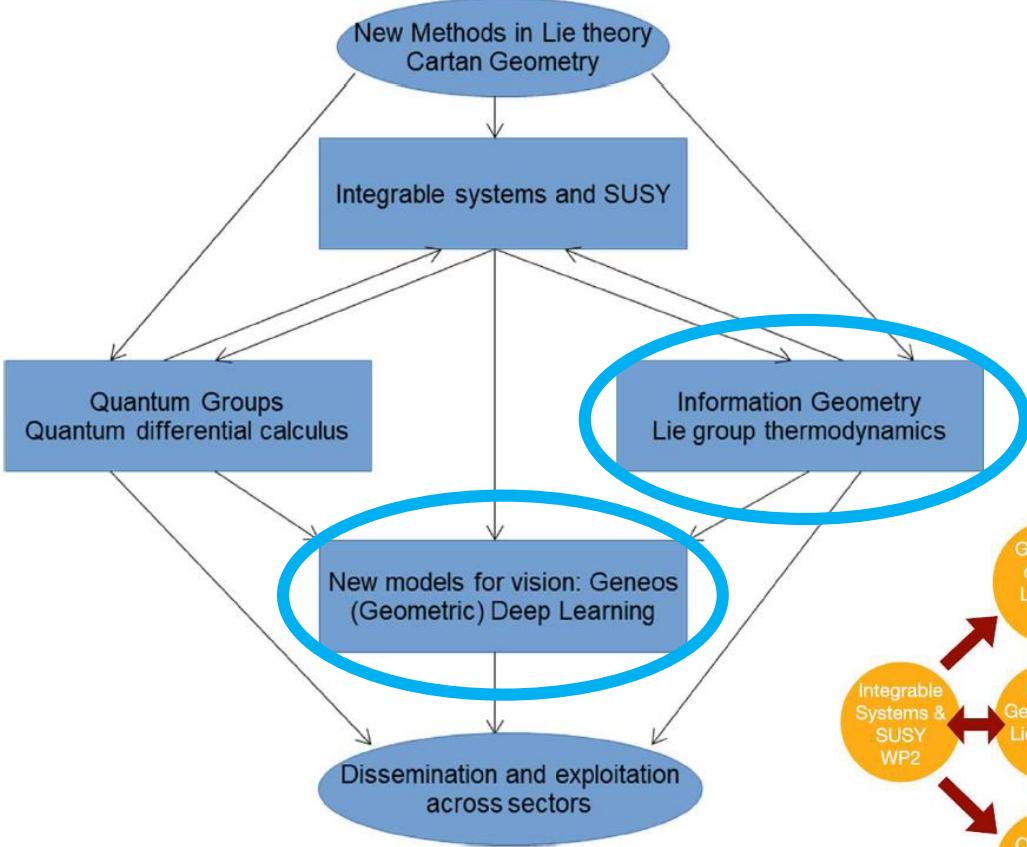
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<https://site.unibo.it/calista/en>

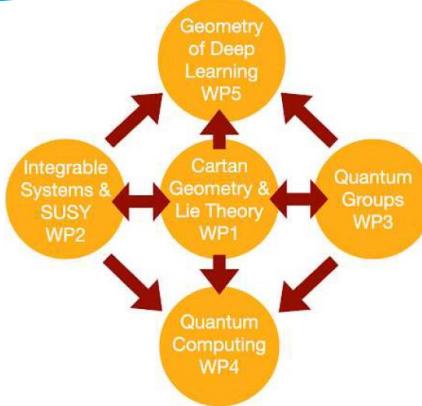
WG4 VISION: T4.3: Interpret SGD and the metric structure of the model space with Souriau Lie Thermodynamics. Interpret the coadjoint orbits of the symmetry group action as level set of entropy; exploit their symplectic structure to construct further symmetries (group equivariant GDL).

Nous parfîmes cinq cents; mais par un prompt renfort, nous nous vîmes trois mille en arrivant au port, tant, à nous voir marcher avec un tel visage, les plus épouvantés reprenaient leur courage !

MARIE SKŁODOWSKA-CURIE ACTIONS CaLIGOLA



Work Package no.	5		Start/end month		4/48			
Work Package title	The geometry of Deep Learning							
Lead beneficiary	Fiorese (UNIBO) [Barbaresco (Thales)]							
Participating organisation short name	UNIBO	CU	MU	UVEG	UC	ISI	ETH	
Total person months per participating organisation:	5	9	4	6	6	0	0	
Objectives:	Develop new vision models and algorithms with the use of persistent homology, Lie Group statistics to boost the machine learning techniques. Understand (Geometric) Deep Learning with the new invariance mathematical tools and quantum group techniques.							
Description of Work and role of specific beneficiaries/associated partners broken down and listed into numbered tasks including the following details:								
Task 5.1. Lie groups thermodynamics.	Use the techniques of WP1 and WP2 to explore the geometry of the information manifold and its Lie structure (i.e. foliation induced by the coadjoint action). Key expertise: UNIBO, Fiorese; Thales, Barbaresco.							
Task 5.2 Persistent Homology and Geneos:	going beyond topology and endow the space of all equivariant non-expansive operators (Geneo) a suitable Lie (differentiable) structure. Key expertise: UNIBO, Frosini; UC Otter.							
Task 5.3 The geometry of (Geometric) Deep Learning.	Differential calculus on graph neural networks, via quantum calculus. New theoretical foundations for the geometric deep learning algorithm. Key expertise: UNIBO (Fiorese, Latini), CU (O'Buachalla), UC IPAM Institute, Sturmels.							



CaLIGOLA: Cartan geometry, Lie and representation theory, Integrable Systems, quantum Groups and quantum computing towards the understanding of the geometry of deep Learning and its Application



7th International Conference on
**GEOMETRIC SCIENCE
OF INFORMATION**
GSI'25
Saint-Malo, France
29th to 31st October 2025

From Classical to Quantum Information Geometry

CALL FOR PAPERS

CONFERENCE CO-CHAIRS:

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Sony Computer Science Laboratories Inc, Japan
Frédéric Barbaresco
President of SEE ISIC Club
Thales Land & Air Systems, Vélizy, France

LOCAL ORGANIZING COMMITTEE:

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TOPICS OF INTERESTS INCLUDE BUT ARE NOT LIMITED TO:

- Geometric Learning and Differential Invariants on Homogeneous Spaces
- Statistical Manifolds and Hessian information geometry
- Rényi Entropy & Information
- Geometric Foliation Structures of Dissipation and Machine Learning
- Geometric Structures of Quantum Information & Processing
- Applied Geometric Learning
- Probability, Information and Topology (fundamentals & applications)
- Divergences in Statistics and Machine Learning
- Geometric Statistics
- Geometric Methods in Hybrid Classical/Quantum Systems
- Computational Information Geometry and Divergences
- Geometric Methods in Thermodynamics
- The Geometry of Classical & Quantum States
- Geometric Mechanics
- Geometric Green & Quantum Machine Learning

FIRST ANNOUNCEMENT AND CALL FOR PAPERS:

As for GSI'13/GSI'15/GSI'17/GSI'19/GSI'21/GSI'23, the objective of this 7th SEE GSI'25 conference, hosted in Saint-Malo, is to bring together pure/applied mathematicians and engineers, with common interest for Geometric tools and their applications for Information analysis and Learning. It emphasizes an active participation of young researchers to discuss emerging areas of collaborative research on "Geometric Science of Information and their Applications".

GSI presents a panorama of Geometric Tools emphasizing mathematical theory, physical model, computational methods, and applications in mathematics, physics, statistics, engineering, signal/image processing, machine learning and data science.

The Conference will be therefore held in areas of topics of mutual interest with the aim to:

- Provide an overview on the most recent state-of-the-art
- Exchange mathematical information/knowledge/expertise in the area

This conference will be an interdisciplinary event and will unify skills from Physics, Geometry, Probability and Information Theory. Proceedings are published in Springer's Lecture Note in Computer Science series.

Gala Diner will take place in Mont Saint-Michel

IMPORTANT DATES:

- CMT (Microsoft Research) opening: February 17th 2025
- Deadline for 8 pages SPRINGER LNCS format: April 2nd 2025
- Notification of acceptance: June 2nd 2025
- Final paper submission: July 1st, 2025
- Registration opening: August 4th, 2025
- Registration deadline for hardcopy of proceedings : September 15th

29 th
October
31 st
October
2025

GSI25 - 7th International Conference on **Geometric Science of Information**

↗ Saint-Malo, Palais du Grand Large, France

Geometric Structures of Statistical & Quantum Physics, Information Geometry, and Machine Learning

FROM CLASSICAL TO QUANTUM INFORMATION GEOMETRY

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Michel Serres: new epistemology of Thermodynamics & Information (La Traduction, Hermès III, 1974)



MICHEL SERRES
HERMÈS III
LA TRADUCTION



LES EDITIONS DE MINUIT



Michel SERRES

In "Cahiers de formation" of **Michel Serres** edited in 2024, we discover that he was influenced by Léon Brillouin book "**The Science and Theory of Information**" (Bertrand 2023). He claims that "**greatest discovery of history is that entropy and information are connected, in epistemology as well as in the theory of matter.**". For Michel Serres, Prometheus, who went to seek fire from the gods, is the divinity of the arts of fire, that of blacksmiths and then of the construction of steam engines. Our world was that of Atlas and became with Sadi Carnot, that of the fire of Prometheus. Our world was cold, silent and clean like that of Atlas, Mechanics and Carnot the father and it has become hot, noisy and polluted like that of Prometheus, Thermodynamics and Carnot the son. Michel Serres wrote in "Turner translates Carnot" (Serres, 1974):

[...] From Garrard to Turner, the path is very simple is **the path that runs from Lagrange to Carnot, from simple machines to steam engines, from mechanics to thermodynamics**, by the way of industrial Revolution. (**Serres 1974**)

[...] Let us reread the preface where Jean Perrin draws his own genealogy. He comes from two lines: that of thermodynamicists, whom he calls analogists, and who arrived, without hypothesis, at **the two principles - observe carefully that the first, that of equivalence, is a principle of invariance, that the second, that of the increase in entropy, is a principle of irreversible evolution** –; and that of the atomists, whom he calls the intuitives, and who interpret the previous results. Put the closed box down and look inside. Boltzmann forever links the two paths. It is still on the banks of the divine sea, where the song of the bards had been silent for two thousand years, that the drama ends and begins. The great Boltzmann commits suicide there, he who had once again united the story of Prometheus with the story of Pandora. From this torture our time will be born: atomic physics and molecular biochemistry. (**Serres 1974**)

"It is possible that science is the set of messages optimally invariant by any translation strategy]" - Michel Serres

Sadi Carnot by Michel Serres (**La Légende des Sciences - Brûler**):
<https://www.youtube.com/watch?v=J5JSWebXms0&list=PLZlkyegBPndOvTGjchR186h3u gekstwm8&index=5>

« Turner translates Carnot » - LA TRADUCTION (Hermes III) Michel Serres

« Science and the Humanities: The Case of Turner » - Michel Serres



"The Fighting Temeraire" (1838) - Joseph Mallord William Turner

« From Garrard to Turner, the path is very simple. It is the same path that runs from Lagrange to Carnot, from simple machines to steam engines, from mechanics to thermodynamics – by way of the Industrial Revolution... Turner or the introduction of fiery matter into culture. The first true genius in thermodynamics. The era of the wooden ship is dead. The Fighting Téméraire is tugged to her last berth to be broken up.... And here is the true Trafalgar, the true battle, the true clash: the immense division of the heavens and the sea into two zones. One of them is red, yellow, and orange, where the hot colors shout, ignited, burning; the other is violet, blue, green, and sea-green, where the cold and icy hues freeze. Within its own matter, the entire world becomes a steam engine between Carnot's two sources: the cold and the hot. Seawater in the boiler tank. Yes, Turner entered into the boiler. The 1838 painting is inside the tugboat. »

2nd Principle by Henri Poincaré (Bon sens paysan) fable of the grain of oats and the pile of wheat



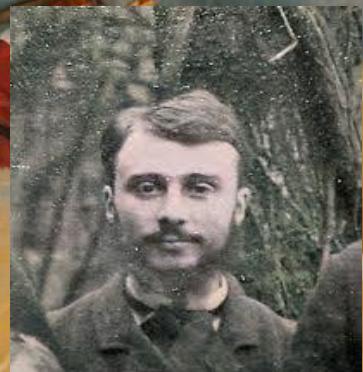
Supposons que nous voulions
placer un grain d'avoine au milieu
d'un tas de blé; cela sera facile;
supposons que nous voulions
ensuite l'y retrouver et l'en retirer;
nous ne pourrons y parvenir.

With his “peasant common sense”, **Henri Poincaré described the 2nd principle** in the form of a fable of the grain of oats and the pile of wheat, in the last sentence of his course:

[...] Suppose we want to place a grain of oats in the middle of a pile of wheat; it will be easy; suppose that we then want to find it there and remove it; we will not be able to achieve this. All irreversible phenomena, according to certain physicists, are built on this model [Supposons que nous voulions placer un grain d'avoine au milieu d'un tas de blé ; cela sera facile; supposons que nous voulions ensuite l'y retrouver et l'en retirer; nous ne pourrons y parvenir. Tous les phénomènes irréversibles, d'après certains physiciens, seraient construits sur ce modèle]. (Poincaré 1908)



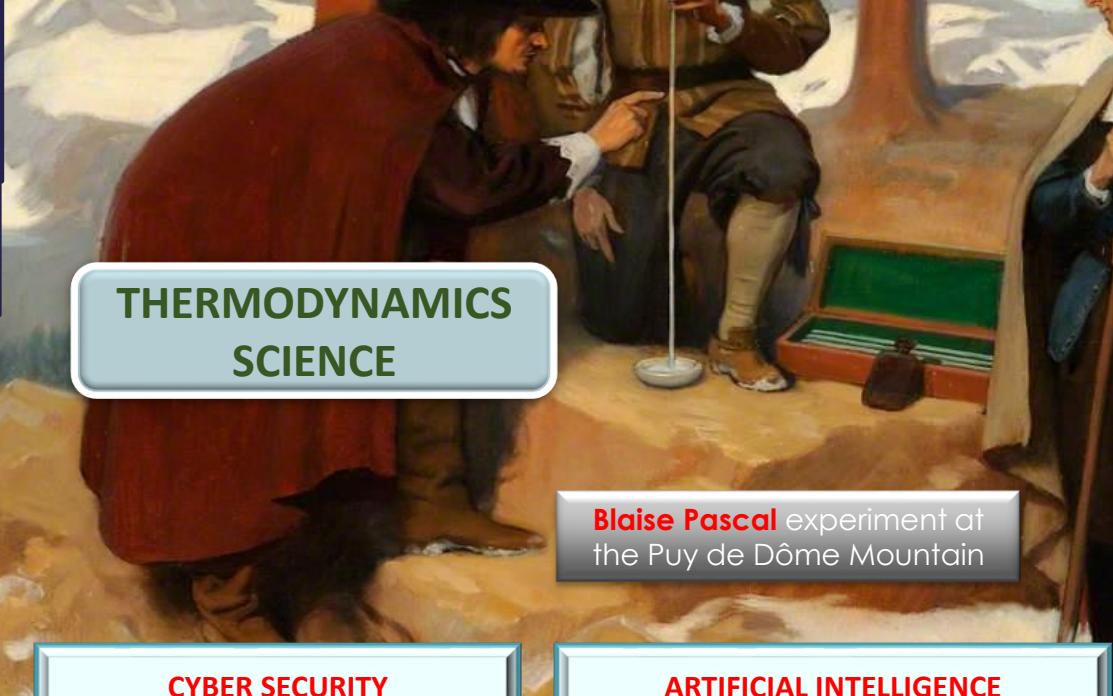
Sadi Carnot
2nd Thermodynamics
Principle



Pierre Duhem
Clausius-Duhem
Equation

QUANTUM DEEPTECH
Quantum Thermodynamics
Error Corr. Quantum Feedback
Quantum Computer Cryostat

CLIMATE CHANGE
Atmosphere Thermodynamics
Earth Global Warming
Contrails Reduction for H₂ Engine

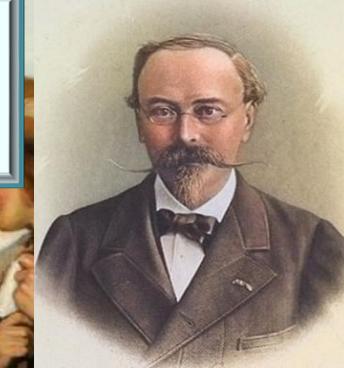


THERMODYNAMICS SCIENCE

Blaise Pascal experiment at
the Puy de Dôme Mountain

CYBER SECURITY
KLJN Noise & Key Distribution
Entropy For Cryptography
KRR Energy Devaluation

ARTIFICIAL INTELLIGENCE
Thermodynamics-Informed NN
Symplectic/Metriplectic Integrator
Non-Equilibrium Thermodynamics



François Massieu
Thermodynamics
Potentials



Jean-Marie Souriau
Lie Groups
Thermodynamics

Information Geometry, Toric Manifold, Delzant Polytope, Symplectic Potentials and Guillemin Metric

Notation.

- $\mathbb{P}_n(c)$ is the complex projective space of complex dimension n and holomorphic sectional curvature $c > 0$.
- $\mathbb{D}(c)$ is the unit disk in \mathbb{C} endowed with the Hyperbolic metric of constant holomorphic sectional curvature $c < 0$.

Similar examples

Toric manifold	Statistical manifold
$S^2 = \mathbb{P}_1(\frac{1}{n})$	Binomial
\mathbb{C}	Poisson
$\mathbb{P}_n(1)$	Categorical
$\mathbb{P}_n(\frac{1}{m})$	Multinomial
$\mathbb{D}(-\frac{1}{r})$	Negative Binomial

Recap

- $\mathcal{B}(n) \cong (0, n)$.
- Fisher metric $h_F(x) = \frac{x}{x(n-x)}$.
- Potential $\phi(x) = x \ln(x) + (n-x) \ln(n-x)$.

Key observation:

$$\phi = 2\phi_G,$$

where ϕ_G is the Guillemin's potential associated to the polytope $\Delta = [0, n]$.

Thus:

- By Delzant's correspondence:
- $[0, n] \rightsquigarrow (S^2, \frac{n}{2}\omega_{round})$
 - Momentum map $J(x, y, z) = \frac{y}{2}(z+1)$.

By Guillemin's results and the fact that $\phi = 2\phi_G$:

- $J : (S^2, \frac{n}{2}\omega_{round}) \rightarrow ((0, n), \frac{1}{2}h_F)$ is a Riemannian submersion.

Rescale by a factor 2:

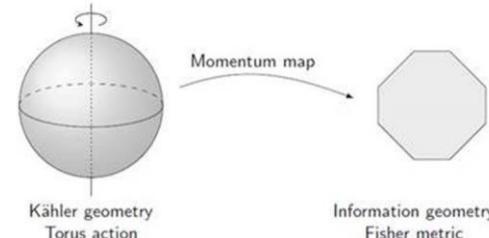
- $J : (S^2, ng_{round}) \rightarrow ((0, n), h_F)$ is a Riemannian submersion.

Summary

- $(\mathcal{B}(n), h_F)$ is isometric to the interior of a Delzant polytope endowed with (a multiple of) the Guillemin metric.
- (S^2, ng_{round}) is the corresponding toric manifold.

Molitor, M.: Kähler toric manifolds from dually flat spaces. arXiv:2109.04839v1

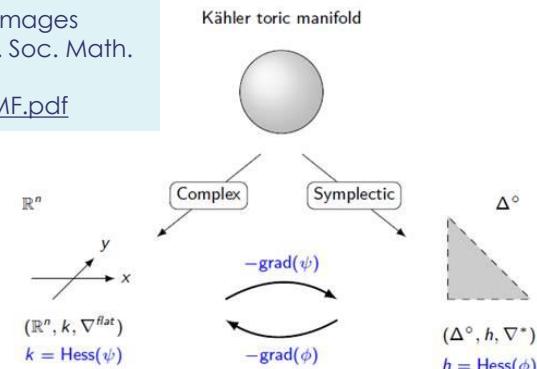
Variables actions-angles non commutatives et exemples d'images convexes de l'application moment, Thomas Delzant PhD, Paris 6 1986, supervised by CM. Marle



Symplectic versus complex points of view

Delzant, T.: Hamiltoniens périodiques et images convexes de l'application moment. Bull. Soc. Math. France 116(3), 315–339 (1988), <https://irma.math.unistra.fr/~delzant/BSMF.pdf>

Fujita, H. The generalized Pythagorean theorem on the compactifications of certain dually flat spaces via toric geometry. Info. Geo. 7, 33–58 (2024). <https://doi.org/10.1007/s41884-023-00123-y>



Proposition

- The map $-\text{grad}(\psi) = (\frac{\partial \psi}{\partial x_1}, \dots, \frac{\partial \psi}{\partial x_n})$ is an isometry from (\mathbb{R}^n, k) onto (Δ°, h) with inverse $-\text{grad}(\phi)$.
- ϕ and ψ are Legendre dual to each other:

$$\phi(x) + \psi(y) + \langle x, y \rangle = 0, \quad x = -\text{grad}(\phi)(y), \quad y \in \Delta^\circ.$$

OPEN



Institut de recherches
Mathématiques
d'Oberwolfach, 1949



René THOM

Jean-Pierre SERRE

Georges REEB

"We haven't yet discovered **the foliations**, but it will come. I am convinced that the fundamental problem posed by **quantum mechanics** is the following problem. ... it will be necessary to take into consideration **more complicated mathematical structures, such as foliations** ... It's a bit my hope that one day or another, we will manage to develop models where a **phenomenon will be defined as a leaf of foliation** in a product of spaces of vision by a space of observer positions." – René Thom in "DETERMINISME ET INNOVATIONS"
<https://www.youtube.com/watch?v=vBXxKQVQFnRo>

"At the start, the theory of structural stability had seemed to me of such breadth and generality, that with it I could hope in some way to **replace thermodynamics by geometry, to geometrize in a certain sense thermodynamics**, eliminate from thermodynamic considerations all aspects of a measurable and stochastic nature to retain only the corresponding geometric characterization of the attractors." – René Thom



Contact

Frédéric BARBARESCO

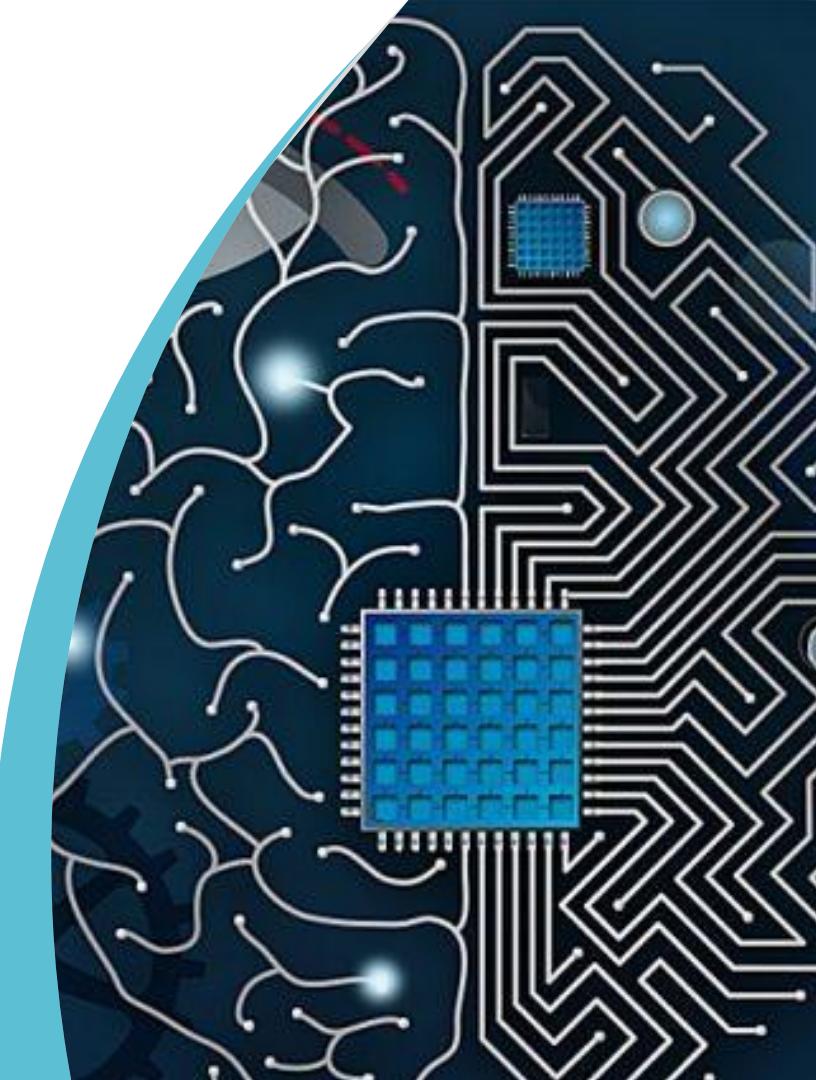
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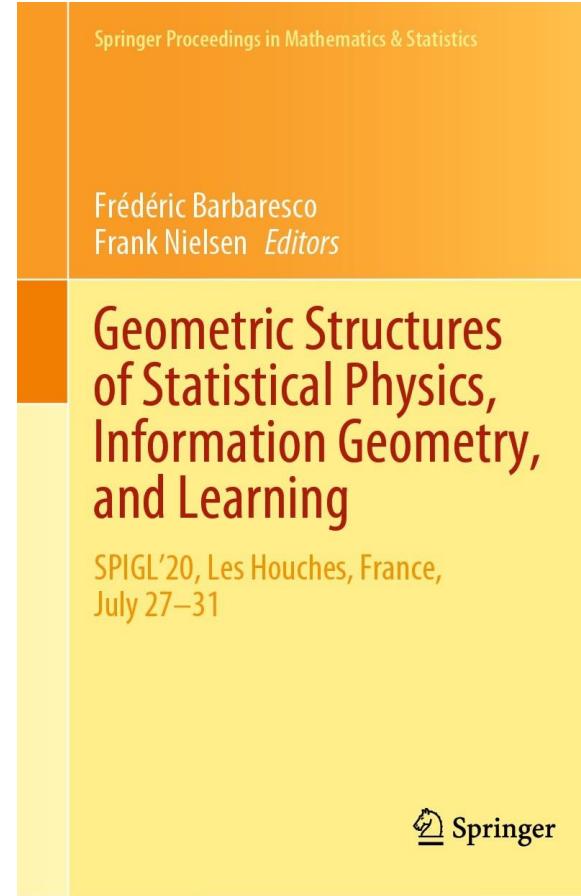
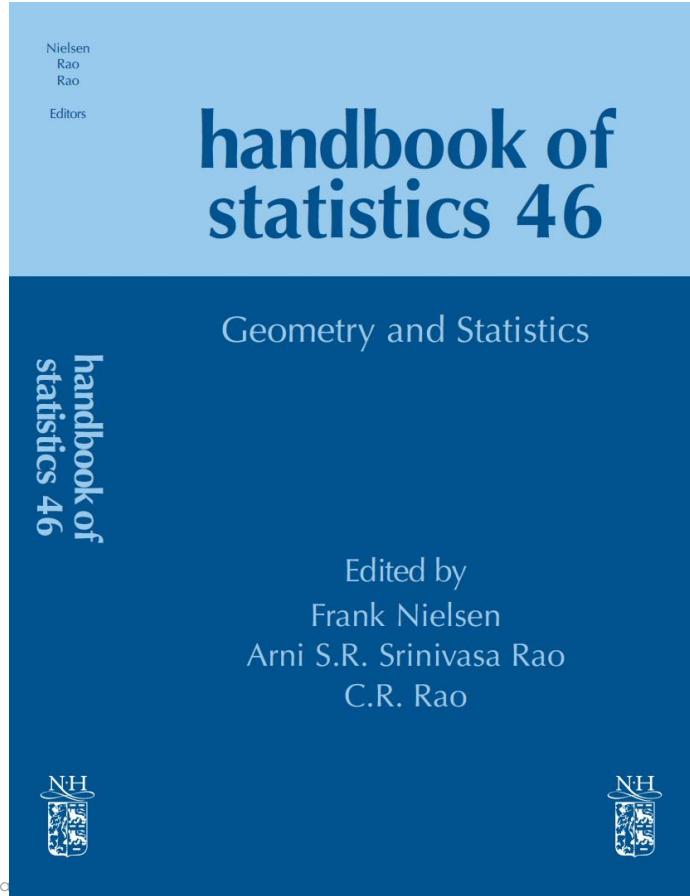
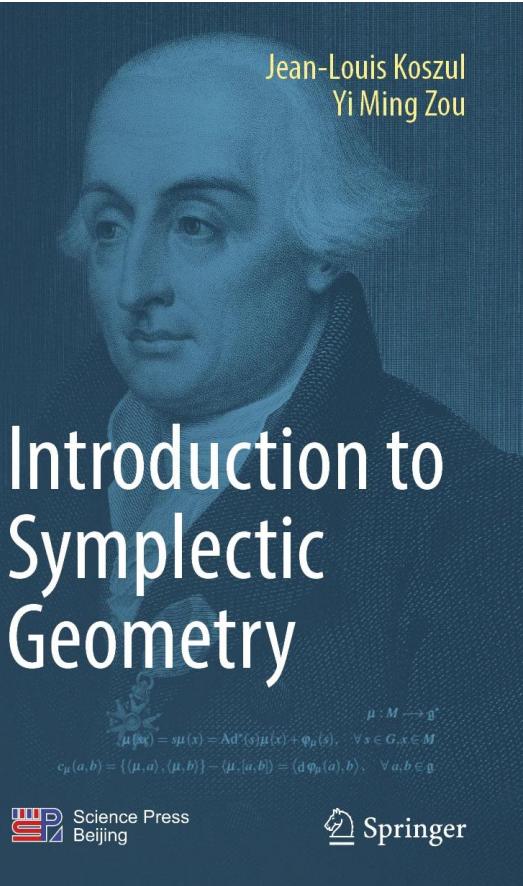
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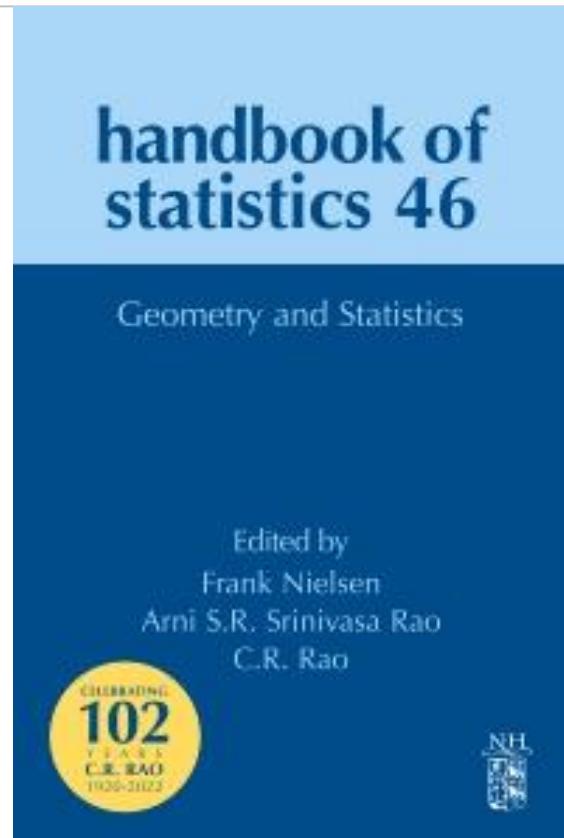
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To Go Further on Information Geometry and Symplectic Geometry



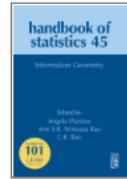
Elsevier Handbook of Statistics n°46 « Geometry and Statistics »



Handbook of Statistics

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Symplectic theory of heat and information geometry

Frédéric Barbaresco [ORCID](#) [✉](#)

THALES Land & Air Systems, Meudon, France

Available online 22 April 2022.

<https://www.sciencedirect.com/science/article/abs/pii/S0169716122000062>

<https://www.elsevier.com/books/geometry-and-statistics/nielsen/978-0-323-91345-4>

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Koszul Book on Souriau Work: The Little Green Book



Koszul Book on Souriau Work: The Little Green Book

Jean-Louis Koszul · Yiming Zou

Introduction to Symplectic Geometry

Forewords by Michel Nguiffo Boyom, Frédéric Barbaresco and Charles-Michel Marle

This introductory book offers a unique and unified overview of symplectic geometry, highlighting the differential properties of symplectic manifolds. It consists of six chapters: Some Algebra Basics, Symplectic Manifolds, Cotangent Bundles, Symplectic G-spaces, Poisson Manifolds, and A Graded Case, concluding with a discussion of the differential properties of graded symplectic manifolds of dimensions (o,n). It is a useful reference resource for students and researchers interested in geometry, group theory, analysis and differential equations.

$$\mu : M \longrightarrow \mathfrak{g}^*$$

$$\mu(sx) = s\mu(x) = \text{Ad}^*(s)\mu(x) + \varphi_\mu(s), \quad \forall s \in G, x \in M$$

$$c_\mu(a, b) = \{\langle \mu, a \rangle, \langle \mu, b \rangle\} - \langle \mu, [a, b] \rangle = \langle d\varphi_\mu(a), b \rangle, \quad \forall a, b \in \mathfrak{g}$$



Jean-Louis Koszul
Yiming Zou



Introduction to Symplectic Geometry

$$\mu : M \longrightarrow \mathfrak{g}^*$$

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Volume 1 · Number 1 · 2018



INFORMATION GEOMETRY, SPRINGER JOURNAL
<https://www.springer.com/mathematics/geometry/journal/41884>

Special Issue:

Affine Differential Geometry and Hesse Geometry: A Tribute and Memorial to Jean-Louis Koszul

Submission Deadline: 30th November 2019

Jean-Louis Koszul (January 3, 1921 – January 12, 2018) was a French mathematician with prominent influence to a wide range of mathematical fields. He was a second generation member of Bourbaki, with several notions in geometry and algebra named after him. He made a great contribution to the fundamental theory of Differential Geometry, which is foundation of Information Geometry. The special issue is dedicated to Koszul for the mathematics he developed that bear on information sciences.

Both original contributions and review articles are solicited. Topics include but are not limited to:

- Affine differential geometry over statistical manifolds
- Hessian and Kahler geometry
- Divergence geometry
- Convex geometry and analysis
- Differential geometry over homogeneous and symmetric spaces
- Jordan algebras and graded Lie algebras
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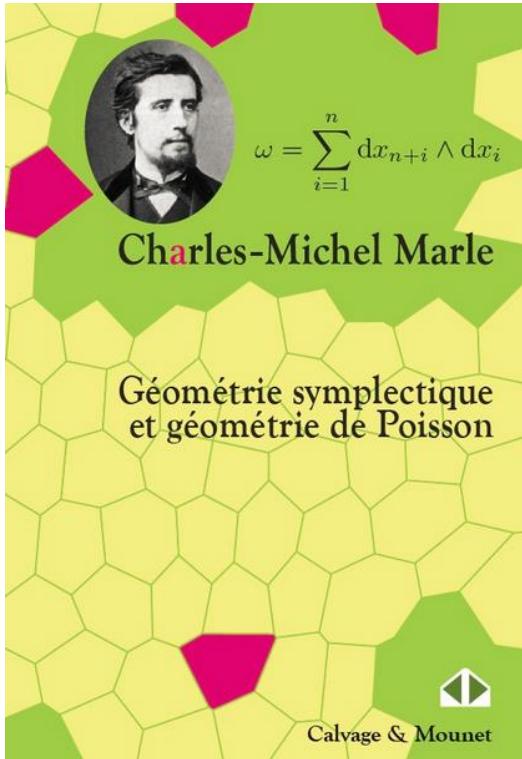
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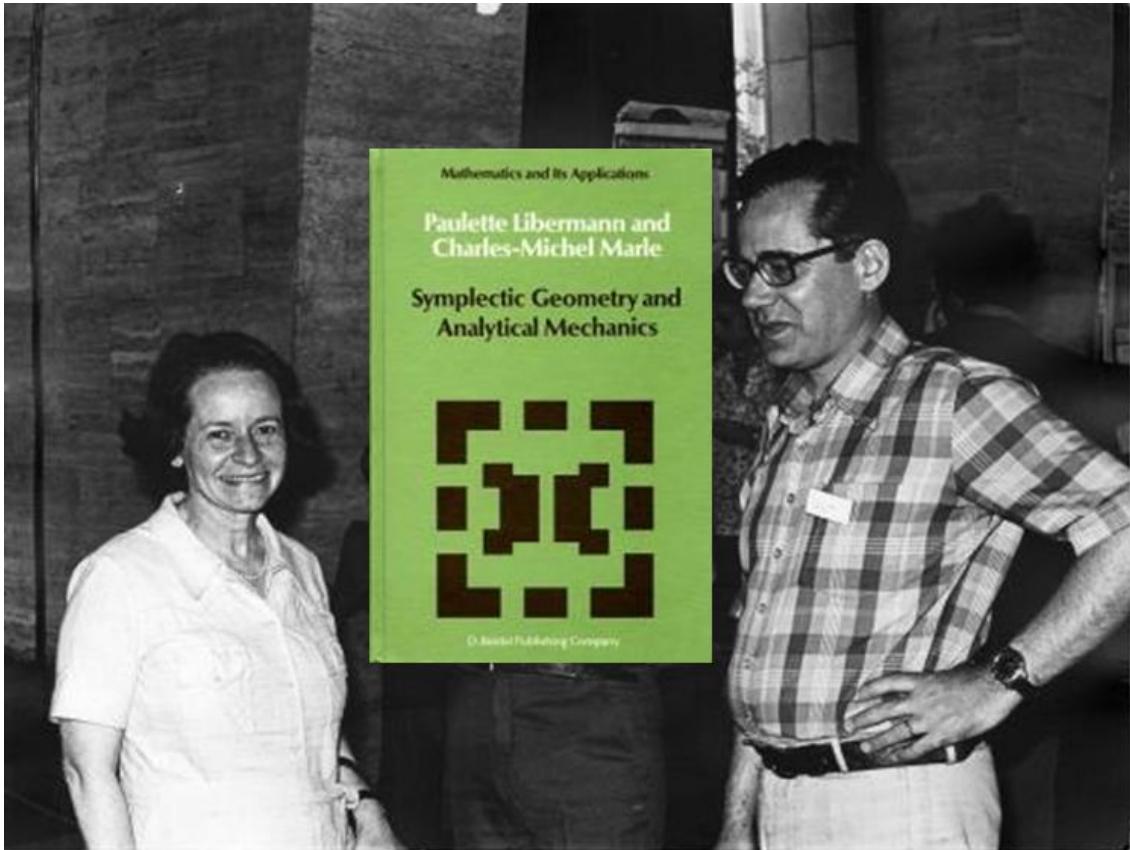
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ESI Seminar "Infinite-dimensional Geometry: Theory and Applications"



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Introduction to Lie Groups Representation Theory , Symplectic and Poisson Geometries (Université Paris Saclay & Sorbonne Université Lectures)

Introduction aux groupes de Lie
pour la physique

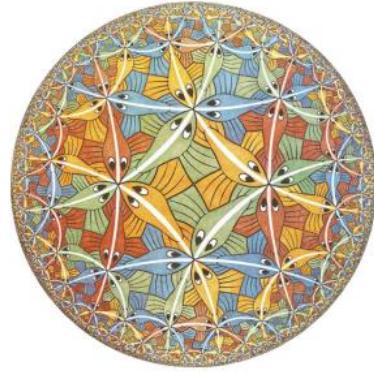
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Professeur à l'Université Paris-Saclay (Faculté des sciences d'Orsay)

Cours de troisième année de Centrale-Supélec
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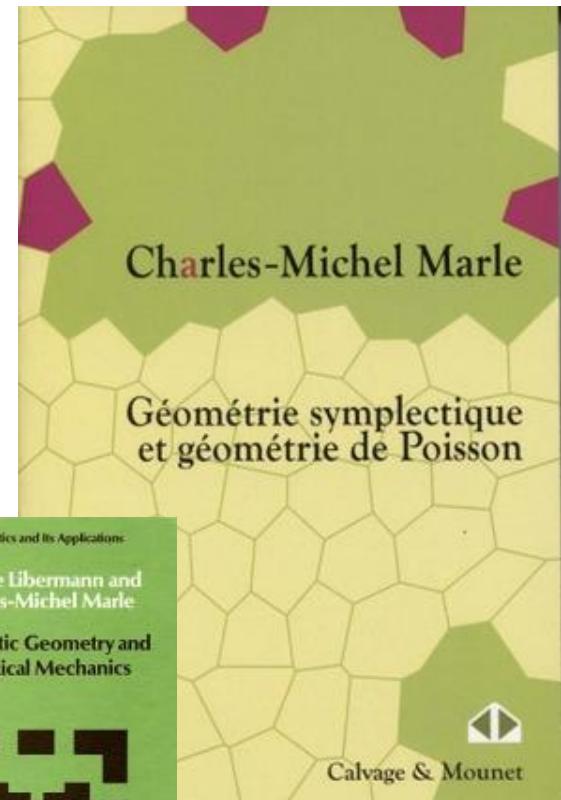
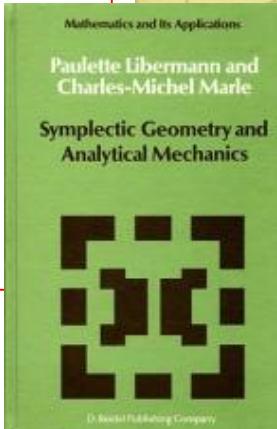
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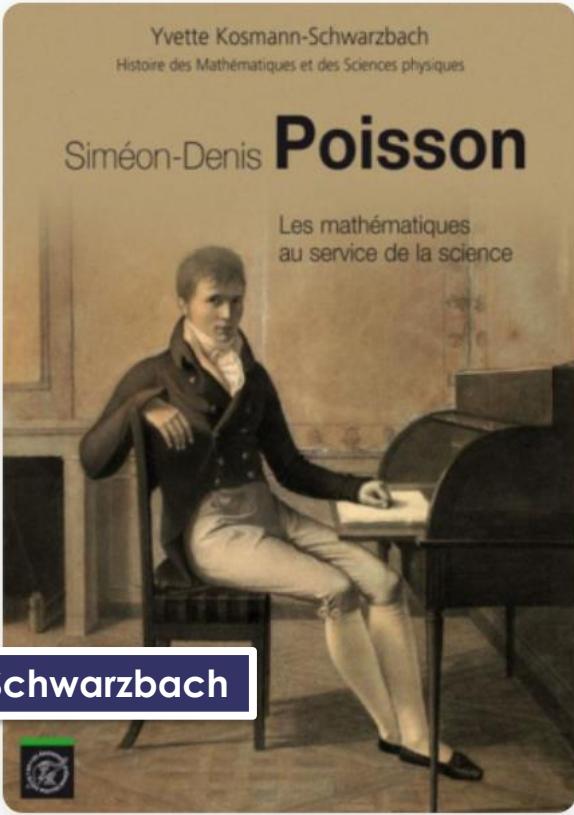
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Parcours de Physique Théorique

Invariances en physique
et théorie des groupes

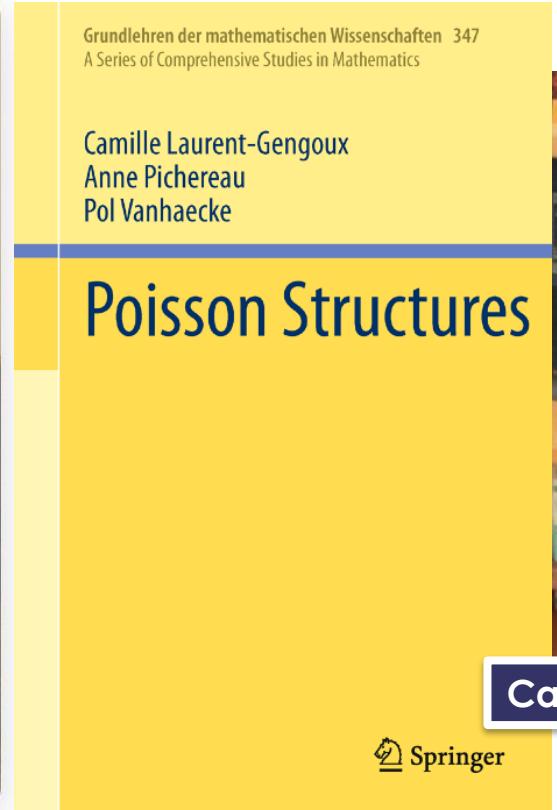
Jean-Bernard Zuber



Poisson Geometry



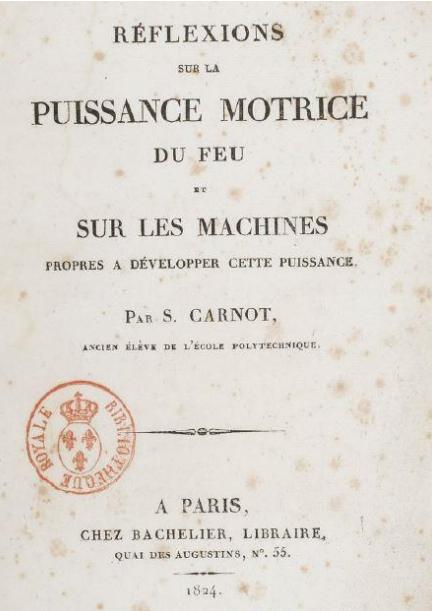
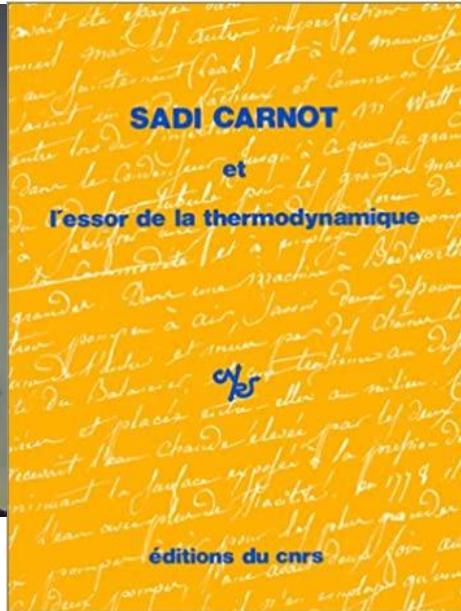
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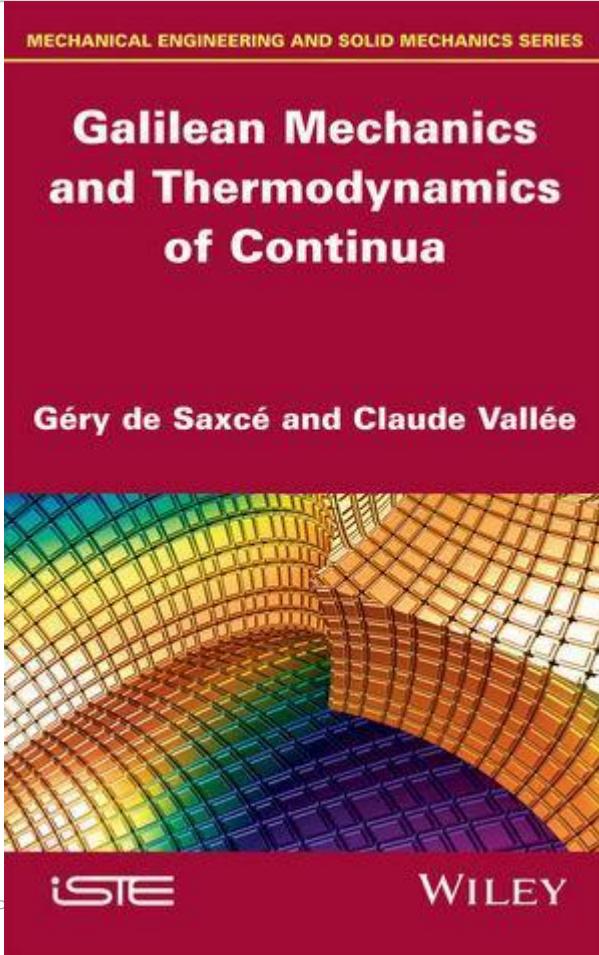
150th birthday of Thermodynamics – Ecole Polytechnique



1st & 2nd Thermodynamics Principles

1824-2024: 200 years
publication of Carnot's Book

Gery de Saxcé & Claude Vallée

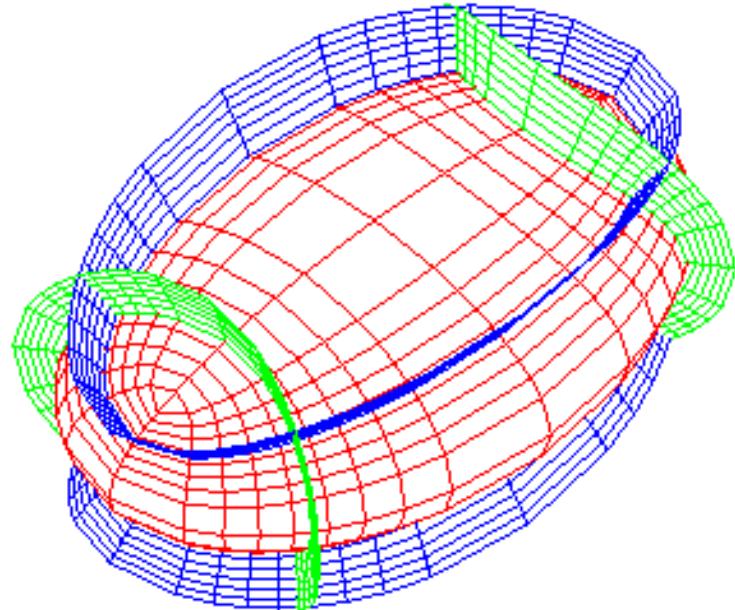
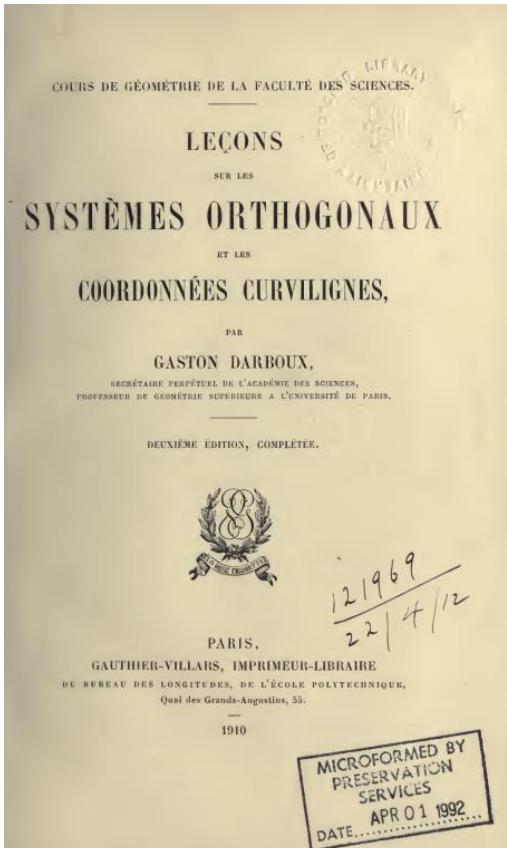
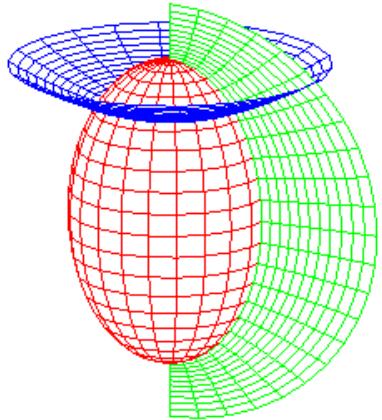


This title proposes a unified approach to continuum mechanics which is consistent with Galilean relativity. Based on the notion of affine tensors, a simple generalization of the classical tensors, this approach allows gathering the usual mechanical entities — mass, energy, force, moment, stresses, linear and angular momentum — in a single tensor.

Starting with the basic subjects, and continuing through to the most advanced topics, the authors' presentation is progressive, inductive and bottom-up. They begin with the concept of an affine tensor, a natural extension of the classical tensors. The simplest types of affine tensors are the points of an affine space and the affine functions on this space, but there are more complex ones which are relevant for mechanics – torsors and momenta. The essential point is to derive the balance equations of a continuum from a unique principle which claims that these tensors are affine-divergence free.

<https://www.wiley.com/en-us/Galilean+Mechanics+and+Thermodynamics+of+Continua-p-9781848216426>

Triple Orthogonal Systems : Darboux and Lamé



Large Dimensions: Mathematics and Applications.(Unfinished and Unedited)

Misha Gromov

December 14, 2018

This is a guide to my lectures in the Fall 2018 at CIMS.

□ ◻

SYMMETRY AND ENTROPY: ARCHIMEDES → BOLTZMANN → FISHER.

By the above (iv), the quotient space $\mathbb{C}^{N+1}/\mathbb{T}^{N+1}$ can be identified with the positive cone \mathbb{R}_+^{N+1} , and the complex Archimedean map $A : \mathbb{C}^{N+1} : \mathbb{C}^{N+1} \rightarrow \mathbb{R}_+^{N+1}$ factors through the real one,

$$\mathbb{R}A : \mathbb{R}_+^{N+1} \rightarrow \mathbb{R}_+^{N+1} \text{ for } (x_0, x_1, \dots, x_N) \mapsto (x_1^2, x_2^2, \dots, x_N^2).$$

Since the quotient metric in $\mathbb{C}^{N+1}/\mathbb{T}^{N+1}$ is equal to the standard Euclidean metric in $\mathbb{R}_+^{N+1} = \mathbb{C}^{N+1}/\mathbb{T}^{N+1}$, the Riemannian metric in the receiving positive cone \mathbb{R}_+^{N+1} which is induced from \mathbb{C}^{N+1} equals the transport of the Euclidean (regarded as Riemannian) metric $\sum_i dx_i^2$ from $\mathbb{R}_+^{N+1} = \mathbb{C}^{N+1}/\mathbb{T}^{N+1}$ to $\mathbb{R}_+^{N+1} = \mathbb{R}A(\mathbb{R}_+^{N+1})$ by the map $\mathbb{R}A$, where this metric is

$$\mathbb{R}A_*(g_{Eu}) = \sum_{i=0}^N \frac{1}{2x_i} dx_i^2,$$

since the differential of the map $\mathbb{R}A$ is given by the diagonal matrix with the entries $2x_i = \frac{dx_i^2}{dx_i}$.

Remarkably – only exceptional quadratic forms have this property –

there exists a smooth function, say Φ on \mathbb{R}_+^{N+1} , the Hessian of which is equal to the (differential) quadratic form $\mathbb{R}A_*(g_{Eu})$,

$$\frac{\partial_{ij}\Phi}{\partial x_i \partial x_j} dx_i dx_j = \mathbb{R}A_*(g_{Eu}).$$

In fact, the function $\frac{1}{2} \sum_i x_i \log x_i$ can be taken for Φ , since $\frac{d^2}{dx^2}(x \log x) = \frac{1}{x}$. It follows that

the entropy function on the simplex $\Delta^N \subset \mathbb{R}_+^{N+1}$,

$$ent(x_0, x_1, \dots, x_N) = -(x_0 \log x_0 + x_1 \log x_1 + \dots + x_N \log x_N)$$

is concave, where, moreover - this, apparently, goes back to Ronald. Fisher –

$$Hess(ent)) = -2\mathbb{R}A_*(g_{Eu}).$$

In particular, the Riemannian metric defined by $-Hess(ent)$ has constant positive curvature.⁵⁵

It is also clear that the entropy is the only function which satisfies this equality and vanishes at the vertices $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$, ..., $(0, 0, \dots, 1)$ of the simplex Δ^N . Therefore one can take the equality $Hess(ent)) = -2\mathbb{R}A_*(g_{Eu})$ for the definition of entropy⁵⁶ entropy by Boltzmann-Shannon computational(!) formula $-\sum_i p_i \log p_i$ is as inappropriate as defining (rather than computing) the area of the disk as the limit of the areas of the inscribed regular n -gons. (The original Boltzmann's definition, translated to the 21st century mathematical language, is described in "search for a structure-entropy" on my page <https://cims.nyu.edu/gromov/>).

Conversely, by taking the Hessian of $\sum_i x_i \log x_i$, one arrives from the clumsy simplex to the beautifully round sphere with the huge symmetry group.

(This, possibly, may explain the "unreasonable effectiveness" of entropy in mathematical physics and in math generated by physics, and which points toward "quantum nature" of entropy. But exactly this beautiful hidden symmetry makes one wary of transplanting the idea of entropy from physics to mathematical models of Life.)

Fisher Metric. Recall (Archimedes, 287-212 BCE) the *real moment map* from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ to the probability simplex $\Delta^n \subset \mathbb{R}^{n+1}$ for

$$(x_0, \dots, x_n) \mapsto (p_0 = x_0^2, \dots, p_n = x_n^2)$$

and observe following R. Fisher that the spherical metric (with constant curvature +1) thus transported to Δ^n , call it ds^2 on Δ^n , is equal, up to a scalar multiple, to the *Hessian of the entropy*

$$\text{ent}\{p_0, \dots, p_n\} = - \sum_i p_i \log p_i.$$

$$ds^2 = \text{const} \frac{\partial^2 \text{ent}(p_i)}{\partial p_i \partial p_j}.$$

OPEN

-
- I. What is probability?
 - II. What is it good for?
 - III. What are limitations of the concept of probability?
 - IV. What alternatives are desirable and what is available?

If, accordingly, we take the "inverse Hessian" – a kind of double integral " $\iint ds^2$ " for the *definition* of entropy – we arrive at

Question 2. Are there *interesting* "entropies" associated to (real and complex) moment maps of general toric varieties? Is there a *meaningful* concept of "generalised probability" grounded in positivity encountered in algebraic geometry?

Gromov question: Are there « entropies » associated to moment maps

Bernoulli Lecture - What is Probability?

27 March 2018 - CIB - EPFL - Switzerland

Lecturer: Mikhail Gromov

[https://bernoulli.epfl.ch/images/website/What is Probability v2\(2\).mp4](https://bernoulli.epfl.ch/images/website/What is Probability v2(2).mp4)

<http://forum.cs-dc.org/uploads/files/1525172771489-alternative-probabilities-2018.pdf>

Fisher Metric. Recall (Archimedes, 287-212 BCE) the *real moment map* from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ to the probability simplex $\Delta^n \subset \mathbb{R}^{n+1}$ for

$$(x_0, \dots, x_n) \mapsto (p_0 = x_0^2, \dots, p_n = x_n^2)$$

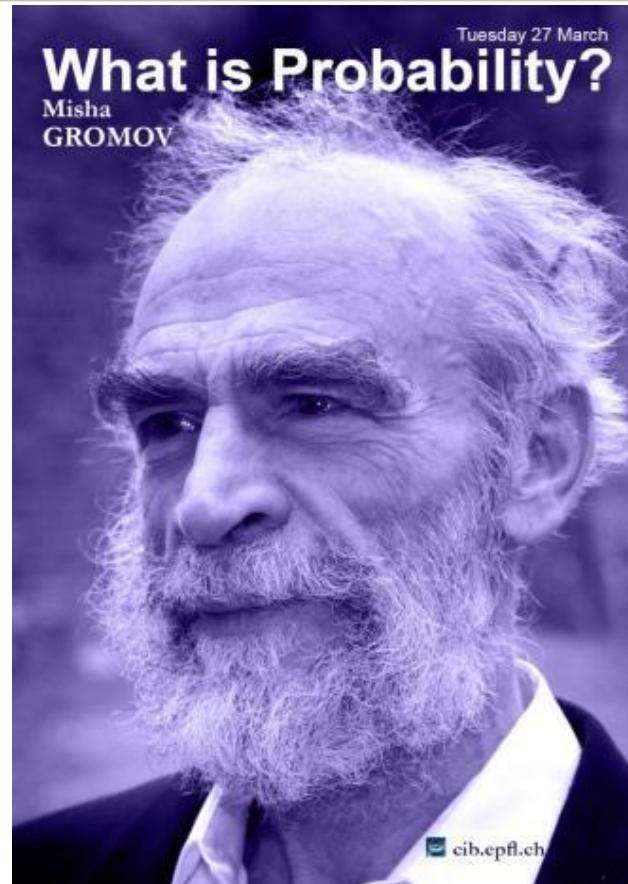
and observe following R. Fisher that the spherical metric (with constant curvature +1) thus transported to Δ^n , call it ds^2 on Δ^n , is equal, up to a scalar multiple, to the *Hessian of the entropy*

$$\text{ent}\{p_0, \dots, p_n\} = -\sum_i p_i \log p_i.$$

$$ds^2 = \text{const} \frac{\partial^2 \text{ent}(p_i)}{\partial p_i \partial p_j}.$$

If, accordingly, we take the "inverse Hessian" – a kind of double integral " $\int \int ds^2$ " for the *definition* of entropy – we arrive at

Question 2. Are there *interesting* "entropies" associated to (real and complex) moment maps of general toric varieties? Is there a *meaningful* concept of "generalised probability" grounded in positivity encountered in algebraic geometry?



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