Direct connections on jet groupoids

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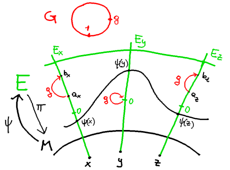
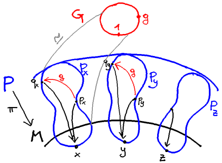
Geometric model underlying field and gauge theories

- space-time manifold M
- vector/spinor bundle $E \to M$ with fibres V
- (matter) field $\psi : M \to E$ section of E
- dynamics via action $S[\psi]$
  need covariant derivative $D_X\psi$ i.e. linear connection on E



- symmetries by Lie group $G \subset GL(V)$ acting on fibres of E
- principal G-bundle $P \to M$ with associated bundle $E = P \times_G V$
- gauge bosons (force carriers) $A \in \Omega^1_{\mathrm{loc}}(M, \mathfrak{g})$
  local form of principal connection on P
- gauge group $\widehat{G} = \mathrm{Aut}_M(P) \cong \Gamma(M, G) \cong \mathrm{Aut}_M(E)$
  $\Rightarrow$ acts on ψ and A
- dynamics via $\widehat{G}$-inv. action $S[\psi, A]$ (minimal coupling)
  $\Rightarrow$ new covariant derivative $D^A_X\psi$



Idea: replace
1) Lie algebra $\mathfrak{g}$
2) gauge group $\hat{G}$
3) connection on P
4) need jet bundles

by
- Lie algebroids (Weinstein, Cattaneo, Strobl,...)
- gauge groupoid of P (Forger recently)
- direct connection (new)
- jet groupoids

.

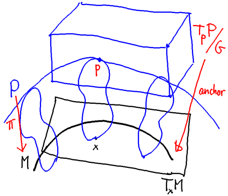
1) Infinitesimal gauge transformations with values in Lie algebroids

- **Lie algebroid**: vector bundle $\boxed{A \to M}$ with **Lie bracket** $[\ ,\ ]_A$ and
  anchor map $\boxed{a : A \to TM}$ s.t. $[X, fY]_A = f[X,Y]_A + a(X)(f)Y$

Ex. Lie algebra bundle: $A = M \times \mathfrak{g} \xrightarrow{0} TM$

> **Generalized gauge theories and sigma models:** Lie algebroids used for infinitesimal actions or as target of gauge transformations giving new degrees of freedom
> e.g. Algebroid Gauge theory, Curved Yang-Mills Gauge theory [Kotov, Strobl et al.] etc

- Atiyah Lie algebroid of a principal G-bundle $\pi : P \to M$:
  $\boxed{A(P) = TP/G \to M}$ with fibres $A_x(P) \cong T_{p_x}P$,
  anchor $A(P) \to TM$ induced by $d\pi : TP \to TM$
  via quotient map $\natural : TP \to TP/G$
  and Lie bracket of G-invariant vector fields on P.

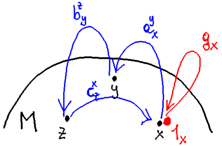


Ex. Trivial G-bundle: $A(M \times G) = TM \oplus (M \times \mathfrak{g}) \xrightarrow{id+0} TM$

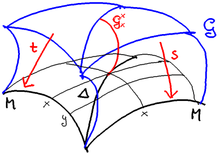
Ex. Frame bundle of a vector bundle $E \to M$ of rank r: $F(E) = \bigcup_{x \in M} \mathrm{Iso}(\mathbb{R}^r, E_x) \to M$

$\boxed{A(F(E)) = \mathrm{Der}(E) \to TM}$ bundle of derivative endomorphisms
s.t. $\Gamma(\mathrm{Der}(E)) =$ derivations of $\Gamma(E)$.

2) Gauge transformations by Lie groupoid actions



- **Lie groupoid** $\boxed{\mathcal{G} \rightrightarrows M}$: bi-fibred manifold $\mathcal{G} = \bigcup_{(y,x)\in M\times M} \mathcal{G}_y^x$
- contains **arrows** $a_{yx} \in \mathcal{G}_y^x$ with $\begin{cases} \textbf{source} & s(a_{yx}) = x \\ \textbf{target} & t(a_{yx}) = y \end{cases}$
- some arrows can be **composed**: $b_{zy}a_{yx} \in \mathcal{G}_z^x$ (composition is associative),
- there are **units** $u(x) = 1_x \in \mathcal{G}_x^x$ and $M \equiv u(M) \subset \mathcal{G}$,
- each arrow $a_{yx} \in \mathcal{G}_y^x$ has an **inverse** $a_{yx}^{-1} \in \mathcal{G}_x^y$.



- Induced **anchor** map $(t,s) : \mathcal{G} \to M \times M$.
- Each $\mathcal{G}_x^x$ is a Lie group, called the **vertex group** or **isotropy**.
  But vertex groups may not be isomorphic one to each other!

Ex. Trivial Lie groupoid with vertex group G: $\boxed{M \times G \times M \rightrightarrows M}$

- **Infinitesimal structure of Lie groupoid = Lie algebroid:** $\boxed{A(\mathcal{G}) = \bigcup_{x\in M} T_{1_x}\mathcal{G}^x \to TM}$
- **Lie groupoids naturally act on vector bundles** with action $\boxed{\mathcal{G}_y^x \times E_x \to E_y}$

Gauge groupoids: single vertex group!

- **Gauge groupoid of principal G-bundle $P \to M$:** $\boxed{\mathcal{G}(P) = P \times_G P \rightrightarrows M}$
  contains equivalence classes $[p, q]$ under $(p, q) \sim (p\,g, q\,g)$ for $g \in G$.
- $\boxed{A(\mathcal{G}(P)) = A(P) \to TM}$

Ex. Pair groupoid: $\boxed{\mathrm{Pair}(M) = M \times M \rightrightarrows M}$ for $P = M \times \{1\} \to M$
$A(\mathrm{Pair}(M)) = TM \xrightarrow{id} TM$

Ex. Frame groupoid of $E \to M$: $\boxed{\mathrm{Iso}(E) = \bigcup_{x,y} \mathrm{Iso}(E_x, E_y)}$ for $P = F(E) \to M$
$A(\mathrm{Iso}(E)) = \mathrm{Der}(E) \to TM$

If structure gp $GL_r(\mathbb{R})$ reduces to G and $P \subset F(E)$ then $\boxed{\mathcal{G}(P) \hookrightarrow \mathrm{Iso}(E)}$

- $\boxed{\mathcal{G}(P) \textbf{ acts on } P}$ $\quad \mathcal{G}(P)_y^x \times P_x \to P_y, \quad [p_y, q_x] \cdot r_x = p_y g$ if $g \in G$ s.t. $r = qg$ (principal action)

  $\boxed{\mathcal{G}(P) \textbf{ acts on } E = P \times_G V}$ $\quad \mathcal{G}(P)_y^x \times E_x \to E_y, \quad [p_y, q_x] \cdot [r_x, \mathrm{v}] = [p_y, g\mathrm{v}]$

- **Gauge groupoids contain gauge transformations:**
  $\boxed{\widehat{G} = \mathrm{Aut}_M(P) \subset \mathcal{G}(P) = \mathrm{Aut}(P)}$ given by $\boxed{\Phi \mapsto [\Phi(p), p]}$ for any $p \in P$.

Generalized gauge theories: gauge transformations replaced by Lie groupoids action. M. Forger et al. proved Noether's theorem (arXiv:1508.04632), Minimal Coupling and Utiyama's theorem (arXiv:1806.01329).

3) Principal connections, gauge fields and covariant derivative

- **Principal connection on P:** five equivalent presentations

1) G-equivariant **horizontal subbundle** $\boxed{HP \subset TP \to P}$ s.t. $TP = HP \oplus VP$, where the **vertical bundle** VP (spaces tangent to the fibres) is canonical.

2) G-equivariant **connection 1-form** $\boxed{\omega : TP \to P \times \mathfrak{g}}$ s.t. $\omega|_{VP}$ is an isomorphism onto $\mathfrak{g}$.

3) **Infinitesimal connection** $\boxed{\delta : TM \to A(P)}$ section of the anchor, then $HP = \natural^{-1}\delta(TM)$.

4) **Parallel transport** $\boxed{\tau_\gamma(y,x) : P_x \xrightarrow{\simeq} P_y}$ horizontal lift of a curve γ of M from x to y.

5) (Local) **gauge fields** $\boxed{\{A : T_U M \to U \times \mathfrak{g}\}}$ pull back of ω along local sections of P.

- **Covariant derivative on sections of E:** bundle map $\boxed{D^A : TM \to \mathrm{Der}(E)}$ equivalent to $C^\infty(M)$-derivation $D^A_X : \Gamma(E) \to \Gamma(E)$ given locally by

$$\boxed{D^A_X(\psi)_{|U} = \sum_{\mu,i,j} \left(X^\mu \partial_\mu \psi^i + X^\mu A^i_{\mu j} \psi^j \right) e_i}$$

if $X = X^\mu \partial_\mu$ in coordinates x^μ on $U \subset M$
$\psi = \psi^i e_i$ on a local basis (e_i) of E_U
and $A^i_j = A^i_{\mu j} dx^\mu$ are the components of the gauge field A in terms of generators of $\mathfrak{g}$.

Direct connections on Lie groupoids

- **Direct connection** on $\mathcal{G} \rightrightarrows M$: local right inverse of the anchor which preserves units, i.e. $\boxed{\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}}$ defined on an open n. $\mathcal{U}_\Delta$ of the diagonal $\Delta \subset \mathrm{Pair}(M)$ s.t.

$$\boxed{\Gamma(y,x) \in \mathcal{G}_y^x} \text{ for all } (y,x) \in \mathcal{U}_\Delta \quad \text{and} \quad \boxed{\Gamma(x,x) = 1_x \in \mathcal{G}_x^x} \text{ for all } x \in M.$$

  [Teleman 2004 in the linear case, Kock 2007 similar, ABFP general]

- A Lie groupoid with a direct connection is a gauge groupoid.

- If $\mathcal{G} \times_M E \to E$ is a linear action, then a direct connection Γ on $\mathcal{G}$ induces a transport on fibres $E_x \to E_y$ which is not necessarily a parallel displacement!

Ex. By the Inverse Function Theorem, a smooth parallelism on E [Dahlqvist-Diehl-Driver 2019] is a direct connection on $\mathrm{Iso}(E)$.

- **Groupoid invariants:**

- Γ **natural** if $\boxed{\Gamma(x,y)\,\Gamma(y,x) = 1_x}$ for all $x \in M$ and nearby y.

- **Curvature of Γ at x**: $\boxed{K_x^\Gamma(z,y) = \Gamma(z,x)^{-1}\Gamma(z,y)\,\Gamma(y,x) \in \mathcal{G}_x^x}$ for nearby y, z.

- Γ is **flat** if $K_x^\Gamma(\_,\_) = 1_x$ for any x, i.e. Γ **is a groupoid morphism**.

Relationship to usual connections

Assume affine connection ∇^M and local geodesics on M.

- Parallel displacement τ on P along small geodesic $x(t)$ from x to y defines a (natural) direct connection Γ on $\mathcal{G}(P)$ by
$$\Gamma(y,x) = \left[\tau(y,x)(p), p\right] \quad \text{for any choice of } p \in P_x$$

  Same for $\mathrm{Iso}(E)$ and a parallel transport on $E \to M$ [Teleman 2004].

- Direct connection Γ on $\mathcal{G}(P)$ induces an infinitesimal connection δ on $A(P)$ by
$$\delta(\dot{x}(0)) = D\Gamma_{|M}(\dot{x}(0)) = \frac{d}{dt}\Gamma(x(t),x)\big|_{t=0}.$$
  If $\mathcal{G}(P)$ acts on E, Γ induces a linear connection ∇ on E by
$$\nabla_{\dot{x}(0)}\psi = \frac{d}{dt}\Gamma(x(t),x)^{-1}\psi(x(t))\big|_{t=0}.$$

  Derivative of Γ-curvature K^Γ give usual curvature tensors R^δ.

- There are many more direct connections on $\mathcal{G}(P)$ then parallel displacements on P!

Ex. $\nabla^E_{\partial_x}\psi(x) = \psi'(x) + f(x)\psi(x)$ then $\boxed{\tau(y,x) = e^{F(y)-F(x)}}$ with $F = -\int f dx$

Instead, the following direct connections **are not parallel transports**:

$\boxed{\Gamma(y,x) = e^{y-x+(y-x)^2}}$ non natural $\left(\Gamma(x,y)\Gamma(y,x) = e^{2(y-x)^2} \neq 1_x\right)$,

$\boxed{\Gamma(y,x) = e^{y-x+(y-x)^3}}$ natural, non-flat.

4) Jet groupoids

Consider **(local) sections** $\psi : M \to E$ with **partial derivatives** $\partial^\alpha \psi(x)$ in local coordinates.

- **Jets** are coordinate-free equiv. classes of $\partial^\alpha \psi(x)$'s w.r.t. "contact of given order at x".
  $\Rightarrow$ They form a tower of affine bundles $J^n E \to J^{n-1} E$, ultimatly "projective" over M.
- **Taylor polynomials** are the coordinate-dependent "**difference**" steps from $J^{n-1}E$ to $J^n E$
  $\Rightarrow$ They form a graded vector bundle $T_{\leqslant n} E = \bigoplus_{k=0}^n S^k(T^*M) \otimes VE$ over M.
- They contain **the same informations** but only jets build up a functor and are prolonged.
  E.g. need $J^1 E = TE \to E \to M$ to fix a full configuration space $\mathcal{E} = \Gamma(M, J^1 E)$
  but $TE \cong T^*M \otimes VE$ only w.r.t. connections on M and E.

- [Kolář-Michor-Slovak 1993] The **jet prolongation** of P is not just $J^n P$ (not principal!) but
  $$\boxed{W^n P = F^n M \times_M J^n P} \qquad \boxed{F^n M = \mathrm{inv} J^n_0(\mathbb{R}^d, M)}.$$
  with structure group
  $$\boxed{W^n_d G = GL^n_d \ltimes T^n_d G} \qquad \boxed{\begin{array}{l} GL^n_d = \mathrm{inv} J^n_0(\mathbb{R}^d, \mathbb{R}^d) \\ T^n_d G = J^n_0(\mathbb{R}^d, G) \end{array}}$$
- **Jet groupoid** $\boxed{J^n\mathcal{G} \rightrightarrows M}$: jet set of **local bisections** $\sigma : U \to \mathcal{G}$ s.t. $\sigma(x) \in \mathcal{G}^x_{\varphi_\sigma(x)}$
  where $\boxed{x \mapsto \varphi_\sigma(x) \text{ is a diffeomorphism on } U}$.
- [Kolář 2008] $\boxed{J^n\mathcal{G}(P)) \cong \mathcal{G}(W^n P)}$ but $\boxed{J^n\mathrm{Iso}(E)) \subsetneq \mathrm{Iso}(J^n E)}$

Jet prolongations of direct connections

- An **affine connection** ∇^M **on** M can be given as

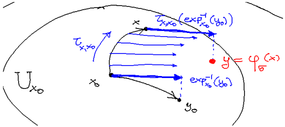
- infinitesimal connection $\delta_M : TM \to \mathrm{Der}(TM) = A(\mathrm{Iso}(TM))$
- parallel transport $\tau^M(y,x) : T_x M \to T_y M$ along unique small geodesic.

Apply jets: $\boxed{\mathrm{Der}(TM) = J^1 TM}$ and $\boxed{\mathrm{Iso}(TM) = J^1 \mathrm{Pair}(M)}$.

For (y_0, x_0) nearby points joint by a unique geodesic $\gamma(t)$, we have

$$\boxed{\tau^M(y_0,x_0) = j^1_{x_0}\varphi^{y_0}_{x_0} = d(\varphi^{y_0}_{x_0})}$$

$$\boxed{\varphi^{y_0}_{x_0}(x) = \exp_x\left(\tau^M(x,x_0)(\exp^{-1}_{x_0}(y_0))\right)}$$



- [Mikulski 2007, Kolář 2009] If ∇^M is torsion-free, δ_M can be prolonged $\delta^{(n)}_M : TM \to J^n TM$.
- [ABFP 2020] 1) The map $\boxed{x \mapsto \sigma^{y_0}_{x_0}(x) = \left(\varphi^{y_0}_{x_0}(x), x\right)}$ is a local bisection of $\mathrm{Pair}(M)$.

2) Its jet $\boxed{\Delta^{(n)}_M(y_0,x_0) = j^n_{x_0}\sigma^{y_0}_{x_0}}$ gives a direct connection $\Delta^{(n)}_M : \mathrm{Pair}(M) \ast\!\!\to J^n\mathrm{Pair}(M)$ which integrates $\delta^{(n)}_M$.

3) Any direct connection $\Gamma : \mathrm{Pair}(M) \ast\!\!\to \mathcal{G}$ can be prolonged to $\Gamma^{(n)} : \mathrm{Pair}(M) \ast\!\!\to J^n\mathcal{G}$ by setting

$$\boxed{\Gamma^{(n)}(y_0,x_0) = j^n_{x_0}\left(\Gamma \circ \sigma^{y_0}_{x_0}\right) = J^n\Gamma \circ \Delta^{(n)}(y_0,x_0)}.$$

Direct connections on jet groupoids which are not jet prolongations

- [ABFP 2021] **Geometric polynomial structure** on $E \to M$: $\boxed{\Big([[0,n]], J^n E, J^n\mathcal{G}(P)\Big)}$

  with $P \subset F(E)$ and $J^n\mathcal{G}(P) \subset J^n\mathrm{Iso}(E) \subset \mathrm{Iso}(J^n E)$ acting on $J^n E$.

  Model from any given $\Gamma : \mathrm{Pair}(M) \ast\!\!\!\twoheadrightarrow \mathcal{G}(P)$: $\boxed{(\Pi^n, \widehat{\Gamma}^n)}$ with

  $\boxed{\Pi^n_{x_0} : J^n_{x_0} E \to \mathcal{D}'(U_{x_0}, E)}$ $\quad U_{x_0}$ normal open n. of x_0, $\quad \gamma(t)$ geodesic $x_0 \rightsquigarrow x$

$$\boxed{\left(\Pi^n_{x_0} j^n_{x_0} f\right)(x) = \Gamma(x, x_0) \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k}{dt^k} \Gamma(\gamma(t), x_0)^{-1} f(\gamma(t))\Big|_{t=0}}$$

  $\boxed{\widehat{\Gamma}^n : \mathrm{Pair}(M) \ast\!\!\!\twoheadrightarrow J^n\mathcal{G}(P)}$ given by $\boxed{\widehat{\Gamma}^n(y_0, x_0)\, j^n_{x_0} f = j^n_{y_0}\Big(x \mapsto \left(\Pi^n_{x_0} j^n_{x_0} f\right)(x)\Big)}$

- $\widehat{\Gamma}^n$ is **not a jet prolongation**, the action **is not prounipotent** and Hairer's equality $\Pi^n_{y_0} \widehat{\Gamma}^n(y_0, x_0) = \Pi^n_{x_0}$ **does not hold** if ∇^M and Γ have curvature.

Ex. $d = 1$, $n = 1$, call $g(x) = \left(\Pi^1_{x_0} j^1_{x_0} f\right)(x)$ and $h = y_0 - x_0$, then

$$\begin{pmatrix} g(y_0) \\ g'(y_0) \end{pmatrix} = \begin{pmatrix} \Gamma(y_0, x_0)\big(1 + a(x_0)h\big) & \Gamma(y_0, x_0)h \\ b(y_0)\big(1 + a(x_0)h\big) + \Gamma(y_0, x_0)a(x_0) & b(y_0)h + \Gamma(y_0, x_0) \end{pmatrix} \begin{pmatrix} f(x_0) \\ f'(x_0) \end{pmatrix}$$

hence $g(y_0)$ **is not** the value $f(x_0)$ transported above y_0!

Conclusion:

- There is a surjective functor

Gauge groupoids with direct connections	$\longrightarrow$	Principal bundles with connections

which admits an inverse, but **it is not an equivalence of categories**.

- Direct connections allow generalized local symmetries (gauge fields).
- Direct connections allow displacement along fibres which is not the solution of a 1st order PDE.

Next:

- Find equations solved by direct connections which are not parallel displacements.
- Adapt to α-Hölder sections of bundles i.e. define distributional direct connections and compare to usual propagators.
- Study the whole geometry of **groupoids with direct connections** and compare with **usual gauge theory**.
- Integrate Algebroid/Curved YM theories to groupoid symmetries (with Simon-Raphael Fischer).

Thank you for the attention!