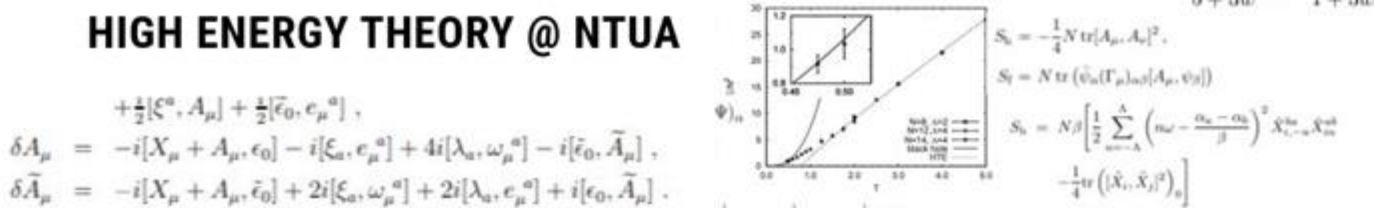
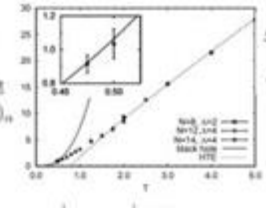


SCHOOL OF APPLIED MATHEMATICAL AND PHYSICAL SCIENCES
NATIONAL TECHNICAL UNIVERSITY OF ATHENS

Recent progress in the numerical studies of the Lorentzian IKKT model

Konstantinos N. Anagnostopoulos
Physics Department
School of Applied Mathematical and Physical Sciences
National Technical University of Athens

HIGH ENERGY THEORY @ NTUA



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In collaboration with:

Takehiro Azuma (Setsunan U), Kohta Hatakeyama (Hirosaki U), Mitsuaki Hirasawa (INFN Milano-Bicocca), Yuta Ito (NIT, Tokuyama C), Jun Nishimura (KEK & SOKENDAI), Stratos Papadoudis (NTUA), Asato Tsuchiya (Shizuoka U)

8-hour time zone meeting

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Type IIB matrix model (IKKT: Ishibashi, Kawai, Kitazawa, Tsuchiya, 1996)

- Proposed as a nonperturbative definition of superstring theory in the large-N limit

$$S = -\mathrm{N}\mathrm{Tr}\left(\frac{1}{4}[\mathrm{A}_\mu, \mathrm{A}_\nu][\mathrm{A}^\mu, \mathrm{A}^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[\mathrm{A}_\mu, \psi]\right)$$

N $\times$ N Hermitian matrices
A_μ: 10D Lorentzian vector ($\mu = 0, 1, \ldots, 9$)
ψ: 10D Majorana-Weyl spinor

- Spacetime *emerges* from the Bosonic matrix degrees of freedom: the eigenvalues of the A_μ can be thought of as spacetime coordinates

Type IIB matrix model (IKKT: Ishibashi, Kawai, Kitazawa, Tsuchiya, 1996)

- This identification allows us to study questions like:
  - The dynamical emergence of time
  - The dynamical emergence of space
  - The dynamical compactification of extra dimensions
  - The time evolution of the large dimensions of the universe

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- Nontrivial dynamical properties of the model:
  - Time must be homogeneous and of infinite extent in the large-N limit
  - The number of large dimensions of space must be 3, and expand in a way consistent with cosmological models at late times
  - Time and space must be real, and the signature of the spacetime geometry Lorentzian, at least at late times

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These are the questions that we will try to address in this talk!

Nonperturbative effects, will resort to numerical computations...

Lattice String Theory

- Using "Lattice string theory", those questions have been studied since 1999 (Hotta-Nishimura-Tsuchiya, Ambjorn-Anagnostopoulos-Bietenholz-Hotta-Nishimura)
- Studied related Euclidean 4D, 6D and 10D simplified matrix models, attempting to understand the mechanism of the dynamical compactification of the extra dimensions.
- Extra dimensions are compactified via the SSB of the SO(D) rotational invariance of the model
- The dynamics of the fermions are crucial for the realization of the scenario
- Numerical computations are hard because of the complex action problem

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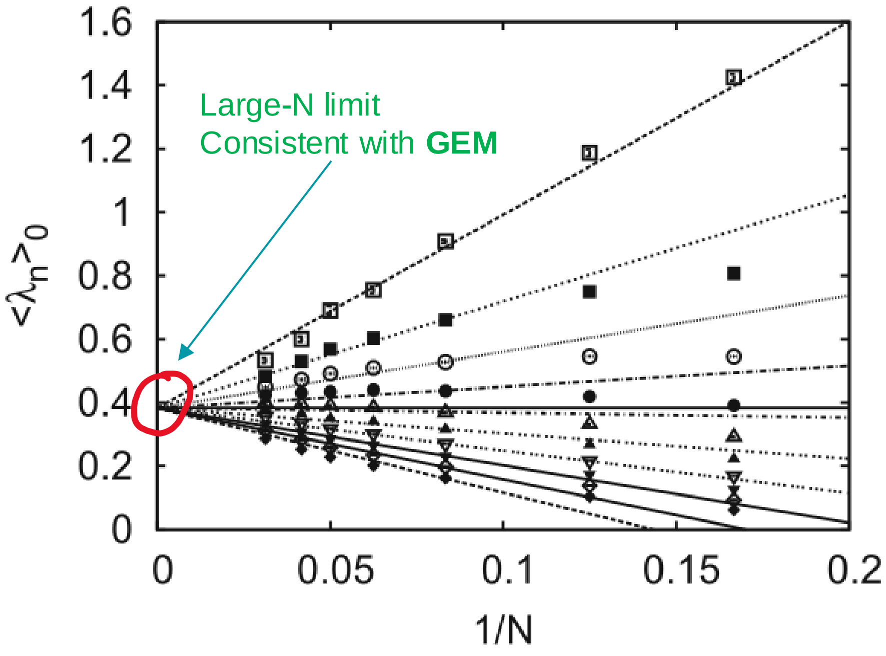
- Numerical computations are hard because of the complex action problem

- Using the GEM, Nishimura-Okubo-Sugino (2011) suggested that SO(10) breaks down to SO(3)
- Using the Complex Langevin method (CLM), we were able to produce results consistent with the GEM (KNA-Azuma-Ito-Nishimura-Okubo-Papadoudis, 2020)

$$T_{\mu\nu} = \frac{1}{N}\mathrm{tr}(A_\mu A_\nu)$$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{10}$$



10D Euclidean IKKT, phase-quenched model
(KNA-Azuma-Nishimura, 2015)

$$\mathrm{Z_f(A) = Pf\mathcal{M}(A) = |Pf\mathcal{M}(A)|\, e^{i\Gamma}}$$

ignored

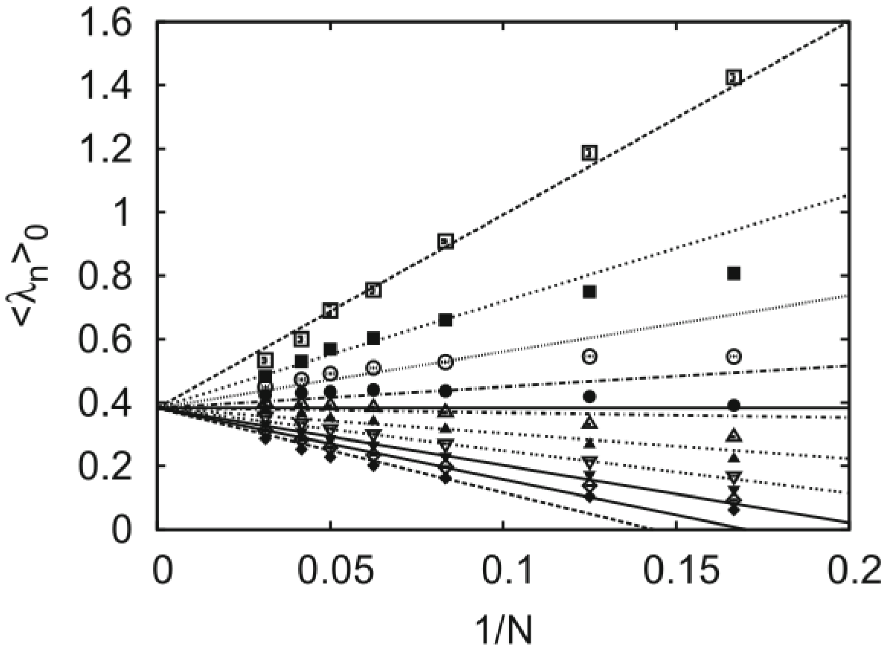
The fluctuations of the phase of the Pfaffian is crucial for the occurrence of the SSB

- No SSB when fermions are quenched ("Bosonic model")
- No SSB in the 4D model where $\mathrm{Pf}\mathcal{M}(\mathrm{A}) > 0$
- No SSB when Γ is quenched

$$T_{\mu\nu} = \frac{1}{N}\mathrm{tr}(A_\mu A_\nu)$$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{10}$$

$$\rho_\mu(m_\mathrm{f}, \varepsilon, N) = \frac{\langle \lambda_\mu \rangle_{m_\mathrm{f},\varepsilon,N}}{\sum_{\nu=1}^{10} \langle \lambda_\nu \rangle_{m_\mathrm{f},\varepsilon,N}}$$



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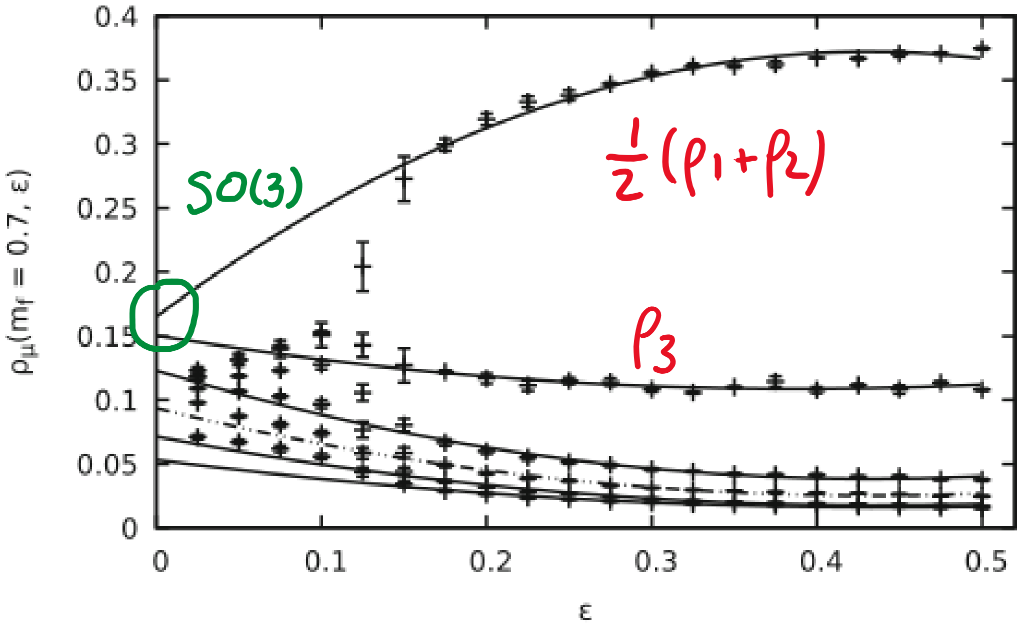


10D Euclidean IKKT
(KNA-Azuma-Ito-Nishimura-Okubo-Papadoudis, 2020)

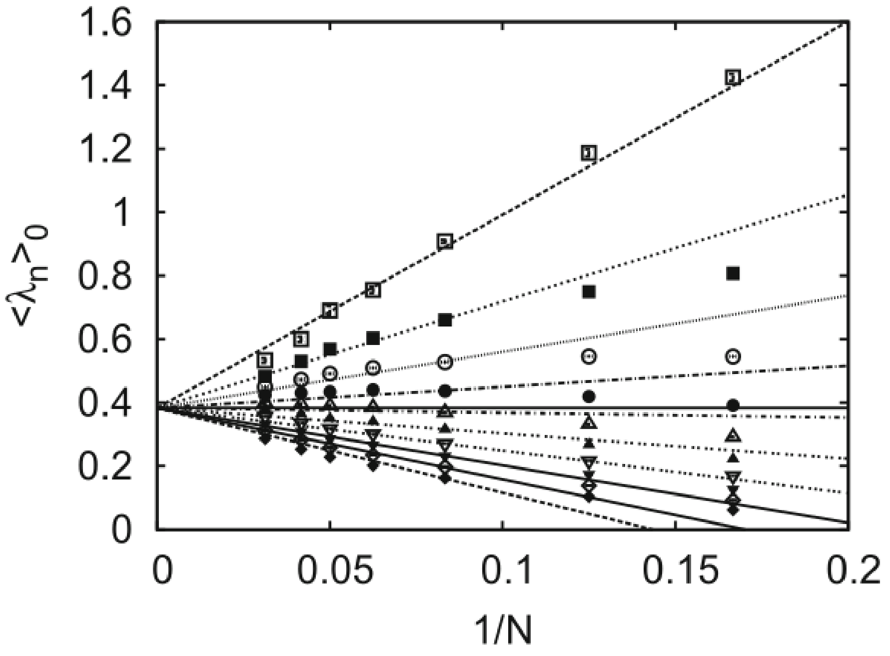
$$\lambda_\mu = \frac{1}{N}\mathrm{tr}(A_\mu)^2$$

$$\Delta S_\mathrm{b} = \frac{N}{2}\varepsilon \sum_{\mu=1}^{10} m_\mu \mathrm{tr}(A_\mu)^2 \qquad \Delta S_\mathrm{f} = -i m_\mathrm{f} \frac{N}{2}\mathrm{tr}(\psi_\alpha (\mathcal{C}\Gamma_8 \Gamma_9^\dagger \Gamma_{10})_{\alpha\beta} \psi_\beta)$$

$$T_{\mu\nu} = \frac{1}{N}\mathrm{tr}(A_\mu A_\nu)$$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{10}$$

$$\rho_\mu(m_\mathrm{f}, \varepsilon, N) = \frac{\langle \lambda_\mu \rangle_{m_\mathrm{f},\varepsilon,N}}{\sum_{\nu=1}^{10} \langle \lambda_\nu \rangle_{m_\mathrm{f},\varepsilon,N}}$$



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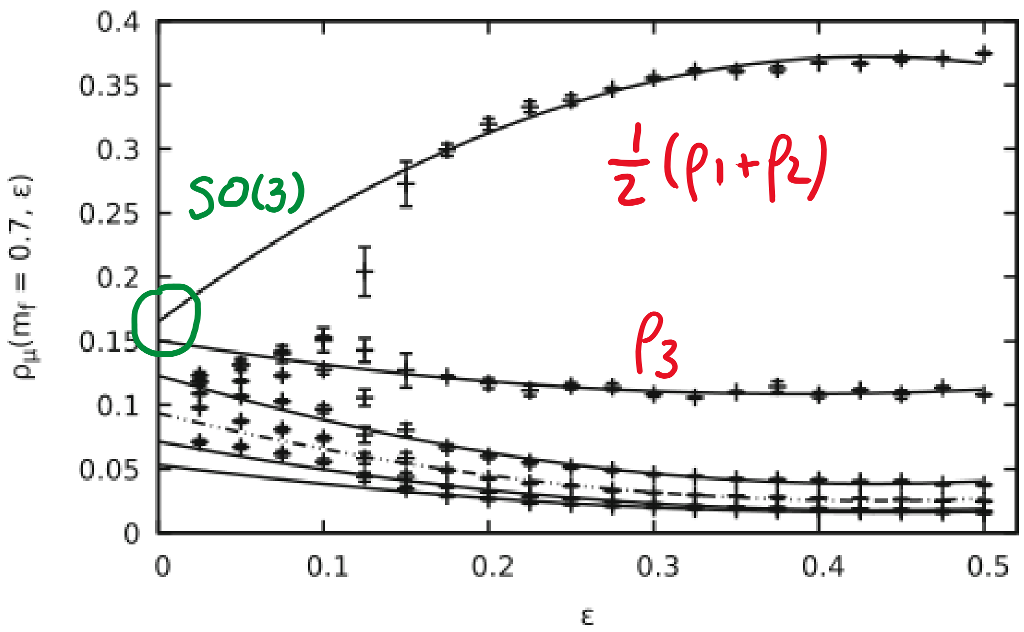


10D Euclidean IKKT
(KNA-Azuma-Ito-Nishimura-Okubo-Papadoudis, 2020)

3-dimensional space emerges in the Euclidean model

The Lorentzian model

$$F_{\mu\nu} = -i[A_\mu, A_\nu]$$

$$\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = -2\mathrm{tr}(F_{0I})^2 + \mathrm{tr}(F_{IJ})^2$$

Not bounded from below

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \dots, 1)$$

$$\mathrm{S} = -\mathrm{NTr}\left(\frac{1}{4}[\mathrm{A}_\mu, \mathrm{A}_\nu][\mathrm{A}^\mu, \mathrm{A}^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[\mathrm{A}_\mu, \psi]\right)$$

$$Z_{\mathrm{L}} = \int dA d\psi e^{i(S_{\mathrm{b}}+S_{\mathrm{f}})} = \int dA e^{iS_{\mathrm{b}}} \mathrm{Pf}\, \mathcal{M}$$

Real polynomial in A

- Equivalent to the Euclidean model (Jun Nishimura, previous talk)

Adding a Lorentz invariant mass term

- We add the following term to the action (Steinacker 2018, Hatakeyama-Matsumoto-Nishimura-Tsuchiya-Yosprakob 2020):

$$S_\gamma = -\frac{1}{2}N\gamma\mathrm{tr}(A^\mu A_\mu) = \frac{1}{2}N\gamma\{\mathrm{tr}(A_0)^2 - \mathrm{tr}(A_I)^2\}$$

- We define the model in the limits: $\mathrm{N} \to \infty$, then $\gamma \to 0^+$ $(\gamma > 0)$

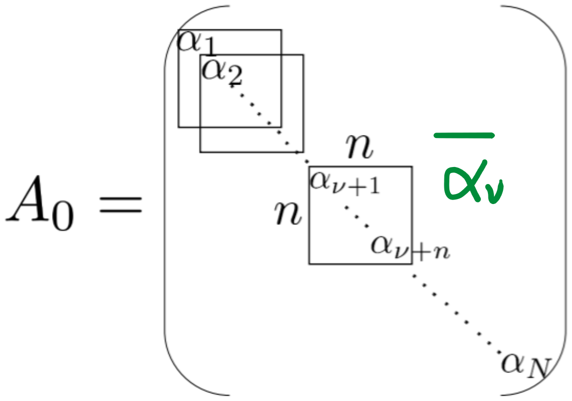
$$\mathrm{Z} = \int \mathrm{dA}\, \mathrm{e}^{\mathrm{i}(\mathrm{S_b}+\mathrm{S}_\gamma)}\ \mathrm{Pf}\mathcal{M}(\mathrm{A})$$

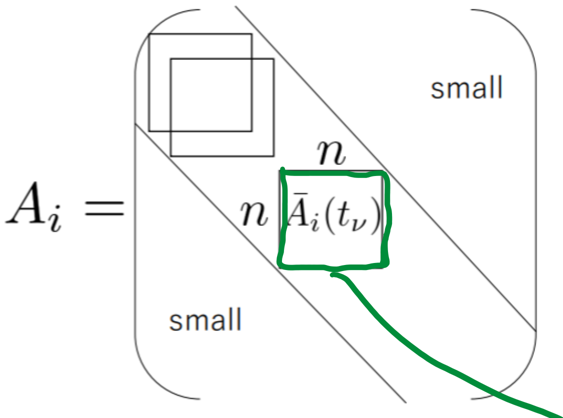
Simulations

- First, using the gauge symmetry of the model, we diagonalize the matrix

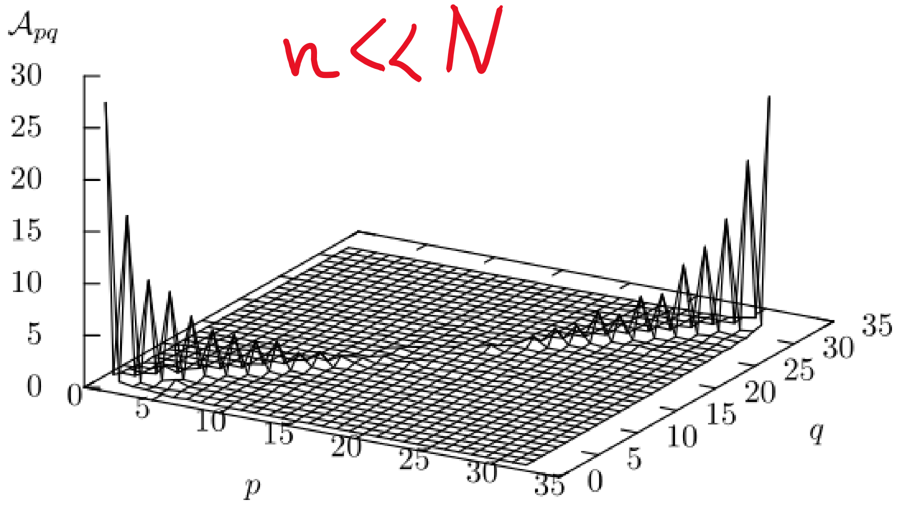
$$A_0 = \mathrm{diag}(\alpha_1, \ldots, \alpha_N) \quad \text{with } \alpha_1 = 0 \quad \alpha_1 \le \alpha_2 \le \cdots \le \alpha_N$$

- We use the nontrivial property of typical spatial configurations to have a narrow band diagonal structure to define space and time





space at time t_ν



$$t_0 = 0, \quad t_\rho = \sum_{k=1}^{\rho} |\bar{\alpha}_{k+1} - \bar{\alpha}_k| \qquad \bar{\alpha}_{k+1} = \frac{1}{n}\sum_{i=1}^{n} \alpha_{k+i}$$

$$\mathcal{A}_{pq} = \frac{1}{9}\sum_{I=1}^{9} |(A_I)_{pq}|^2$$

Simulations

- First, using the gauge symmetry of the model, we diagonalize the matrix

$$A_0 = \mathrm{diag}(\alpha_1, \ldots, \alpha_N) \quad \text{with } \alpha_1 = 0 \qquad \alpha_1 \le \alpha_2 \le \cdots \le \alpha_N$$

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$$t_0 = 0, \quad t_\rho = \sum_{k=1}^{\rho} |\bar{\alpha}_{k+1} - \bar{\alpha}_k| \qquad \bar{\alpha}_{k+1} = \frac{1}{n}\sum_{i=1}^{n} \alpha_{k+i} \qquad \rho = 1, 2, \ldots, N-n \text{ and } k = 0, 1, \ldots, N-n.$$

- We define auxiliary variables that automatically impose the $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_N$ condition (Nishimura-Tsuchiya, 2019)

$$\alpha_i = \sum_{k=1}^{i-1} e^{\tau_k} \quad (i = 2, 3, \ldots, N) \qquad -\infty < \tau_k < \infty \qquad 1 \le k < N$$

Simulations

- Then the model can be rewritten in the form:

$$S^b_{\rm eff} = -iN\left(\frac{1}{2}{\rm Tr}[A_0,A_I]^2 - \frac{1}{4}{\rm Tr}[A_I,A_J]^2\right) - \frac{i}{2}N\gamma\left({\rm Tr}A_0^2 - {\rm Tr}A_I^2\right) - \log\prod_{1\le k<l\le N}(\alpha_k-\alpha_l)^2 - \sum_{k=1}^{N-1}\tau_k$$

$$Z = \int d\tau\, dA_I\, e^{-S^b_{\rm eff}(A)}\quad {\rm Pf}\mathcal{M}(A)$$

gauge fixing
Fadeev-Popov det

Jacobian of
$\alpha_k \to \tau_k$

Simulations

- Then the model can be rewritten in the form:

complex action

$$S^{b}_{eff} = -iN\left(\frac{1}{2}\mathrm{Tr}[A_0, A_I]^2 - \frac{1}{4}\mathrm{Tr}[A_I, A_J]^2\right) - \frac{i}{2}N\gamma\left(\mathrm{Tr}A_0^2 - \mathrm{Tr}A_I^2\right) - \log\prod_{1\le k<l\le N}(\alpha_k - \alpha_l)^2 - \sum_{k=1}^{N-1}\tau_k$$

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Simulations

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$$S_{\rm eff}^{\rm b} = -{\rm i}N\left(\frac{1}{2}{\rm Tr}[A_0, A_I]^2 - \frac{1}{4}{\rm Tr}[A_I, A_J]^2\right) - \frac{\rm i}{2}N\gamma\left({\rm Tr}A_0^2 - {\rm Tr}A_I^2\right) - \log \prod_{1\le k<l\le N}(\alpha_k - \alpha_l)^2 - \sum_{k=1}^{N-1}\tau_k$$

$$Z = \int d\tau\, dA_I\, e^{-S_{\rm eff}^{\rm b}(A)} \quad {\rm Pf}\mathcal{M}(A)$$

- We avoid the complex action problem by employing the Complex Langevin method (CLM)

$$\frac{d(A_I)_{kl}}{d\sigma} = -\frac{\partial S_{\rm eff}}{\partial (A_I)_{lk}} + (\eta_I)_{kl}(\sigma) \qquad \frac{d\tau_a}{d\sigma} = -\frac{\partial S_{\rm eff}}{\partial \tau_a} + \eta_a(\sigma)$$

$$S_{\rm eff} = S_{\rm eff}^{\rm b} - \log {\rm Pf}\mathcal{M}(A)$$

Simulations

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Hermitian/Real Gaussian noise

Simulations

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drift terms

Simulations

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Complex action

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$$\tau_a \in \mathbb{C}$$

$$A_I \in SL(N, \mathbb{C})$$

complexified

Simulations

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$$\tau_a \in \mathbb{C}$$

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$$\langle \mathcal{O}(A_\mu)\rangle = \frac{1}{T}\int_{\sigma_0}^{\sigma_0+T} \mathcal{O}[A_\mu(\sigma)]\, d\sigma$$

Holomorphic in A

Simulations

- We avoid the wrong convergence problem by monitoring whether the distribution of the drift norm has a subexponential asymptotic behavior (Nagata-Nishimura-Shimasaki, 2016)

$$u = \sqrt{\frac{1}{10N^3} \sum_{\mu=1}^{10} \sum_{k,l=1}^{N} \left| \frac{\partial S_{\text{eff}}}{\partial (A_\mu)_{lk}} \right|^2}$$

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$$\tau_{\mathrm{a}} \in \mathbb{C}$$

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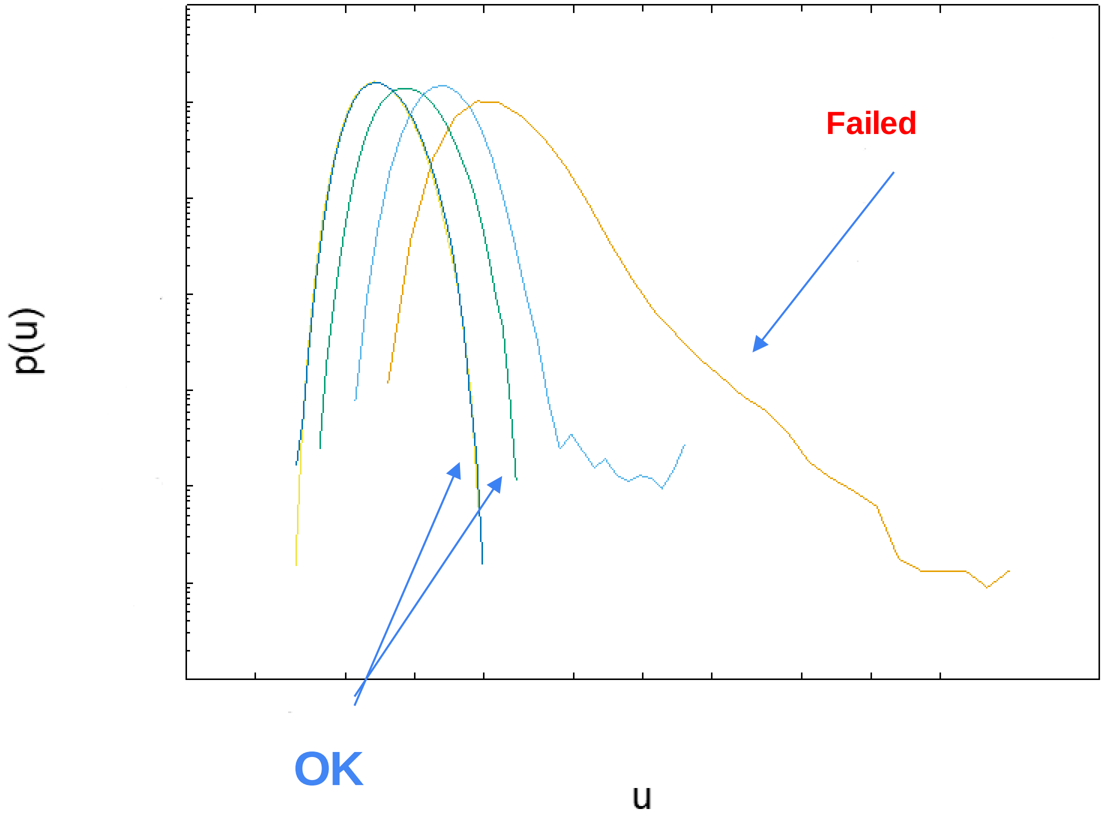

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- When the eigenvalues of $\mathcal{M}$ accumulate near the origin we have the singular drift problem

$$-\frac{\partial}{\partial (A_\mu)_{lk}} \log \operatorname{Pf} \mathcal{M} = -\frac{1}{2} \operatorname{Tr}\left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{lk}} \mathcal{M}^{-1} \right)$$

Simulations

- We avoid the singular drift problem by adding a deformation to the action, which shifts the eigenvalues away from the origin (Ito-Nishimura 2016)

$$S_{m_f} = iNm_f \mathrm{Tr}\left[\bar{\Psi}_\alpha(\Gamma_7\Gamma_8^\dagger\Gamma_9)_{\alpha\beta}\Psi_\beta\right]$$

↳ must send $m_f \to 0$ in the end

- When the eigenvalues of $\mathcal{M}$ accumulate near the origin we have the singular drift problem

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$$S_{m_\mathrm{f}} = iNm_\mathrm{f}\mathrm{Tr}\left[\bar{\Psi}_\alpha(\Gamma_7\Gamma_8^\dagger\Gamma_9)_{\alpha\beta}\Psi_\beta\right]$$

↳ $m_f \to \infty$ gives the Bosonic model

Simulations

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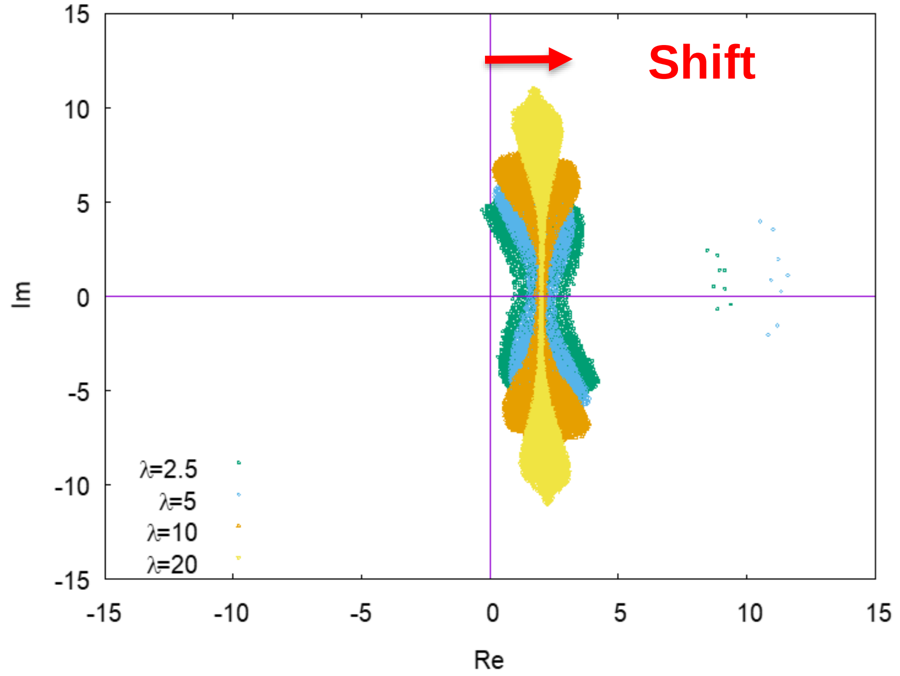

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- The Langevin equation is discretized to 1st order (Euler), or 2nd order Runge-Kutta

$$(A_I)_{kl}(\sigma+\Delta\sigma) = (A_I)_{kl}(\sigma) + \sqrt{\Delta\sigma}(\tilde{\eta}_I)_{kl}(\sigma) \; - \Delta\sigma\left\{\beta_1\left[\frac{\partial S_{\mathrm{eff}}}{\partial(A_I)_{lk}}(A(\sigma))\right] + \beta_2\left[\frac{\partial S_{\mathrm{eff}}}{\partial(A_I)_{lk}}(A'(\sigma))\right]\right\}$$

$$(A'_I)_{kl}(\sigma) = (A_I)_{kl}(\sigma) + \sqrt{\Delta\sigma}(\tilde{\eta}_I)_{kl}(\sigma) \; - \Delta\sigma\left[\frac{\partial S_{\mathrm{eff}}}{\partial(A_I)_{lk}}(A(\sigma))\right] \qquad \beta_1 = \beta_2 = \tfrac{1}{2}\left(1 + \tfrac{N}{6}\Delta\sigma\right)$$

$$\tilde{\eta}_I(\sigma) \propto \exp\left(-\tfrac{1}{4}\textstyle\sum_{\sigma}\mathrm{tr}\tilde{\eta}_I^2(\sigma)\right)$$

Simulations

- The fermionic drift is computed using a noise estimator, using Gaussian noise $\langle \chi_k^* \chi_l \rangle = \delta_{kl}$

$$\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{lk}} \mathcal{M}^{-1} \right) = \left\langle \chi^* \frac{\partial \mathcal{M}}{\partial (A_\mu)_{lk}} \mathcal{M}^{-1} \chi \right\rangle_\chi$$

- The Langevin equation is discretized to 1st order (Euler), or 2nd order Runge-Kutta

$$(A_I)_{kl}(\sigma + \Delta\sigma) = (A_I)_{kl}(\sigma) + \sqrt{\Delta\sigma}(\tilde{\eta}_I)_{kl}(\sigma) \; - \Delta\sigma \left\{ \beta_1 \left[\frac{\partial S_{\text{eff}}}{\partial (A_I)_{lk}} (A(\sigma)) \right] + \beta_2 \left[\frac{\partial S_{\text{eff}}}{\partial (A_I)_{lk}} (A'(\sigma)) \right] \right\}$$

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$$\beta_1 = \beta_2 = \tfrac{1}{2}\left(1 + \tfrac{N}{6}\Delta\sigma\right)$$

$$\tilde{\eta}_I(\sigma) \propto \exp\left(-\tfrac{1}{4} \textstyle\sum_\sigma \mathrm{tr}\tilde{\eta}_I^2(\sigma)\right)$$

Simulations

- The fermionic drift is computed using a noise estimator, using Gaussian noise $\langle \chi_k^* \chi_l \rangle = \delta_{kl}$

$$\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{lk}} \mathcal{M}^{-1}\right) = \left\langle \chi^* \frac{\partial \mathcal{M}}{\partial (A_\mu)_{lk}} \mathcal{M}^{-1} \chi \right\rangle_\chi$$

- Most intensive part of the calculation is to compute $\zeta = \mathcal{M}^{-1}\chi$

→ use conjugate gradient method

$$\mathcal{M}^\dagger \mathcal{M} \zeta = \mathcal{M}^\dagger \chi$$

positive definite

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$$\mathcal{M}^\dagger \mathcal{M} \zeta = \mathcal{M}^\dagger \chi$$

- Compute matrix products using

$$\psi_\alpha \to (\mathcal{M}^\dagger \psi)_\alpha = (\Gamma_\mu^\dagger)_{\alpha\beta} [A_\mu^\dagger, \psi_\beta] \qquad \mathrm{O}(N^3) \text{ in CPU time}$$

Simulations

- We also include a stabilization parameter (Attanasio-Jagger, 2019). Typically, η=0.010, 0.005.

$$A_I \mapsto \frac{A_I + \eta A_I^\dagger}{1+\eta} \quad \text{for } I = 1, \ldots, 9$$

$\eta \to 1$

$A_I \to$ Hermitian

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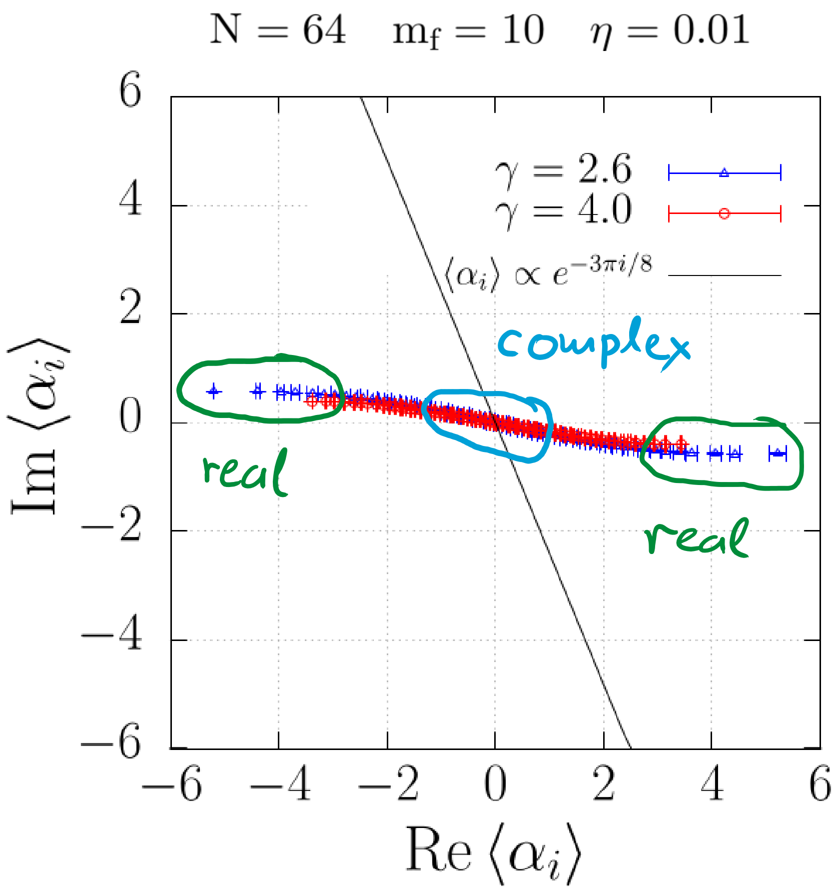
Some results:



- Time is complex near the origin
- Becomes real at large times
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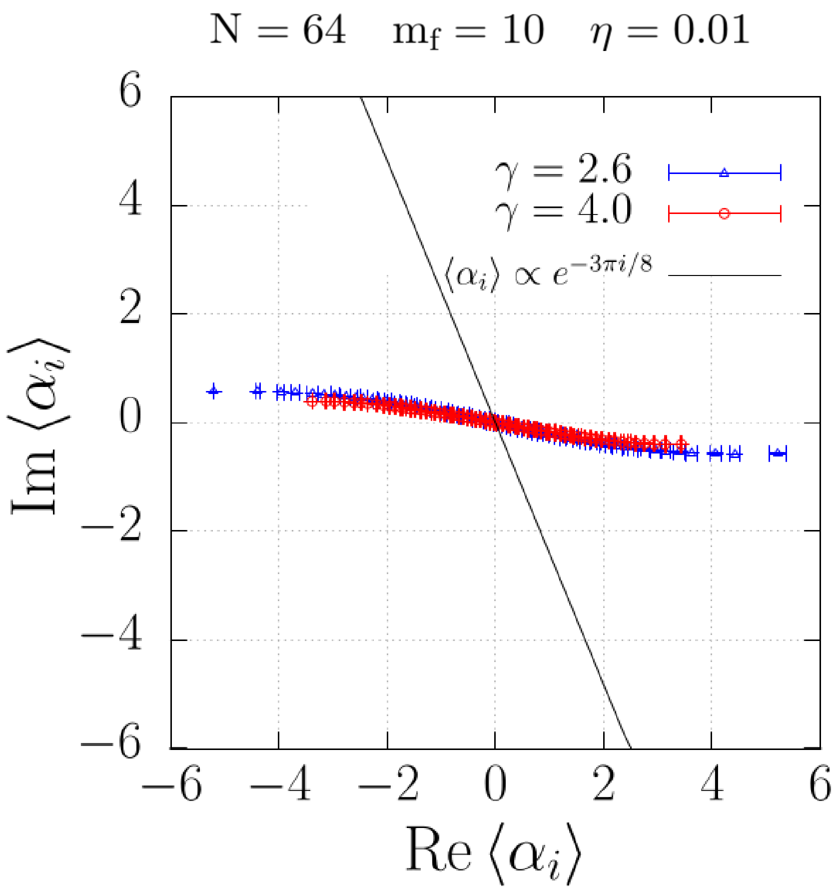

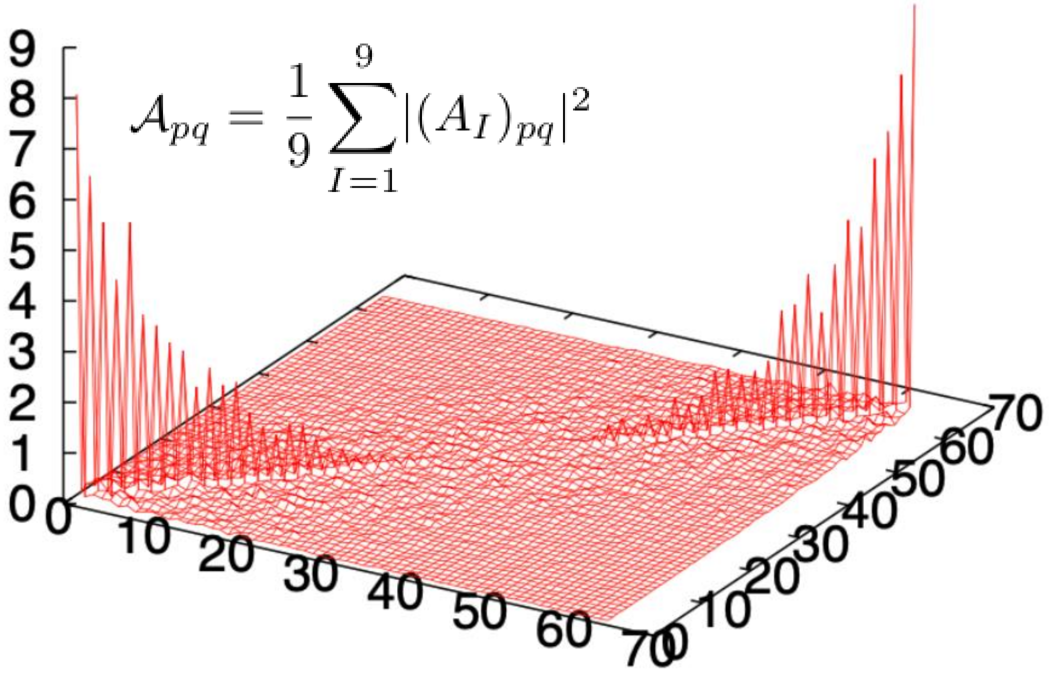

- Time is complex near the origin
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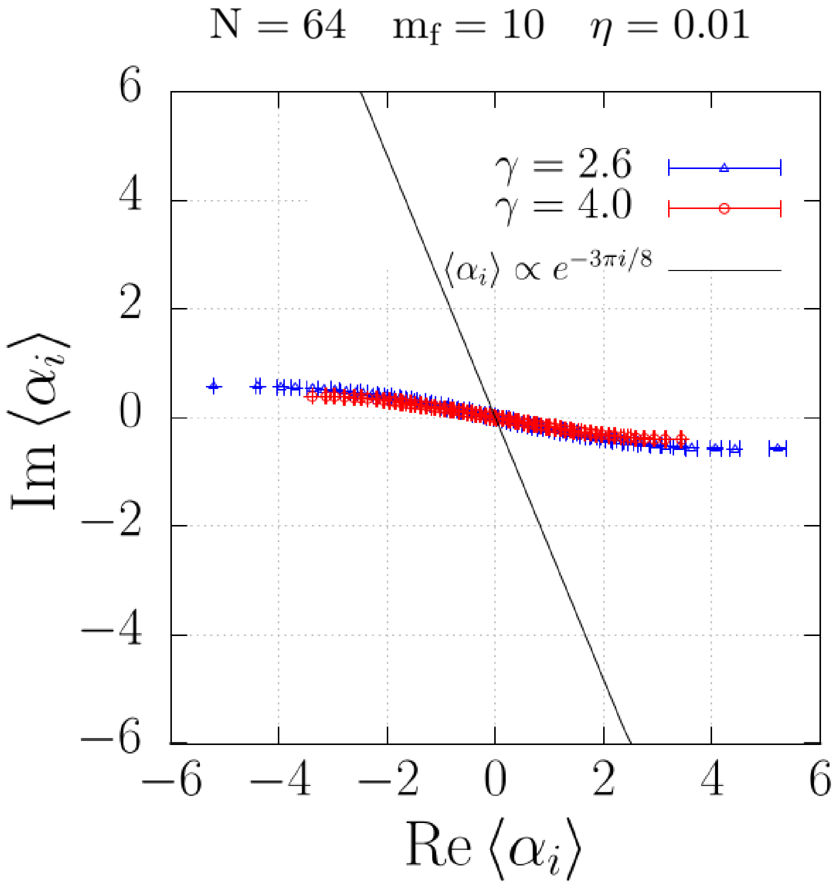
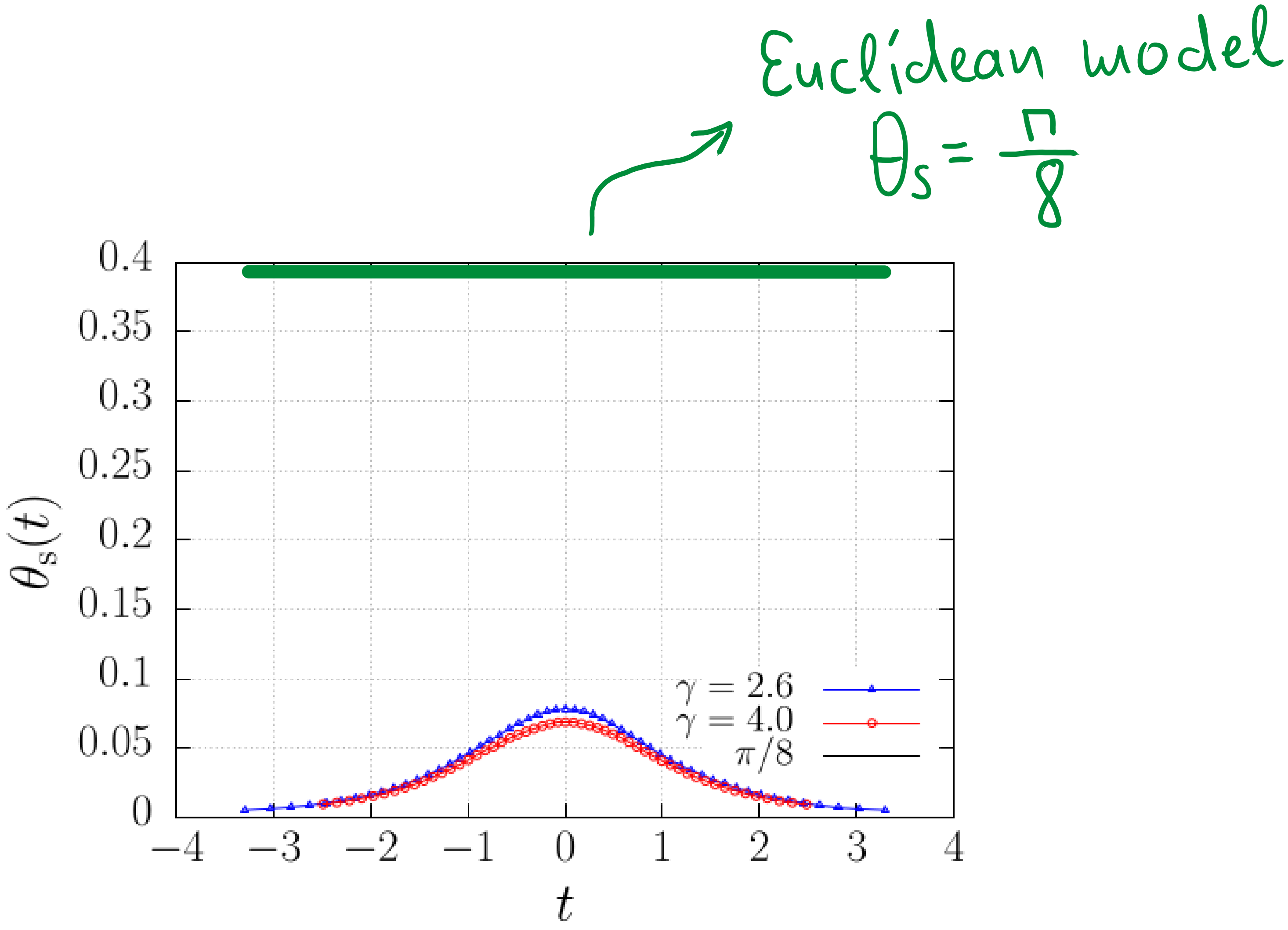
- Narrow band structure

Some results:





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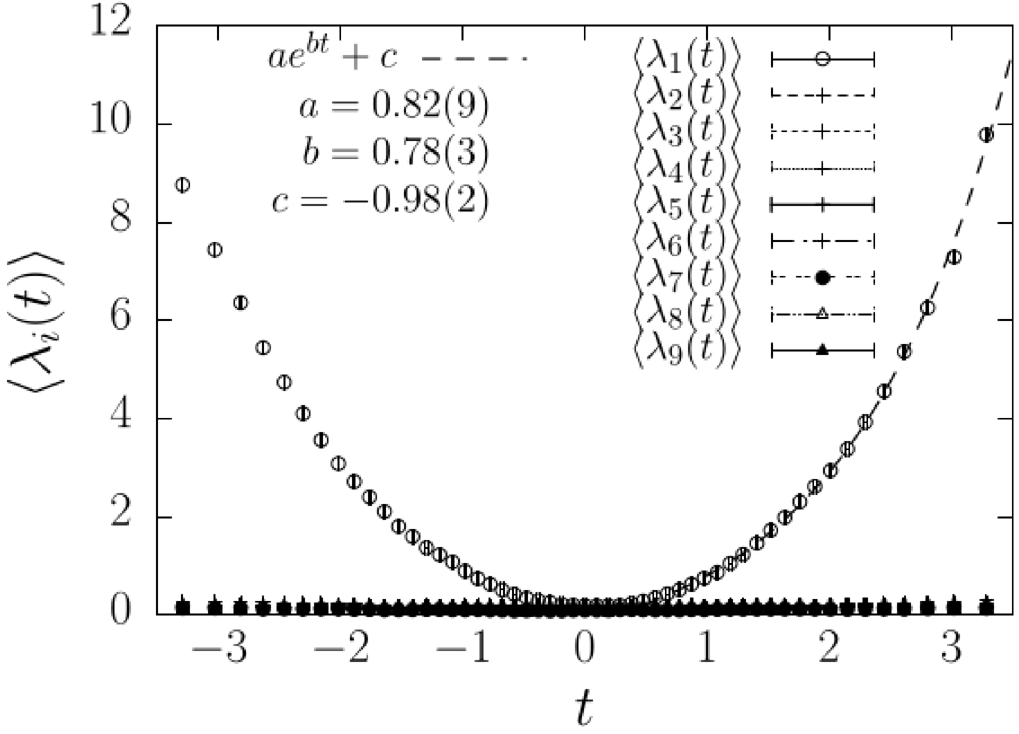
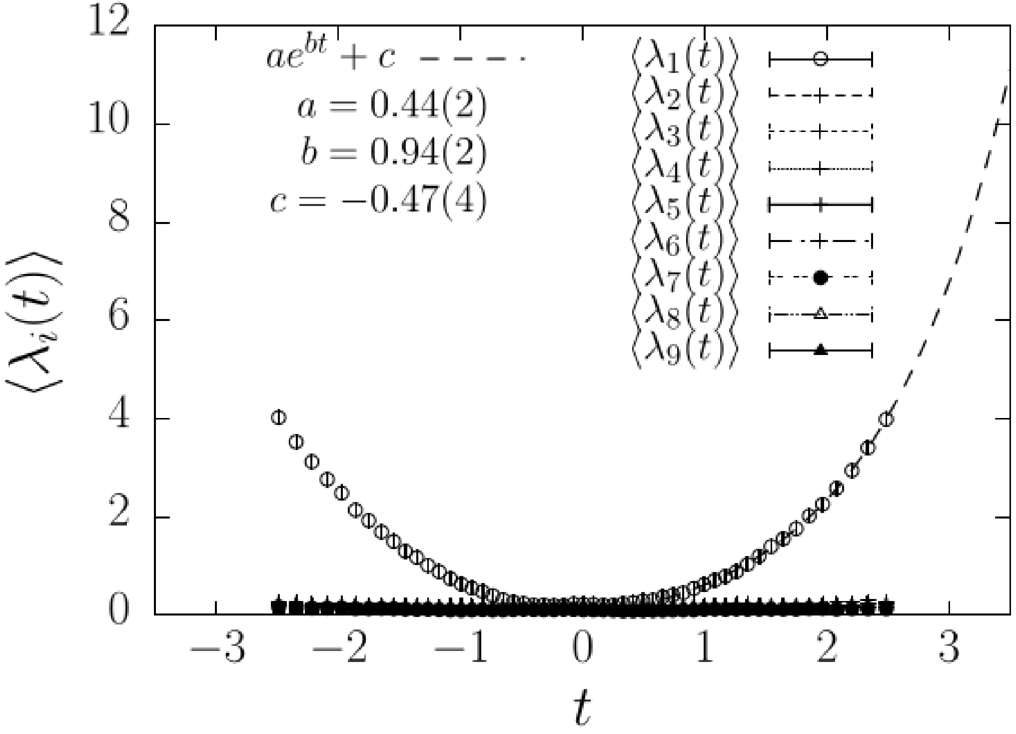
- Reality of space $\mathrm{tr}(\bar{A}_I(t))^2 = e^{2i\theta_S(t)}|\mathrm{tr}(\bar{A}_I(t))^2|$
- $\theta_s(t)$ is small for small t, vanishes at large t

Some results:

$N = 64 \quad m_f = 10 \qquad \gamma = 2.6$



$N = 64 \quad m_f = 10 \qquad \gamma = 4.0$



- The eigenvalues of the moment of inertia tensor
- Lines are exponential fittings
- Extent of time larger at smaller γ

$$T_{IJ}(t) = \frac{1}{n}\mathrm{tr}\{X_I(t)X_J(t)\} \quad X_I(t) = \frac{1}{2}\left(A_I + A_I^\dagger\right)$$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{10}$$

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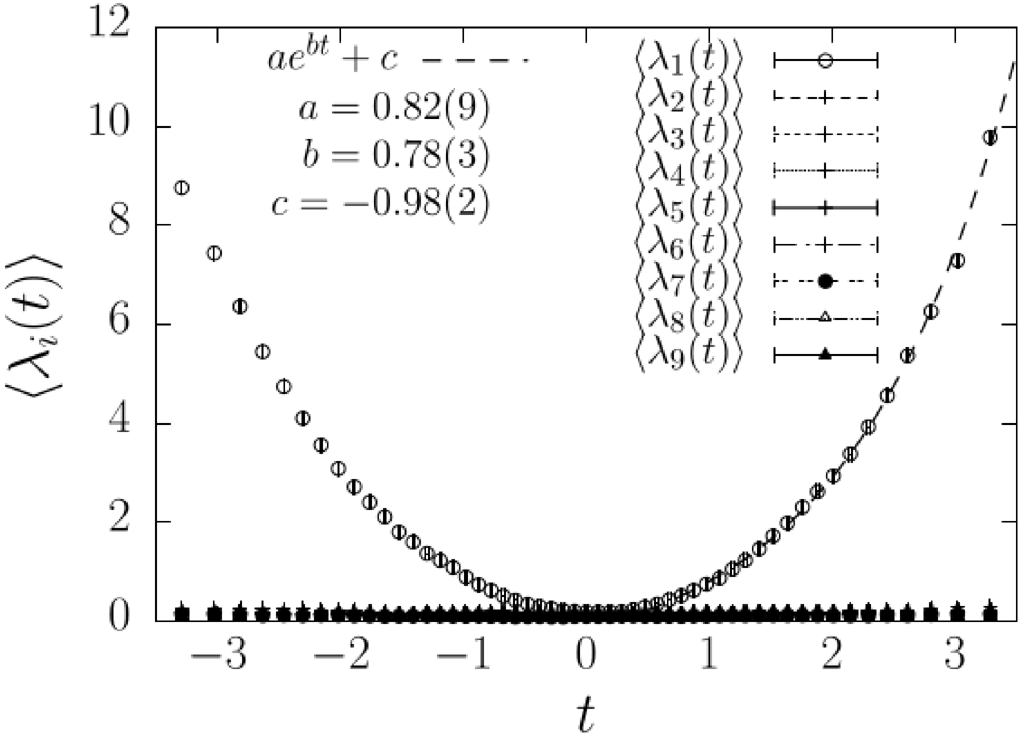
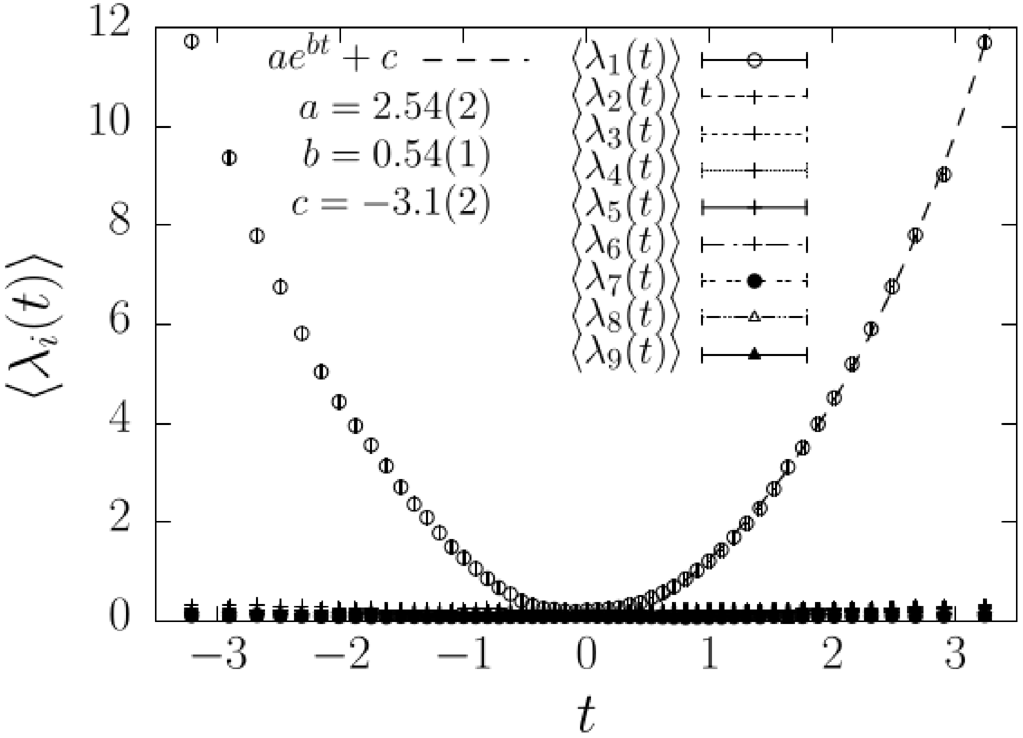


$N = 64 \quad m_f = 5 \qquad \gamma = 2.6$



- This behavior is similar to the Bosonic model, we are still in the "Bosonic phase"

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- Furthermore, if the spatial directions are expanding exponentially with time, then time remains "small". Therefore, we hope to see a 3-dimensional expanding universe, when the effect of the fermions kicks in

Restrict to lower dimensional configurations

- To avoid the problem of not being able to reduce m_f further, and to enhance the effect of the Pfaffian, we have performed simulations that favor lower dimensional configurations

$$\tilde{d} < D - 1$$

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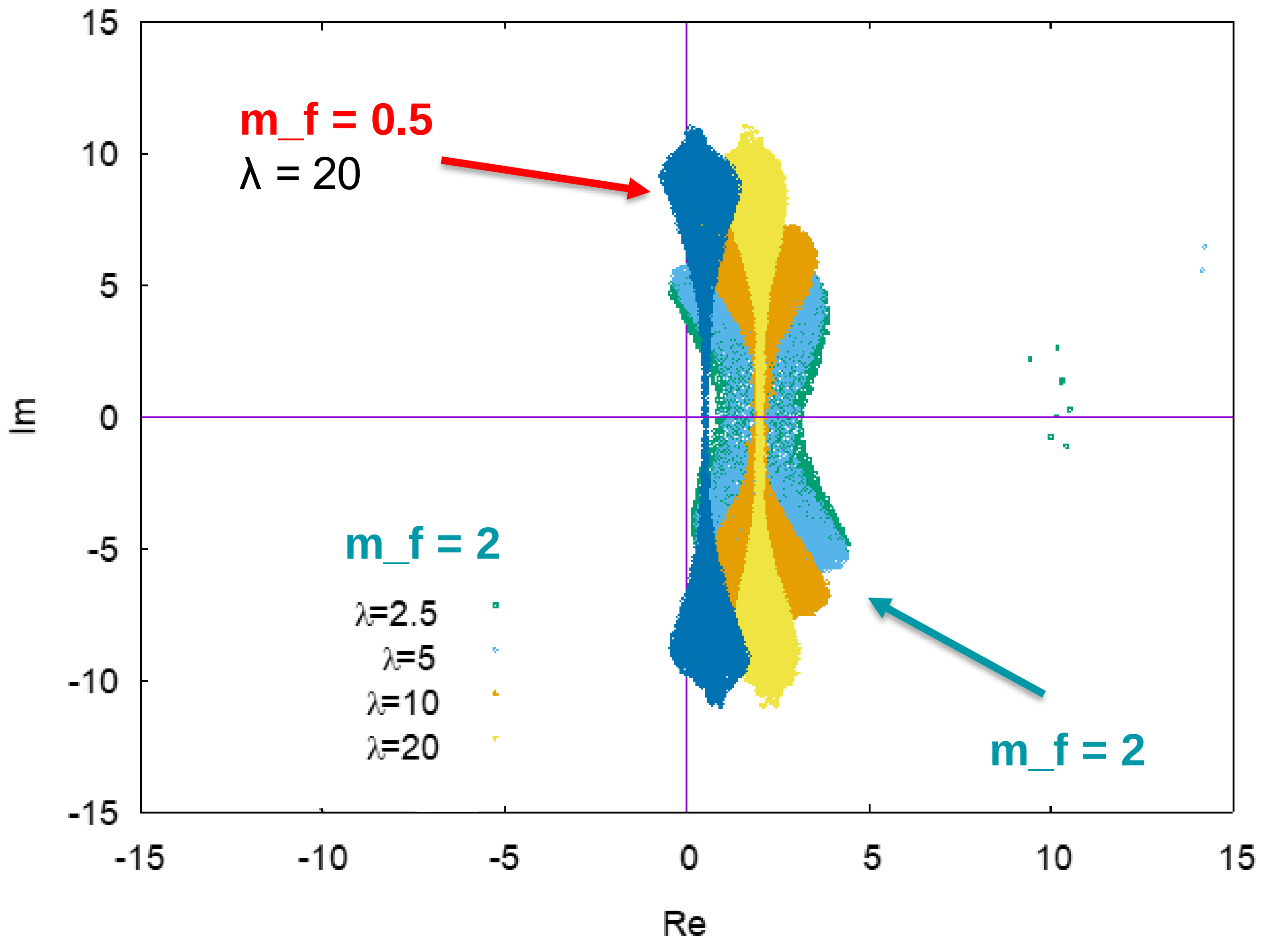
- We hope that, if a low dimensional universe emerges, the bias will be small

Restrict to lower dimensional configurations

- We were able to simulate

$\tilde{d} = 5 \qquad m_f = 0.5 \qquad N = 22 \qquad \gamma = 4$



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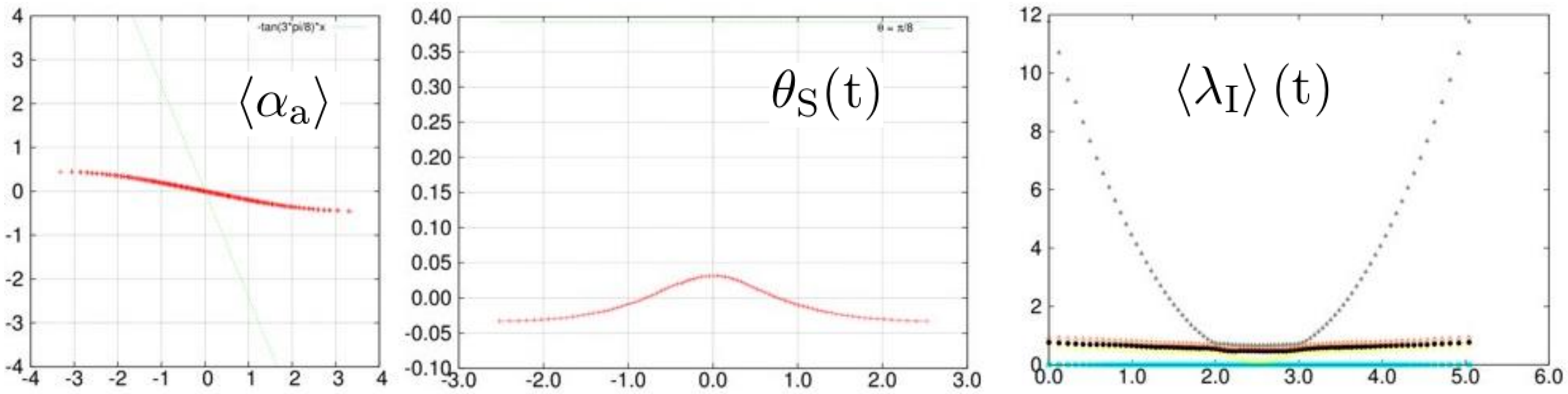
- We can't see the SUSY effect yet, but we can see a transition from 1d-expanding to $\tilde{\mathrm{d}}$ -expanding, with real spacetime. The transition occurs as we lower γ

$\tilde{\mathrm{d}} = 4$ $\quad$ $\mathrm{m_f} = 2.0$ $\quad$ $\mathrm{N} = 96$ $\quad$ $\gamma = 2$ $\quad$ $\eta = 0.005$ $\quad$ $\lambda = \infty$



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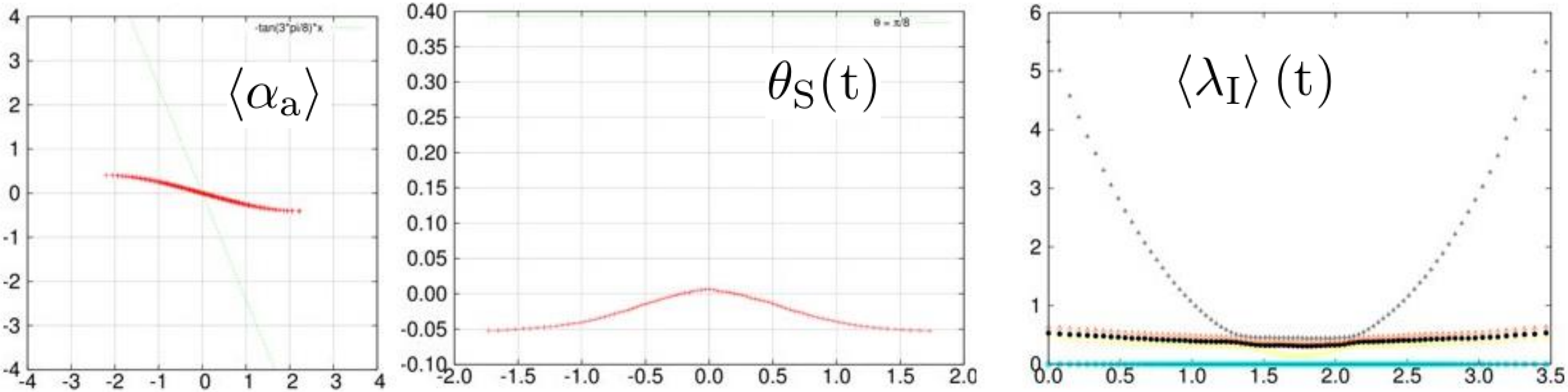

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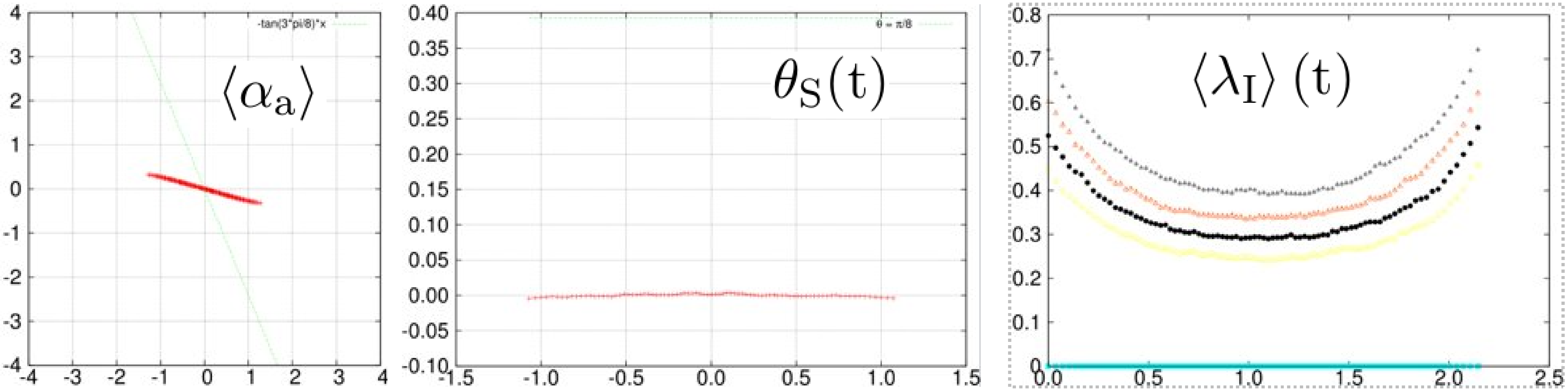

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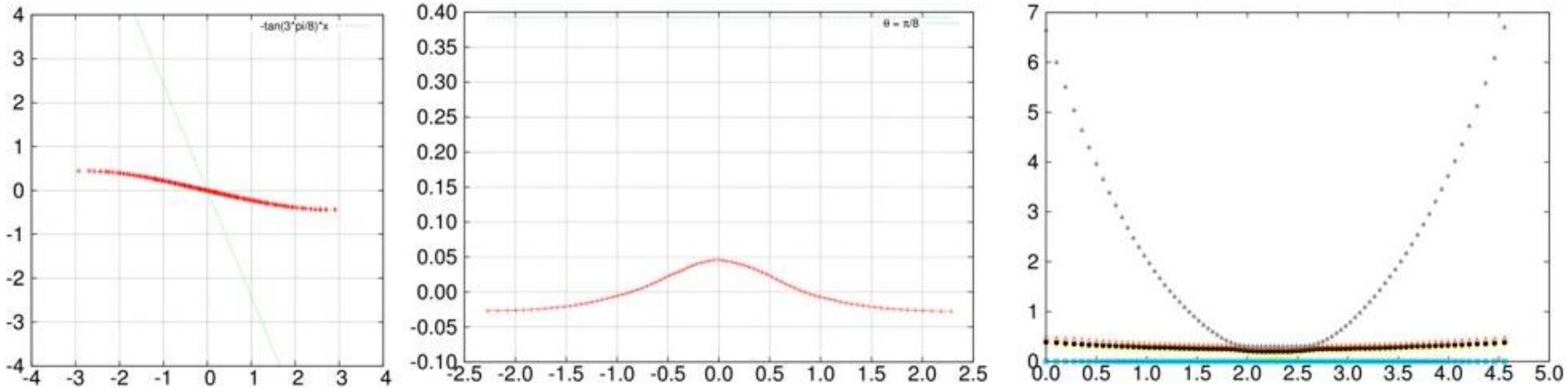

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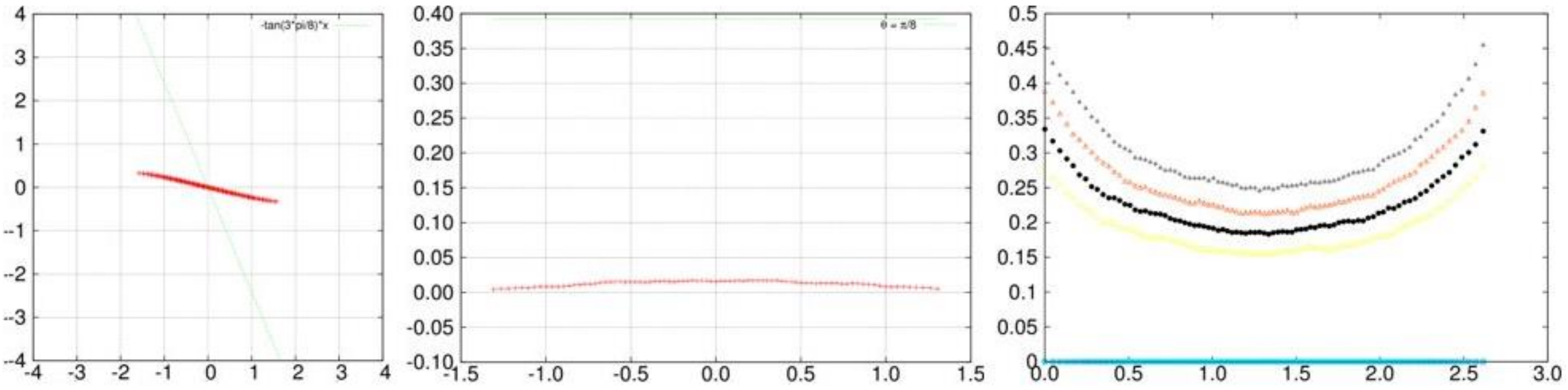

Summary

- The Lorentzian IKKT model can be defined by a contour deformation from the Euclidean model. Then the two models are equivalent, and the emergent spacetime is complex, Euclidean and three dimensional
- We introduced a Lorentz invariant mass term, using a parameter γ>0. We define the model in the limit

  $$\mathrm{N} \to \infty \text{ , then } \quad \gamma \to 0$$

- We simulated the IKKT model, using the CLM to avoid the complex action problem
- To avoid the singular drift problem, we introduced a "fermionic mass term", with the IKKT model obtained when $\mathrm{m_f} \to 0$
- We performed simulations for $\mathrm{m_f} \geq 5$, and we obtained one dimensional, exponentially expanding space
- Time is emerging dynamically: It is homogeneous, complex at small times, becoming real at larger times. Space is also real at late times
- Signature of spacetime is changing from being Euclidean at small times to being Minkowskian at later times
- In order to simulate the model for smaller $\mathrm{m_f}$, and enhance the effect of the Pfaffian, we simulated the model with a $\tilde{\mathrm{d}} < \mathrm{D} - 1$ constrain. We found a transition from the 1-d expanding behavior to $\tilde{\mathrm{d}}$ -expanding behavior
- Since the Pfaffian is zero for 2d configurations, 1-d expansion must be strongly suppressed in the IKKT model. We hope that by further reducing $\mathrm{m_f}$, we can obtain a 3-d expanding universe