Maximally symmetric spacetimes and their lower dimensional theories

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ESI Thematic Programme: Geometry for Higher Spin Gravity
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LEVERHULME TRUST

Based on and with

- Homogeneous spacetimes
  - José Figueroa-O'Farrill, Ross Grassie
  - 1809.01224, 1905.00034
- $2+1$ dimensional gravity
  - Javier Matulich, Jakob Salzer
  - 1903.09165
- $2+1$ dimensional higher spin theories
  - Eric Bergshoeff, Daniel Grumiller, Jan Rosseel, Max Riegler
  - 1612.02277, 1710.11105
- $1+1$ dimensional JT gravity
  - Daniel Grumiller, Jelle Hartong, Jakob Salzer
  - 2011.13870
- Please see these papers for references
- Feel free to ask questions

Plan

- Mainly talk about maximally symmetric spaces
- Algebraic considerations for lower dimensional theories
- Set the stage for more detailed discussion of the theories
- Aug. 26, 2021, 11:00 — 12:00: Daniel Grumiller, *Higher spins and non-AdS holography in lower dimensions*

Outline

Maximally symmetric spaces

- Maximally symmetric spaces
  - Riemannian: Euclidean, Sphere, Hyperbolic
  - Lorentzian: Minkowski, de Sitter, anti-de Sitter
- Properties (intuitive)
  - Maximal amount of symmetry (e.g., Killing vectors)
  - Every point looks the same
  - Symmetry connects each point
- Properties have nice consequences
  - Because every point of the space is "the same" a lot can be learned by just analyzing any specific point
  - More complicated problems can be reduced to linear algebra
    - Similar to Lie group $G \to$ Lie algebra $\mathfrak{g}$
- Why are they important and/or interesting?
  - Backgrounds for physics
  - Vacuum for general relativity (empty universe)
  - Model spaces, c.f., A. Čap, *Cartan geometry*

However...

- What happened to, e.g., galilean space (classical mechanics)?
- Same amount of symmetries, but not necessarily a nondegenerate metric
- Usual definition:

  > the number of linearly independent Killing vector fields. We therefore refer to an n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors as a **maximally symmetric space**. The most familiar examples of maximally symmetric spaces are

- Implicit: Lorentzian

Q: What are they?

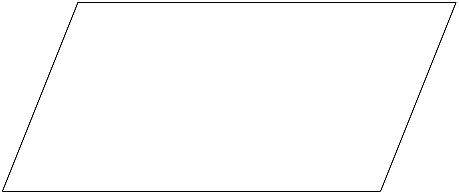
- Q: What are all spaces with the same amount of symmetry, but not necessarily lorentzian/riemannian?
- Useful way to analyze: Homogeneous spaces
  - More focus on symmetries, less focus on metric
- The following discussions are related to the theme of the workshop
  - Homogeneous spaces are the flat models underlying the Cartan geometries
    - In this sense in might be interesting to understand what is "out there"
    - General concept: What is interesting?
  - Shares similarities with higher spin theories: Beyond (pseudo)riemannian geometry

Outline

What is a homogeneous space?

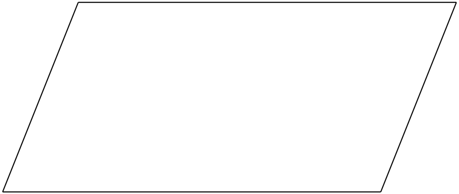
- Two elements
  1. Smooth space (Manifold)
  2. Continuous symmetries acting on the space such that every point is connected continuously to every other point using such a symmetry (transitive Lie group action)
- Show two familiar examples to
  1. set the ground for possibly not so familiar examples
  2. transition to a different but equivalent way to think about homogeneous spaces

Example: (Plane, rotations + translations)



- Space: Two dimensional plane ($\mathbb{R}^2$)
- Symmetries: Rotations and translations ($ISO(2)$)
  - Connect all points ✓
- Space + Symmetries → Homogeneous space
- Often interested in invariants
  - Nondegenerate riemannian metric $ds^2 = dx^2 + dy^2$

Example: (Plane, rotations + translations)



- Equivalent way: Symmetries first
- Symmetries: Rotations and translations ($ISO(2)$)

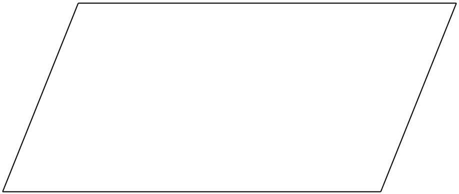
$$(R_1, \vec{a}_1) \cdot (R_2, \vec{a}_2) = (R_1 R_2, R_1 \vec{a}_2 + \vec{a}_1)$$

- Subset of symmetries that close: Rotations

$$\text{Space} = \frac{\text{Rotations + translations}}{\text{“ignore” rotations}} \left(= \frac{ISO(2)}{SO(2)} \right)$$

- “Ignore” it is important that the rotations are a subgroup
- Translations $\Longleftrightarrow$ Points

Example: (Plane, rotations + translations)



- Something interesting has happened
- To specify the homogeneous space specify:
  - Symmetries
  - Subgroup of symmetries
- Only symmetries!
- Homogeneous space:
  - (Manifold, transitive Lie group action) $\rightarrow$ Klein pair (Lie group, closed Lie subgroup)

Minkowski = (Poincaré, Lorentz)

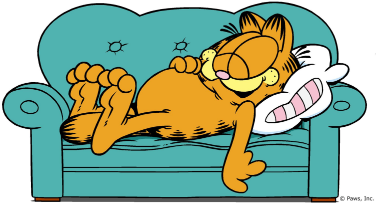
- Try to understand Minkowski space using this concept
- Symmetries: Rotations, boosts, spatial and time translations (Poincaré)
- Subgroup of Symmetries: Rotations, boosts (Lorentz)
- Homogeneous space specified: $(\text{Poincaré}, \text{Lorentz})$
- Minkowski space

$$\frac{\text{Poincaré}}{\text{Lorentz}} = \frac{\text{Rotations, boosts, spatial and time translations}}{\text{Rotations, boosts}}$$

- Sanity check: Spatial and time translations = dimension of spacetime
- Invariant: Nondegenerate lorentzian metric

$$g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Simplify our life



- (Lie group, closed Lie subgroup) $\Longrightarrow$ (Lie algebra, Lie subalgebra)
- We investigate the simply connected spaces
- Much can be said without ever introducing coordinates
- Summary: Homogeneous space characterized by Klein pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$
  - We often implicitly think about the homogeneous spaces, e.g., Poincaré group

Outline

Maximally symmetric spacetimes

- Maximally symmetric spacetimes/spatially isotropic or kinematical spacetimes are a specific class of homogeneous spaces
- How can they be characterized?
- Understand how they are connected and where they are relevant

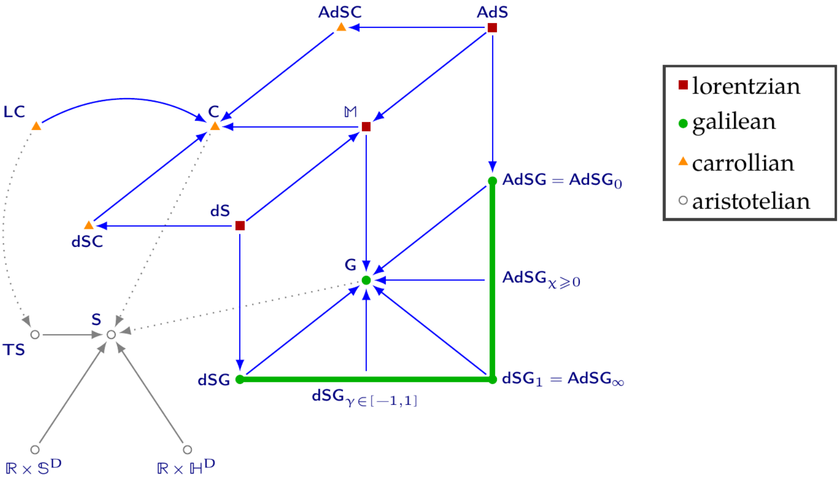
What are spatially isotropic spacetimes?

1. Restrict the Lie algebra to the following generators
   - Rotations $\boldsymbol{J}$ (J_{ab})
   - "Boosts" $\boldsymbol{B}$ (B_a)
   - "Spatial translations" $\boldsymbol{P}$ (P_a)
   - Time translations H

$$[\boldsymbol{J},\boldsymbol{J}]=\boldsymbol{J} \quad [\boldsymbol{J},\boldsymbol{B}]=\boldsymbol{B} \quad [\boldsymbol{J},\boldsymbol{P}]=\boldsymbol{P} \quad [\boldsymbol{J},H]=0$$

2. Classify all possible Lie algebras satisfying this commutation relations
   - This is possible due the rotational invariance
3. Classify all homogeneous spaces for each Lie algebra
   - Search Lie subalgebras $\{\boldsymbol{J},\boldsymbol{B}\}$
   - At this step the "boosts" $\rightarrow$ boosts
4. Search for invariants $\rightarrow$ characterize the spacetime
5. Check that boosts are noncompact

Generic dimensions: $[\boldsymbol{J},\boldsymbol{J}]=\boldsymbol{J}$, $[\boldsymbol{J},\boldsymbol{B}]=\boldsymbol{B}$, $[\boldsymbol{J},\boldsymbol{P}]=\boldsymbol{P}$



Summary

- Maximally symmetric $\rightarrow$ Homogeneous spaces with Klein pair $(\mathfrak{g}, \mathfrak{h})$
  - Well suited to go beyond lorentzian and riemannian cases
  - Generalizes them
- Fall into 4 classes, according to invariants
  - Lorentzian/riemannian: Nondegenerate metric
  - Carrollian: Degenerate metric, vector field
  - Galilean: Degenerate co-metric, one-form
  - Aristotelian: No boosts
- Q2: Can we write general relativity-like models for these spacetimes?
  - Analog to Minkowski space $\rightarrow$ general relativity?

Anecdote: Freeman Dyson – Missed opportunities

- Workshop at ESI with the mission "*advance research in mathematics, physics and mathematical physics through fruitful interaction between scientists from these disciplines*"
- "... *some examples of missed opportunities, occasions on which mathematicians and physicists lost chances of making discoveries by neglecting to talk to each other.*"

Anecdote: Freeman Dyson – Missed opportunities

- Missed opportunity:
  - It was realized that the symmetries underlying of the Maxwell equations is not the Galilean, but the Poincaré group.
  - Is there a group that the Poincaré group is a limit of? *"Following Minkowski's argument, a pure mathematician might easily have conjectured in 1908 that the true invariance group of the universe should be D rather than P."*

Suppose that somebody had been bold enough in 1908 to take this idea seriously. He would have correctly predicted the expansion of the universe twenty years before it was discovered observationally by Hubble. More importantly, he would have been led to postulate the curvature of space-time, and so he would have considerably eased the path which led to general relativity. Luckily, Einstein was able to reach general relativity the hard way, without having his path eased for him by anybody. DeSitter in fact discovered his model of an expanding universe a year after he learned of Einstein's theory.

Outline

From homogenous spaces to theories

- Given a Klein pair $(\mathfrak{g}, \mathfrak{h})$ one might now want to construct curved geometries (Cartan geometries) of it and theories for these curved geometries
- Similarly to Minkowski space $\rightarrow$ general relativity

From homogenous spaces to theories

- We have looked at lower dimensional theories which can be written either as
  - Chern–Simons theories in $2+1$ dimension

$$\mathscr{L}^{\mathrm{CS}}[A] = \langle A \wedge dA + \frac{1}{3}[A, A] \wedge A \rangle$$

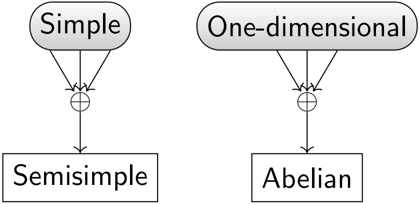
  - (Metric) BF theories (or Poisson sigma models) in $1+1$ dimensions

$$\mathscr{L}^{\mathrm{mBF}}[\mathcal{X}, A] = \langle \mathcal{X}, F \rangle = g_{LK} X^L \Big(dA^K + \frac{1}{2} c_{IJ}{}^K A^I \wedge A^J \Big)$$

- Two ingredients:
  - Gauge algebra $\mathfrak{g}$: $A = e + \omega$ and $\omega \in \mathfrak{h}$
  - Invariant metric $\langle \quad \rangle$
    - Nondegenerate, symmetric, invariant bilinear form on Lie algebra ("Trace")
    - Generically: After limit nonsemisimple $\rightarrow$ degenerate $\rightarrow$ not all components have a kinetic term
    - There exist nonsemisimple Lie algebras with invariant metrics

Outline

Lie algebras with Invariant Metric [Medina, Revoy '85; Figueroa-O'Farrill, Stanciu '95]



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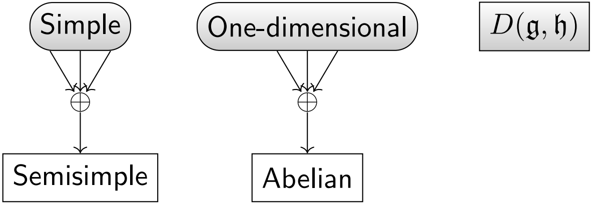

Double Extension $D(\mathfrak{g}, \mathfrak{h})$ [Medina, Revoy '85]

$D(\mathfrak{g}, \mathfrak{h})$ defined on the vector space $\mathfrak{g} \dot{+} \mathfrak{h} \dot{+} \mathfrak{h}^*$ by

$$[\mathtt{G}_i, \mathtt{G}_j] = f_{ij}{}^k \mathtt{G}_k$$

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$$[\mathtt{G}_i, \mathtt{G}_j] = f_{ij}{}^k \mathtt{G}_k$$
$$[\mathtt{H}_\alpha, \mathtt{G}_i] = f_{\alpha i}{}^j \mathtt{G}_j$$
$$[\mathtt{H}_\alpha, \mathtt{H}_\beta] = f_{\alpha\beta}{}^\gamma \mathtt{H}_\gamma$$

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$$
\begin{aligned}
[\mathtt{G}_i,\mathtt{G}_j] &= f_{ij}{}^k\mathtt{G}_k + f_{\alpha i}{}^k\Omega^{\mathfrak{g}}_{kj}\mathtt{H}^\alpha\\
[\mathtt{H}_\alpha,\mathtt{G}_i] &= f_{\alpha i}{}^j\mathtt{G}_j\\
[\mathtt{H}_\alpha,\mathtt{H}_\beta] &= f_{\alpha\beta}{}^\gamma\mathtt{H}_\gamma\\
[\mathtt{H}_\alpha,\mathtt{H}^\beta] &= -f_{\alpha\gamma}{}^\beta\mathtt{H}^\gamma\\
[\mathtt{H}^\alpha,\mathtt{G}_j] &= 0\\
[\mathtt{H}^\alpha,\mathtt{H}^\beta] &= 0
\end{aligned}
$$

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[\mathtt{H}^\alpha,\mathtt{H}^\beta] &= 0
\end{aligned}$$

It has the invariant metric

$$\Omega^{\mathfrak{d}}_{ab} = \begin{array}{c} \\ \mathtt{G}_i \\ \mathtt{H}_\alpha \\ \mathtt{H}^\alpha \end{array}\begin{array}{c} \begin{array}{ccc} \mathtt{G}_j & \mathtt{H}_\beta & \mathtt{H}^\beta \end{array} \\ \begin{pmatrix} \Omega^{\mathfrak{g}}_{ij} & 0 & 0 \\ 0 & h_{\alpha\beta} & \delta_\alpha{}^\beta \\ 0 & \delta^\alpha{}_\beta & 0 \end{pmatrix} \end{array}$$

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[\mathtt{H}^\alpha,\mathtt{G}_j] &= 0 \\
[\mathtt{H}^\alpha,\mathtt{H}^\beta] &= 0
\end{aligned}
$$

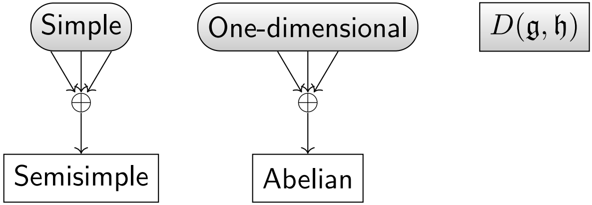
$D(0,\mathfrak{so}(2,1))$

$$
[\mathtt{J}_A,\mathtt{J}_B] = \epsilon_{AB}{}^C\mathtt{J}_C \qquad [\mathtt{J}_A,\mathtt{P}^B] = -\epsilon_{AC}{}^B\mathtt{P}^C \qquad [\mathtt{P}^A,\mathtt{P}^B] = 0
$$

and the invariant metric

$$
\langle\mathtt{J}_A,\mathtt{J}_B\rangle = \eta_{AB} \qquad \langle\mathtt{J}_A,\mathtt{P}^B\rangle = \delta_A{}^B \qquad \langle\mathtt{P}^A,\mathtt{P}^B\rangle = 0\,.
$$

Lie algebras with Invariant Metric [Medina, Revoy '85; Figueroa-O'Farrill, Stanciu '95]



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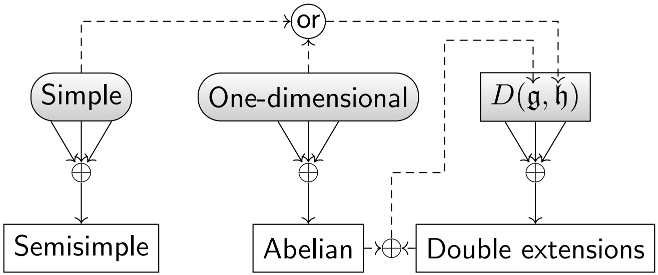

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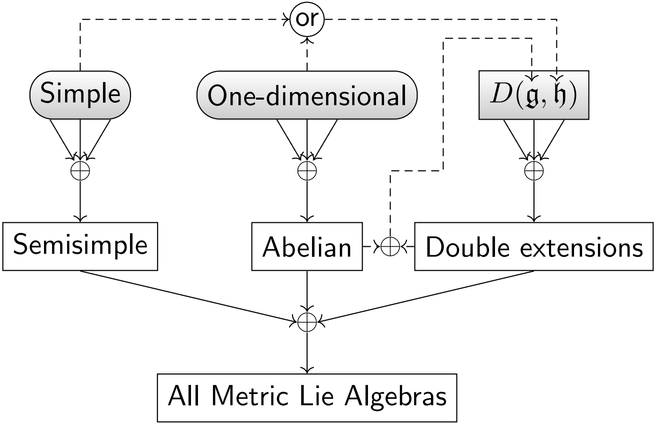


Analyze

- See if the lower dimensional (or their extended version) admit such an invariant metric and analyze theories:
  - Thursday Aug. 26, 2021, 11:00 — 12:00: Daniel Grumiller, *Higher spins and non-AdS holography in lower dimensions*

Outline

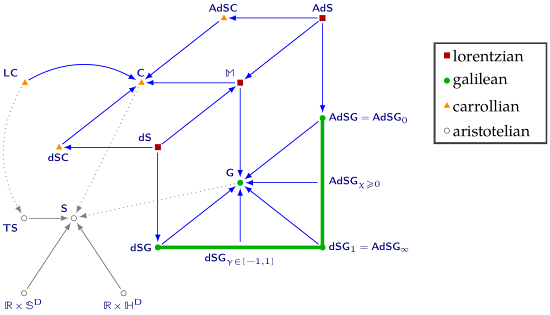
Summary

- Maximally symmetric spacetimes
  - Full classification beyond the lorentzian case
  - Lorentz, Carroll, Galilei and Aristotelian
- Lower dimensional gravitational theories based on these space
  - $2+1$ dimensional Chern–Simons theories
  - $1+1$ BF theories
  - Invariant metric exists
- Analyze if the lower dimensional (or their extended version) admit such an invariant metric and analyze theories

Outlook

- Beyond maximally symmetric spacetimes
  - Adding dilatations, special conformal transformations, central extensions ...
  - A lot of the Lie algebras and their spacetimes fall outside the class of conventional classifications in mathematics (non-semisimple, non-reductive)
- Cartan geometry based on these homogeneous spaces
- Sept. 2, 2021: 11:00 — 12:00 Andrea Campoleoni, *On Carrollian and Galilean higher-spin algebras*

Thank you



AdSC
AdS
LC
C
$\mathbb{M}$
dS
dSC
G
S
TS
dSG
$\mathsf{AdSG} = \mathsf{AdSG}_0$
$\mathsf{AdSG}_{\chi \geqslant 0}$
$\mathsf{dSG}_1 = \mathsf{AdSG}_\infty$
$\mathsf{dSG}_{\gamma \in [-1,1]}$
$\mathbb{R} \times \mathbb{S}^D$
$\mathbb{R} \times \mathbb{H}^D$
lorentzian
galilean
carrollian
aristotelian