The ABCT Variety $V(3,n)$ is a Positive Geometry

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Abstract

The ABCT variety $V(3,n)$ is the image closure of the rational Veronese map. It was studied by **A**rkani-Hamed–**B**ourjaily–**C**achazo–**T**rnka [2] for planar $\mathcal{N}=4$ supersymmetric Yang-Mills scattering amplitudes and Witten's twistor string theory. From this perspective, $V(3,n)$ is conjectured to be a positive geometry by Lam [6].

We study the combinatorial and algebraic geometry aspects of $V(3,n)$ and its subvarieties induced by iteratedly taking analytic boundaries of the totally nonnegative part. We interpret these subvarieties as point configurations on $\mathbb{P}^2$ by the Gelfand-MacPherson correspondence. We construct a top-degree meromorphic form on $V(3,n)$ and show that it is a positive geometry.

The ABCT Variety $V(3,n)$

Definition. Consider the rational Veronese map $\theta : \mathrm{Gr}(2,n) \dashrightarrow \mathrm{Gr}(3,n)$

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix} \mapsto \begin{bmatrix} x_1^2 & x_2^2 & \cdots & x_n^2 \\ x_1y_1 & x_2y_2 & \cdots & x_ny_n \\ y_1^2 & y_2^2 & \cdots & y_n^2 \end{bmatrix}$$

The ABCT variety $V(3,n)$ is the Zariski closure of the image of θ.

***Theorem* 1.** *[1] The following holds for the ABCT variety $V(3,n)$.*

- *$V(3,n)$ is reduced, irreducible, Cohen-Macaulay, and has expected dimension $2(n-2)$.*
- *$V(3,n)$ is cut out as a subscheme of $\mathrm{Gr}(3,n)$ by quartics of the form*
$$[i_1i_2i_3][i_1i_5i_6][i_2i_4i_6][i_3i_4i_5] - [i_2i_3i_4][i_1i_2i_6][i_1i_3i_5][i_4i_5i_6].$$
- *$V(3,n)$ is invariant under column rescaling and set-theoretically, $V(3,n)$ consists of points $[M] \in \mathrm{Gr}(3,n)$ such that its nonzero columns give projective points in $\mathbb{P}^2$ lying on a common conic.*

Motivations

- $V(k,n)$ arises [2] in an approach to compute the planar $\mathcal{N}=4$ SYM scattering amplitude as contour integrals over $\mathrm{Gr}(k,n)$ [4].
- $V(k,n)_{\geq 0}$ maps diffeomorphically to the interior of the momentum amplituhedron and their canonical forms are conjectured to be related by pushforward [5], connecting two approaches of computing the amplitude.
- The amplitude can be expressed as a sum over points in the intersection of $V(k,n)$ and a certain Schubert class [7]. They give soluions to CHY scattering equations [3] and the size of intersection is an Eulerian number.

Main Results

***Theorem* 2.** *The following holds for the ABCT variety $V(3,n)$*

(i) *$(V(3,n), V(3,n)_{\geq 0}, \Omega_{V(3,n)})$ is a normal positive geometry, where $V(3,n)_{\geq 0} := V(3,n) \cap \mathrm{Gr}(3,n)_{\geq 0}$ and $\Omega_{V(3,n)} = \theta_*(\Omega_{\mathrm{Gr}(2,n)})$.*

(ii) *The (analytic) iterated boundaries of $V(3,n)$ are (TTN parts of) $(V(3,n) \cap \Pi_f)_{\mathrm{red}}$ for certain positroid varieties $\Pi_f \subset \mathrm{Gr}(3,n)$.*

(iii) *Poset of iterated boundaries is the induced sub-poset on positroids.*

(iv) *The variety $V(3,n)$ and all its iterated boundaries are reduced, irreducible, rational, normal, Cohen-Macaulay, and have expected dimensions from the perspective of point configurations.*

***Corollary* 3.** *The ABCT variety $V(n-3,n)$ is a positive geometry.*

Normal Positive Geometry

A normal positive geometry consists of the data $(X, X_{\geq 0}, \Omega_X)$ such that

1. X is an irreducible normal projective variety with analytically closed semi-algebraic subset $X_{\geq 0} \subset X(\mathbb{R})$, whose analytic interior $X_{>0}$ is an orientable real manifold of dimension $\dim X$ and is analytically dense in $X_{\geq 0}$.
2. Let $\cup C_i$ be the irreducible decomposition of the Zariski closure C of $X_{\geq 0} \setminus X_{>0}$. Define $C_{i,\geq 0} := C_i \cap X_{\geq 0}$.
3. Ω_X is a top-degree meromorphic form on X. Ω_X has no poles outside C and simple poles along C.
4. For each i, iterate the above on $(C_i, C_{i,\geq 0}, \mathrm{Res}_{C_i}\Omega_X)$ until we reach $(\text{point}, \text{point}, \pm 1)$.

Iterated Boundaries of $V(3,n)$

Definition. Fix a partition $A_0 \sqcup (\sqcup_{i=1}^r I_i)$ of $[n]$ and a partition $[r] = A \sqcup B$, where I_i, A, B are cyclic intervals. Define two families of positroid varieties in $\mathrm{Gr}(3,n)$ as follows.

$$\Pi(A_0; I_1, \cdots, I_r) := \{[M] \in \mathrm{Gr}(3,n) : M_{A_0} = 0,\ M_{I_i} \leq 1 \text{ for all } i\}, \text{ and}$$
$$\Pi(A_0; I_a \text{ for } a \in A | I_b \text{ for } b \in B) := \{[M] \in \mathrm{Gr}(3,n) : M_{A_0} = 0,\ M_{I_i} \leq 1 \text{ for all } i,\ M_{\sqcup_{a\in A} I_a} \leq 2,\ M_{\sqcup_{b \in B} I_b} \leq 2\}.$$

***Theorem* 4.** *Iterated boundaries of $V(3,n)$ are exactly as follows.*

- *Boundaries of collinear configurations $\Pi(A_0; I_a \text{ for } a \in A | I_b \text{ for } b \in B)$.*
- *Boundary of colliding configurations $V(A_0; I_1, \cdots, I_r)_{\mathrm{red}} := (\Pi(A_0; I_1, \cdots, I_r) \cap V(3,n))_{\mathrm{red}}$.*

Constructing New Positive Geometries

***Theorem* 5** (Sketch)**.** *Let $(X, X_{\geq 0}, \Omega_X)$ be a positive geometry. Let $Y \to \mathbb{P}^1$ be a locally trivial fibration with X-fibers such that transition map "preserves positivity" and "glues boundaries of X."*
Then $(Y, Y_{\geq 0}, \Omega_Y)$ is a positive geometry, where Ω_Y is given locally on each chart by $\Omega_X \wedge d\log t$.

***Theorem* 6** (Sketch)**.** *Let $(X, X_{\geq 0}, \Omega_X)$ and $(Y, Y_{\geq 0}, \Omega_Y)$ be as in the previous theorem. Then $(R, R_{\geq 0}, \Omega_R)$ is a positive geometry if there exist the following structure*

- *an iterated boundary S of X that lifts to $Y_S \hookrightarrow Y$ and $Y_S \hookrightarrow Y \to \mathbb{P}^1$ is the projection $S \times \mathbb{P}^1 \to \mathbb{P}^1$.*
- *a closed embedding $Y \hookrightarrow R \times \mathbb{P}^1$ and $\psi : Y \hookrightarrow R \times \mathbb{P}^1 \to R$ is an isomorphism outside of Y_S.*
- *ψ restricted to $Y_S = S \times \mathbb{P}^1$ is the projection onto the first component S.*

where $R_{\geq 0}$ is the image $\psi(Y_{\geq 0})$ and Ω_R is the pushforward $\psi_*(\Omega_Y)$.

Key Step in Induction

***Theorem* 7** (Sketch)**.** *Let $s, t \in I_1$ be distinct. Define the incidence subscheme*

$$\mathcal{I}_{st}(A_0; I_1, \cdots, I_r) = \{([M], [x:y]) : y \cdot M_{\{s\}} = x \cdot M_{\{t\}}\} \subset V(A_0; I_1, \cdots, I_r)_{\mathrm{red}} \times \mathbb{P}^1,$$

where $M_{\{s\}}$ denotes the s-th column of the matrix M. The incidence condition can be rephrased in terms of Plücker coordinates as $[sjk](M)y = [tjk](M)x$ for all j, k. Then

- *projecting to the second component $\mathcal{I}_{st}(A_0; I_1, \cdots, I_r) \to \mathbb{P}^1$ satisfies Theorem 5 with fibers isomorphic to $V(A_0, I_1', \cdots, I_r)_{\mathrm{red}} \subset V(3, n-1)$, where I_1' is obtained from I_1 by "merging" columns s and t.*
- *projecting to the first component $\mathcal{I}_{st}(A_0; I_1, \cdots, I_r) \to V(A_0; I_1, \cdots, I_r)_{\mathrm{red}}$ satisfies Theorem 6 and induces an isomorphism on the structural sheaves $\mathcal{O}_{V(A_0;I_1,\cdots,I_r)_{\mathrm{red}}} \cong (\pi_1)_*\mathcal{O}_{\mathcal{I}_{st}(A_0;I_1,\cdots,I_r)}$.*

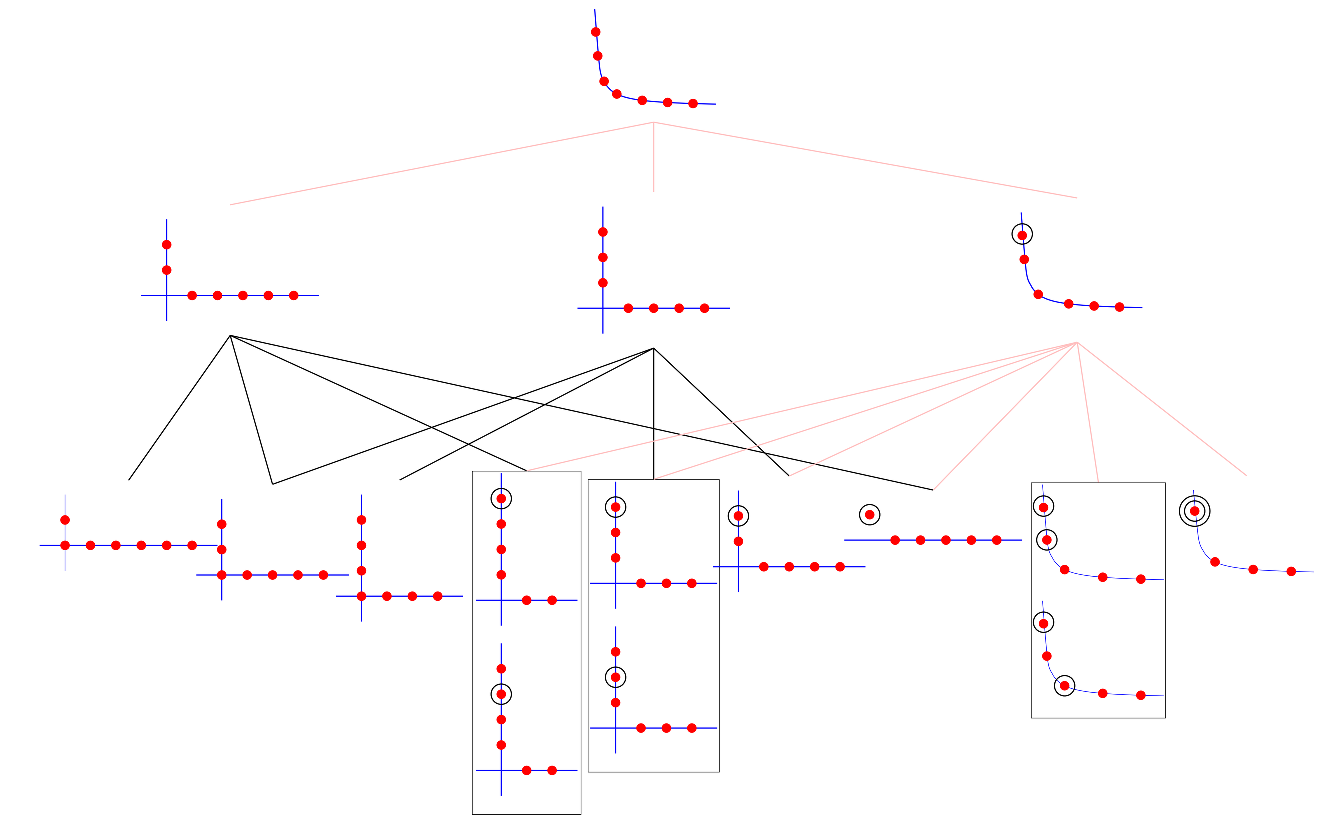
Proof Sketch

We induct on n to show that $V(3,n)$ and all iterated boundaries are positive geometries.

- For $V(3,n)$, we show that $V(3,n)_{\geq 0}$ has analytic interior $V(3,n) \cap \mathrm{Gr}(3,n)_{>0}$. We decompose the boundary into components and verify the residue by direct computation.
- For a boundary of collinear configurations, this is a positroid variety and is known to be a positive geometry.
- For a boundary of colliding configurations, we use Theorem 7 to "merge" two columns, thereby reducing n by 1.

Example of $V(3,7)$

Below are iterated boundaries of codimension $0, 1, 2$ up to dihedral action, omitting the codimension 2 boundary where one column vanishes.



Open Problems

1. Find a manifestly cyclically invariant global expression of the canonical form $\Omega_{V(3,n)}$ instead of in matrix coordinates.
2. Is $V(k,n)$ a positive geometry?

References

[1] D. Agostini, L. Ramesh, and D. Shen. Points on Rational Normal Curves and the ABCT Variety, 2024.

[2] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, and J. Trnka. Unification of Residues and Grassmannian Dualities. *Journal of High Energy Physics*, 01:049, 2011.

[3] F. Cachazo, S. He, and E. Y. Yuan. Scattering in three dimensions from rational maps. *Journal of High Energy Physics*, 2013(10), Oct. 2013.

[4] H. Elvang and Y.-t. Huang. *Scattering Amplitudes in Gauge Theory and Gravity.* Cambridge University Press, Cambridge, 2015.

[5] S. He, C.-K. Kuo, and Y.-Q. Zhang. The momentum amplituhedron of SYM and ABJM from twistor-string maps. *Journal of High Energy Physics*, (2):148, 2022.

[6] T. Lam. Moduli spaces in positive geometry. *Le Matematiche*, 80:17–101, 2025.

[7] R. Roiban, M. Spradlin, and A. Volovich. On the tree level S-matrix of Yang-Mills theory. *Physical Review D*, 70:026009, 2004.

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