Ratio-limit boundaries for random walks on relatively hyperbolic groups.

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Example (Simple RW on $\mathbb{F}_d$)

Let $\Gamma = \mathbb{F}_d = \langle a_1, ..., a_d \rangle$, and $\mu(a_i^{\pm 1}) = \frac{1}{2d}$ for all $i = 1, ..., d$. This is called the simple random walk on $\mathbb{F}^d$.

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The *spectral radius* of P is $\rho := \limsup_n \sqrt[n]{P^n(x,y)}$ for some (all) $x, y \in \Gamma$. In this talk our groups will be non-amenable, so we will always have $\rho < 1$ (Kesten 1959).

Martin boundary

Let $R := \rho^{-1}$ be the inverse of the spectral radius. For $r \in [1, R]$ we define the *Green function* for $x, y \in \Gamma$ by

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The left action of Γ on itself induces an action $\Gamma \curvearrowright \partial_{M,r}\Gamma$, and we have $r^{-1} \cdot K_r(x,\xi) = \sum_{y \in \Gamma} P(x,y) K_r(y,\xi)$ for $\xi \in \partial_{M,r}\Gamma$.

Harmonic functions

We will say that a function $u : \Gamma \to (0, \infty)$ is *t-harmonic* if $t \cdot u(x) = \sum_{y \in \Gamma} P(x, y)u(y)$. The set $\mathcal{H}_1^+(P, t)$ of positive harmonic functions with $u(e) = 1$ is a *compact convex set* with the topology of pointwise convergence.

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Theorem (Poisson–Martin integral representation)

Let u be in $\mathcal{H}_1^+(P, r^{-1})$. Then there is a representing probability measure ν^u on $\partial_{M,r}\Gamma$ such that

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Denote $\partial^m_{M,r}\Gamma$ the points $\xi \in \partial_{M,r}\Gamma$ with $x \mapsto K_r(x, \xi)$ extreme.

Ratio-limits

Definition (Strong ratio-limit property)

Let P be a RW on Γ. We say it has SRLP if for any $x, y \in \Gamma$ the limit $H(x,y) := \lim_n \frac{P^n(x,y)}{P^n(e,y)}$ exists.

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Definition + Proposition (D. 2021 & DDG)

Suppose P is a symm RW on Γ with SRLP, and denote by R_μ the set $R_\mu := \{ \ g \in \Gamma \mid H(x,g) = H(x,e), \ \forall x \in \Gamma \ \}$.

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Definition (D. 2021)

Let P be a symmetric RW on Γ with SRLP. The (reduced) ratio-limit compac$\underline{^n}$ is the smallest compac$\underline{^n}$ $\Delta_{\mathrm{R}}\Gamma$ of Γ/R_μ to which the functions $y \mapsto H(x,y)$ extend continuously.

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Used to show that certain canonical equivariant quotient C*-algebra generally fail to be the unique equivariant quotient, even when such a quotient exists.

Local limit theorems

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When $\Gamma = \mathbb{Z}^3 * \mathbb{Z}^6$, there is a symmetric RW on Γ for which $P^n(x,y) \sim \beta(x,y) R^{-n} n^{-\frac{3}{2}} \log(n)^{-\frac{1}{2}}$.

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Classifying all LLT behaviors is still open, but we can compute $H(x,y) = \frac{\beta(x,y)}{\beta(e,y)}$ in the presence of a LLT.

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On the other hand, hyperbolic groups do not allow for arbitrary subgroups. For instance $\mathbb{Z}^2 * \mathbb{Z}^3$ is not hyperbolic, even though it does admits some "global" hyperbolic behaviour.

Relatively hyperbolic groups

Let Γ be a f.g. group and Ω a finite set of subgroups. The *relative Cayley graph* $\mathrm{Gr}(\Gamma;\Omega)$ is obtained from $\mathrm{Gr}(\Gamma)$ by adding a vertex gP and an edge from h to gP for any $h \in gP$ and $P \in \Omega$.

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Examples include free products $A_1 * A_2$ with A_1, A_2 f.g., as well as fundamental groups of *finite volume* Riemannian manifolds of pinched negative sectional curvature.

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Theorem (GGPY 2021)

Let P be a symm RW on Γ hyperbolic relative to Ω. For any $r \in [1, R]$ the identity on Γ induces a continuous Γ-surjection

$$\pi : \partial_{M,r}\Gamma \to \partial_B(\Gamma; \Omega),$$

and $\pi^{-1}(\xi)$ is a singleton for any conical point $\xi \in \partial_B(\Gamma; \Omega)$.

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4. Suppose wlog $\xi_+, \xi_- \neq \eta_-$. If $\nu \in \mathrm{Prob}(\overline{\partial^m_{M,R}\Gamma})$, then $s^n\nu$ converges to $\nu' := \lambda\delta_{\xi_-} + (1-\lambda)\delta_{\xi_+}$. Then $t^m\nu'$ converges to δ_{η_+}. Thus $\Gamma \curvearrowright \overline{\partial^m_{M,R}\Gamma}$ is strongly proximal.

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1. By Poisson-Martin representation theorem we get a continuous surjection $\varphi : \mathrm{Prob}(\overline{\partial^m_{M,R}\Gamma}) \to \mathcal{H}^+_1(\mu, \rho)$ given by integrating $\varphi(\nu)(x) = \int_{\partial_{M,R}\Gamma} K_R(x,\xi) d\nu(\xi)$.

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3. By strong proximality, $\overline{\Gamma\nu^u}$ intersects $\overline{\partial^m_{M,R}\Gamma}$, and by minimality this intersection is all of $\overline{\partial^m_{M,R}\Gamma}$.

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The measure μ is s.n.d. RW, while μ' is not. We can prove an analogue of our Γ-equivariant embedding result $\partial^{\mu'}_{M,R}\Gamma \to \partial_{\mathrm{R}}\Gamma$ as a unique Γ-minimal subspace for $\Gamma = \mathbb{Z}^5 * \mathbb{Z}$ and μ' as above.

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Theorem (D., Dussaule & Gektman)

Let $\Gamma = \mathbb{Z}^5 * \mathbb{Z}$. Then there exist two random walks μ and μ' for which $\partial^{\mu}_{\mathrm{R}}\Gamma$ and $\partial^{\mu'}_{\mathrm{R}}\Gamma$ are not Γ-equivariantly homeomorphic.

Toeplitz C*-algebras for random walks

Let P be a RW on Γ induced by μ. We let $\mathcal{H}_P$ be the Hilbert space with o.n.b. $\{e_{y,z}^{(m)}\}_{P^m(y,z)>0,m\geq 0}$.

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$$S^{(n)}_{x,y}(e^{(m)}_{y',z}) = \delta_{y,y'} \sqrt{\frac{P^n(x,y)P^m(y,z)}{P^{n+m}(x,z)}} e^{(n+m)}_{x,z}.$$

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It arises from a general *subproduct system* construction of Shalit and Solel (2009), when applied to P as a positive map on $c_0(\Gamma)$. This came about form work of mine with Markiewicz (2017).

Co-universal Toeplitz quotient

This C*-algebra has a *natural* action $\alpha : \mathbb{T} \curvearrowright \mathcal{T}(\Gamma, \mu)$ given on generators by $\alpha_z(S^{(n)}_{x,y}) = z^n S^{(n)}_{x,y}$, but there is also a group action $\beta : \Gamma \curvearrowright \mathcal{T}(\Gamma, \mu)$ given by $\beta_g(S^{(n)}_{x,y}) = S^{(n)}_{gx,gy}$. These actions commute, and give rise to an action $\Gamma \times \mathbb{T} \curvearrowright \mathcal{T}(\Gamma, \mu)$.

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Theorem (D. 2021)

When Γ is hyperbolic and μ is symmetric, the co-universal quotient exists, and coincides with $C(\mathbb{T} \times \partial\Gamma) \otimes \mathbb{K}(\ell^2(\Gamma))$.

The case of RH groups

When defining Toeplitz C*-algebra for RW, for every $z \in \Gamma$ there is a reducing subspace for $\mathcal{T}(\Gamma, \mu)$ which is given by $\mathcal{H}_{P,z} := \overline{\mathrm{Sp}}\{e_{y,z}^{(m)}\}_{P^m(y,z)>0,\ m\geq 0}$.

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Theorem (D., Dussaule & Gekhtman)

Let P be a symmetric aperiodic (s.n.d.) RW on a RH Γ. Then the co-universal quotient is $C(\overline{\partial^m_{M,R}\Gamma} \times \mathbb{T}) \otimes \mathbb{K}(\ell^2(\Gamma))$.

Concluding remarks

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4. Viselter's original quotient C*-algebra is by the ideal $\oplus_{z\in\Gamma}\mathbb{K}(\mathcal{H}_{P,z})$, and it seems to be intimately related to the spacetime Martin boundary of the RW.

Thank you

Thank you for your attention !