Banach manifolds and integrable systems around them

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M. Sato and Y. Sato.
Soliton equations as dynamical systems on infinite dimensional Grassmann manifold.
Lect. Notes in Num. Appl. Anal., 5:259–271, 1982

G. B. Segal and G. Wilson.
Loop groups and equations of KdV type.
Inst. Hautes Études Sci. Publ. Math., 61:5–65, 1985

A. Pressley and G. Segal.
Loop Groups.
Oxford Mathematical Monographs. Clarendon Press, Oxford, 1986

T. Miwa, M. Jimbo, and E. Date.
Solitons: differential equations, symmetries and infinite dimensional algebras.
Cambridge University Press, Cambridge, 2000

T. Wurzbacher.
Fermionic second quantization and the geometry of the restricted Grassmannian.
Infinite Dimensional Kähler Manifolds, DMV Seminar, vol. 31. Birkhäuser, Basel, 2001

- P. Bóna.
  Extended quantum mechanics.
  Acta Physica Slovaca, 50:1–198, 2000
- A. Odzijewicz and T. S. Ratiu.
  Banach Lie–Poisson spaces and reduction
  Comm. Math. Phys., 243:1–54, 2003
- D. Beltiţă, T. Goliński, and A. B. Tumpach.
  Queer Poisson brackets
  J. Geom. Phys., 132:358–362, 2018
- T. Goliński, P. Rahangdale and A. B. Tumpach.
  Poisson structures in the Banach setting: comparison of different approaches
  Proceedings of WGMP 2024, to appear

- D. Beltiţă, T. S. Ratiu, and A. B. Tumpach.
  The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits
  J. Funct. Anal., 247:138–168, 2007
- T. Goliński and A. Odzijewicz.
  Hierarchy of Hamilton equations on Banach Lie–Poisson spaces related to restricted Grassmannian
  J. Funct. Anal., 258:3266–3294, 2010
- A. Odzijewicz and T. Goliński.
  Hierarchy of integrable Hamiltonians describing nonlinear n-wave interaction.
  J. Phys. A Math. Theor. 45(4):045204, 2012
- T. Goliński and A.B. Tumpach.
  Geometry of integrable systems related to the restricted Grassmannian.
  SIGMA 20, 104, 2024

Plan of the talk

1. Restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$
2. KdV equation
3. Banach Lie–Poisson spaces
4. Quantum mechanics and Toda latice
5. Hierarchy of integrable systems on Banach Lie–Poisson spaces related to the restricted Grassmannian

Restricted Grassmannian

We fix a (complex, separable) Hilbert space $\mathcal{H}$ with an orthogonal decomposition (polarization) $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$.

$P_\pm$ is orthogonal projection on $\mathcal{H}_\pm$.

Definition

Restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ is a set of Hilbert subspaces W in $\mathcal{H}$ such that

1. the projection P_+ restricted to W is Fredholm operator;
2. the projection P_- restricted to W is Hilbert–Schmidt operator;

- $\mathrm{Gr}_{\mathrm{res}}$ is Banach symplectic manifold modelled on Hilbert space $L^2(\mathcal{H}_+, \mathcal{H}_-)$.

  Idea behind the construction:
  $X \in L^2(\mathcal{H}_+, \mathcal{H}_-) \longmapsto$ graph of $X \in \mathrm{Gr}_{\mathrm{res}}$
  Transition maps are certain homographic functions.

- Connected components of $\mathrm{Gr}_{\mathrm{res}}$ are $\mathrm{Gr}_{\mathrm{res},k}$, $k \in \mathbb{Z}$, consisting of such subspaces W that index of P_+ restricted to W is k.

Block decomposition of the operator

$$A = \begin{pmatrix} A_{++} & A_{-+} \\ A_{+-} & A_{--} \end{pmatrix}$$

$$A_{++} : \mathcal{H}_+ \to \mathcal{H}_+ \qquad A_{--} : \mathcal{H}_- \to \mathcal{H}_-$$

$$A_{-+} : \mathcal{H}_- \to \mathcal{H}_+ \qquad A_{+-} : \mathcal{H}_+ \to \mathcal{H}_-$$

Definition

We define the Banach Lie group

$$\mathrm{GL}_{\mathrm{res}} := \{g \in GL^\infty(\mathcal{H}) \mid g_{-+} \in L^2(\mathcal{H}_-, \mathcal{H}_+),\ g_{+-} \in L^2(\mathcal{H}_+, \mathcal{H}_-)\} \tag{1}$$

Banach Lie algebra of $\mathrm{GL}_{\mathrm{res}}$ is

$$\mathrm{L}_{\mathrm{res}} := \{A \in L^\infty(\mathcal{H}) \mid A_{-+} \in L^2(\mathcal{H}_-, \mathcal{H}_+),\ A_{+-} \in L^2(\mathcal{H}_+, \mathcal{H}_-)\} \tag{2}$$

with norm:

$$\|A\|_{\mathrm{res}} := \|A_{++}\|_\infty + \|A_{--}\|_\infty + \|A_{-+}\|_2 + \|A_{+-}\|_2 \tag{3}$$

- Since $g \in \mathrm{GL}_{\mathrm{res}}$ is invertible, g_{++}, g_{--} are Fredholm operators.
- Banach Lie group $\mathrm{GL}_{\mathrm{res}}$ is disconnected. Connected components are indexed by index of Fredholm operator g_{++}. $\mathrm{GL}_{\mathrm{res},0}$ denotes connected component containing identity.
- $\mathrm{GL}_{\mathrm{res}}$ and $\mathrm{U}_{\mathrm{res}}$ (unitary counterpart) act transitively on $\mathrm{Gr}_{\mathrm{res}}$

$$\mathrm{Gr}_{\mathrm{res}} = \mathrm{GL}_{\mathrm{res}} \cdot \mathcal{H}_+ = \mathrm{U}_{\mathrm{res}} \cdot \mathcal{H}_+$$

$$\mathrm{Gr}_{\mathrm{res},0} = \mathrm{GL}_{\mathrm{res},0} \cdot \mathcal{H}_+ = \mathrm{U}_{\mathrm{res},0} \cdot \mathcal{H}_+$$

$$\mathrm{Gr}_{\mathrm{res}} = \mathrm{U}_{\mathrm{res}} / (U(\mathcal{H}_+) \times U(\mathcal{H}_-))$$

Essential case:

$\mathcal{H}$ — Hilbert space of complex square integrable functions on the unit circle S^1

$\mathcal{H}_+ = \mathrm{span}\{1, z, z^2, \ldots\}$
$\mathcal{H}_- = \mathrm{span}\{z^{-1}, z^{-2}, \ldots\}$

If $f : S^1 \to \mathbb{C} \setminus \{0\}$ is twice differentiable then it defines a multiplication operator on $\mathcal{H}$ which belongs to $\mathrm{GL}_{\mathrm{res}}$.

We denote group generated by operators of this form by Γ. Subgroups $\Gamma_\pm \subset \Gamma$ — real-analytic functions which extend to holomorphic functions inside/outside of the disc, $f(0) = 1$ (or $f(\infty) = 1$).

KdV equation

Korteweg–de Vries equation:

$$u_t + u_{xxx} + 6uu_x = 0$$

Idea:

solutions of KdV $\leftrightsquigarrow$ flow on (a part of) $\mathrm{Gr}_{\mathrm{res}}$

$$\mathrm{Gr_{res}}^{(n)} = \{W \in \mathrm{Gr_{res}} \mid z^n W \subset W\}$$

$$g = \exp \sum t_k z^k \in \Gamma_+$$

(almost all t_k vanish)

Given $W \in \mathrm{Gr_{res}}^{(2)} \cap \mathrm{Gr_{res,0}}$ one constructs a function u_W s.t. $u_W \longmapsto u_{gW}$ is a result of KdV flow by times $(t_1, t_2, \ldots)$

$$u_W(x) = 2(\partial_x)^2 \log \tau_W(x)$$

$\tau_W(x)$ is a determinant (in a certain sense) of the projection $e^{-xz} W \to \mathcal{H}_+$.

DEFINITION

Banach Poisson manifold is a smooth manifold M modelled on Banach space with (localizable) Poisson bracket

$$\{f, g\} := \pi(df, dg) \tag{4}$$

given by a Poisson tensor $\pi \in \Gamma^\infty \bigwedge^2 T^{**}M$, such that the map $\sharp : T^*M \to T^{**}M$

$$\sharp\mu := \pi(\cdot, \mu)(m) \in T^{**}_m M, \quad \mu \in T^*_m M \tag{5}$$

takes values in tangent bundle

$$\sharp(T^*M) \subset TM.$$

Note 1: assumption of localizability means that one can apply Poisson bracket also to locally defined functions. It isn't the case in general even for Banach spaces due to the lack of smooth bump functions.

Note 2: there are Poisson brackets which are not defined by Poisson tensor and they might include higher order derivatives. In that case the usual approach to mechanics fails. Leibniz property does not imply the existence of Poisson tensor or the map $\sharp : T^*M \to T^{**}M$. Uniqueness is also not guaranteed.

Those pathological (queer) Poisson bracket exist i.e. on l^p spaces for $1 \leqslant p \leqslant 2$.

Note 3: condition $\sharp(T^*M) \subset TM$ ensures that hamiltonian vector fields are really vector fields and not sections of $T^{**}M$.

Definition

Banach Lie–Poisson space is a Banach space $\mathfrak{b}$ with the structure of Banach Poisson manifold such that $\mathfrak{b}^* \subset C^\infty(\mathfrak{b})$ is Banach Lie algebra with respect to Poisson bracket.

Theorem

Banach space $\mathfrak{b}$ is Banach Lie–Poisson space if $\mathfrak{b}^*$ is Banach Lie algebra with the propriety

$$\mathrm{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**} \qquad \textit{for } x \in \mathfrak{b}^*. \tag{6}$$

- Poisson bracket on $\mathfrak{b}$ is

$$\{f,g\}(b) := \langle [Df(b), Dg(b)], b \rangle, \tag{7}$$

where $Df(b), Dg(b) \in \mathfrak{b}^*$ are Fréchet derivatives at b.

- Hamilton equations on $\mathfrak{b}$ with Hamiltonian $h \in C^\infty(\mathfrak{b})$

$$\frac{d}{dt} b = -\operatorname{ad}^*_{Dh(b)} b \tag{8}$$

Model example: trace-class operators $L^1(\mathcal{H})$

- It is predual to bounded operators:

$$(L^1(\mathcal{H}))^* = L^\infty(\mathcal{H})$$

- Since L^1 is an ideal we get:

$$\mathrm{ad}^*_X \rho = [X, \rho] \in L^1(\mathcal{H})$$

  for $X \in L^\infty(\mathcal{H})$ and $\rho \in L^1(\mathcal{H})$

- $\frac{d}{dt}\rho = -[Dh(\rho), \rho]$
- For $\rho = |\psi\rangle\langle\psi|$ we get

$$\frac{d}{dt}\,|\psi\rangle = Dh(\psi)\,|\psi\rangle$$

A. Odzijewicz and T. S. Ratiu.
Induced and coinduced Banach Lie–Poisson spaces and integrability
J. Funct. Anal., 225:1225–1272, 2008

Weak symplectic manifold $\ell^\infty \times \ell^1$, $q \in \ell^\infty$, $p \in \ell^1$.

$$\omega = dq \wedge dp$$

$$H_{Toda} = \frac{1}{2}\sum_{n=0}^{\infty} p_n^2 + \sum_{n=0}^{\infty} \nu_n e^{2(q_{n+1}-q_n)} \quad \nu \in \ell^1$$

gives semi-infinite Toda lattice.

The momentum map:

$$J:\ \ell^\infty \times \ell^1 \ni (q,p) \ \mapsto \ \begin{pmatrix} p_0 & 0 & 0 & \cdots \\ \nu_0 e^{q_1-q_0} & p_1 & 0 & \\ 0 & \nu_1 e^{q_2-q_1} & p_2 & \\ \vdots & & \ddots & \ddots \end{pmatrix} \in \ L^1$$

is a Poisson map, i.e.

$$\{F,G\}_{L^1} \circ J = \{F\circ J, G\circ J\}_{\ell^\infty\times\ell^1}$$

(note that $\{\,\cdot\,,\,\cdot\,\}_{\ell^\infty\times\ell^1}$ is only a weak Poisson bracket)

- Predual space to $\mathfrak{u}_{\text{res}}$ is

$$\mathfrak{u}^1_{\text{res}} = \{\mu \in \mathfrak{u}_{\text{res}} \mid \mu_{++} \in L^1(\mathcal{H}_+)\ ,\ \mu_{--} \in L^1(\mathcal{H}_-)\} \tag{9}$$

  with pairing

$$\langle \mu, A \rangle = \mathrm{Tr}_{\text{res}}(\mu A) \tag{10}$$

  for $\mu \in \mathfrak{u}^1_{\text{res}}$, $A \in \mathfrak{u}_{\text{res}}$.
- Restricted trace Tr_{res} is defined as follows

$$\mathrm{Tr}_{\text{res}}\, \mu := \mathrm{Tr}(P_+ \mu P_+ + P_- \mu P_-) \tag{11}$$

- $\mathfrak{u}^1_{\text{res}}$ is Banach Lie–Poisson space.

- homogeneous polynomials

$$H^n_k(\mu) := \sum_{\substack{i_0, i_1, \dots i_n \in \{0,1\} \\ i_0 + \dots + i_n = k}} P_+^{i_0} \mu P_+^{i_1} \mu \dots \mu P_+^{i_n}$$

of the degree $n \in \mathbb{N}$ in the operator variable $\mu \in \mathfrak{u}^1_{\mathrm{res}}$ and degree k in P_+, where $k \leqslant n + 1$.

- hierarchy of commuting equations (Lax form)

$$\frac{\partial}{\partial t^n_k} \mu = i^{n+1} [\mu, H^n_k(\mu)]$$

Proposition

The diagonal blocks μ_{++} and μ_{--} are constant

$$\frac{\partial}{\partial t_k^n}\mu_{++} = 0 \qquad \frac{\partial}{\partial t_k^n}\mu_{--} = 0$$

Proposition

In the case $\mu_{++} = 0$ the modulus $|\mu_{-+}|$ is constant along the bihamiltonian flows for all t_k^n, $n \in \mathbb{N}$, $k \leqslant n+1$.

Consider the polar decomposition of $\mu_{-+} = uB$.

Proposition

Assume that $\mu_{++} = 0$ and $|\mu_{-+}|$ is partially invertible. The equations for the evolution of the partial isometry u assume the form

$$\frac{\partial}{\partial t_k^n} u = i^{n+1} (\mu H_{k-1}^{n-1})_{--} u$$

for $n \in \mathbb{N}$, $k \leqslant n+1$.

For $k = 1$

$$\frac{\partial}{\partial t_1^n} u = i^{n+1} (\mu^n)_{--} u.$$

$$\frac{\partial}{\partial t_1^1} u = -Du$$

$$\frac{\partial}{\partial t_1^2} u = i(uB^2 - D^2 u)$$

$$\frac{\partial}{\partial t_1^3} u = -DuB^2 - uB^2 u^* Du + D^3 u$$

$$\frac{\partial}{\partial t_2^3} u = -DuB^2 - uB^2 u^* Du$$

$$\frac{\partial}{\partial t_2^4} u = i(2uB^4 - D^2 uB^2 - uB^2 u^* D^2 u - DuB^2 u^* Du)$$

Affine coadjoint action of $\mathrm{U}_{\mathrm{res}}$ on $\mathfrak{u}^1_{\mathrm{res}}$:

$$\widetilde{\mathrm{Ad}}^*_g(\mu) = g^{-1}\mu g + \gamma(P_+ - g^{-1}P_+ g)$$

Restricted Grassmannian as one of the orbits:

$$\Phi_\gamma : \mathrm{Gr}_{\mathrm{res}} \ni W \longrightarrow \mu = \gamma(P_W - P_+) \in \mathcal{O}_{(0,\gamma)} \subset \mathfrak{u}^1_{\mathrm{res}}, \quad \gamma \in i\mathbb{R}$$

Proposition

An element $\mu \in \mathfrak{u}^1_{res}$ belongs to the coadjoint orbit $\mathcal{O}_{(0,\gamma)}$ if and only if $\frac{1}{\gamma}\mu + P_+$ is an orthogonal projection.

Proposition

For initial conditions in the coadjoint orbit $\mathcal{O}_{(0,\gamma)}$, the equations are linear.

$$\mu^2 = \gamma(\mu - \mu P_+ - P_+\mu) = \gamma(\mu_{--} - \mu_{++})$$

implies $\mu = \mathrm{const}$

In a chart on the restricted Grassmannian:

$$\Phi_\gamma \circ \varphi_{\mathcal{H}_+}^{-1}(A) = \gamma \begin{pmatrix} (1+A^*A)^{-1} - 1 & (1+A^*A)^{-1}A^* \\ A(1+A^*A)^{-1} & A(1+A^*A)^{-1}A^* \end{pmatrix},$$

where $A \in L^2$.