From Feynman diagrams to commutative diagrams

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Physics (QFT)	**Algebra**
1d field theory = mechanics Grady–Li–Li	associative algebras Fedosov quantization
2d chiral CFT Gorbounov–G.–Williams	vertex algebras chiral diff. ops
2d TFT Li–Li	categories (of "branes") B-model
3d TFT Costello–Francis–G.	braided monoidal categories $\mathrm{Rep}(U_\hbar \mathfrak{g})$
⋮	⋮

Q: Is there a systematic relationship?

$$\text{Lagrangian field theories + perturbative quantization} \xrightarrow[\text{Costello–G.}]{\text{observables}} \text{factorization algebras}$$

In current apparatus,
we deal w/ "Euclidean"
field theories in
Batalin-Vilkovisky formalism

<u>Remark</u>: This formalism was inspired & modeled upon Kontsevich-Cattaneo-Felder work on 2d PSM + def. quantization

<u>Slogan</u>: BV quantization provides deformation quantization of the <u>factorization</u> algebra of observables.

I. <u>Factorization Algebras</u>

Let's fix a manifold M.

A <u>prefactorization algebra</u> $\mathcal{F}$ on M with values in $\mathrm{Vect}^{\otimes}$ consists of the following data:

any sym. monoidal category

- for each open set $U \subset M$,

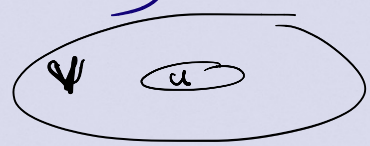
- for each open set U,

$$\mathcal{F}(U) \in \mathrm{Vect}$$

- for each inclusion $U \hookrightarrow V$, a linear map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

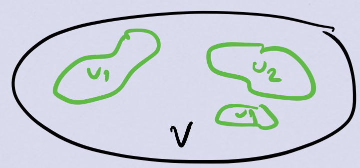
$\mathcal{F}: \mathrm{Open}(M) \to \mathrm{Vect}$ "precosheaf"



- for each finite collection of opens

$$U_1, \ldots, U_n \subset V$$

that are pairwise disjoint, a linear map



"algebra-like"

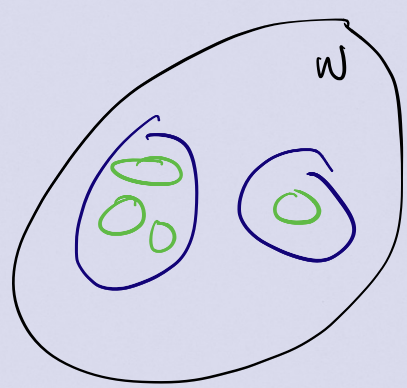
$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V)$$

satisfy

- <u>associativity</u>:

$$U_1 \sqcup U_2 \sqcup \cdots \sqcup U_n \hookrightarrow V_1 \sqcup \cdots \sqcup V_k \hookrightarrow W$$

$$\begin{tikzcd}
\bigotimes_i \mathcal{F}(U_i) \arrow[rr] \arrow[dr] & \circlearrowleft & \bigotimes_j \mathcal{F}(V_j) \arrow[dl] \\
& \mathcal{F}(W) &
\end{tikzcd}$$

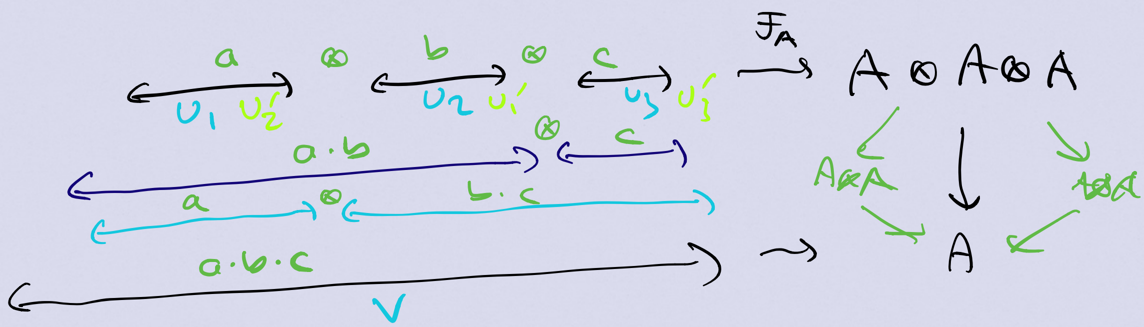


- equivariance: the maps are equivariant under relabeling

Key examples

Every associative algebra A determines a prefactorization algebra on $\mathbb{R}$

$$\mathcal{F}_A(\longleftrightarrow) = A$$



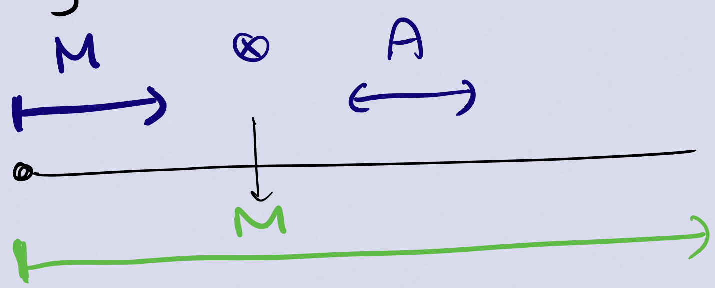
Theorem (Lurie)
There is an equivalence of ∞-categories

$$\left\{ \begin{matrix} \text{locally constant} \\ \text{prefact. algs} \\ \text{on } \mathbb{R}^n \end{matrix} \right\} \simeq \mathrm{Alg}_{\mathcal{E}_n}$$

$\mathcal{E}_n$ = operad of little n-disks

$\mathcal{E}_1 \sim$ associative operad

A right A-module M determines a prefact alg on $\mathbb{R}_{\geq 0}$



Rmk: Claudia has vastly extended Morita theory using this perspective

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A *factorization algebra* is a prefact. alg. such that

① U, V disjoint

$$\mathcal{F}(U) \otimes \mathcal{F}(V) \xrightarrow{\sim} \mathcal{F}(U \sqcup V)$$

"factors"

② cosheaf with respect to Weiss topology on M

For an open set $U \subset M$, a <u>Weiss cover</u> $\{U_\alpha\}$ is a cover such that for any finite collection of points

$$\{x_1, \ldots, x_k\} \subset V$$

there is some $U_\alpha \supset \{x_1, \ldots, x_n\}$.

<u>Consequence</u> $\quad \mathrm{Vect} \rightsquigarrow \mathrm{dgVect}$

A — dg assoc. algebra

$$\mathcal{F}_A(S^1) \simeq A \overset{\mathbb{L}}{\underset{A \otimes A^{op}}{\otimes}} A \simeq \mathrm{Hoch}_*(A, A)$$

factorization homology = global sections of $\mathcal{F}$

$$\mathcal{H}_*(M, \mathcal{F}) \qquad \mathcal{F}(M)$$

is a generalization of Hochschild chains by replacing S^1 with higher-dimensional manifolds

<u>Remark</u>: Ginot et al developed fact. homology by exploring how to

generalize Chen's iterated integrals to higher-dim. mflds //

II. Connection w/ physics

I'll draw a kind of cartoon of physics.

Lagrangian field theory

M — manifold "spacetime"

$$\begin{array}{c} F \\ \downarrow \\ M \end{array} \text{ — bundle \& let } \mathcal{F} : \mathrm{Open}(M)^{op} \to \mathrm{Spaces}, \quad U \longmapsto \Gamma(U,F)$$

"fields"

The equations of motion determine a subsheaf of solutions

$$\mathrm{Sol} : \mathrm{Open}(M)^{op} \longrightarrow \mathrm{Spaces}$$

$$\mathrm{Sol}(U) = \{ \varphi \in \mathcal{F}(U) : EL(\varphi) = 0 \}$$

The observables are functions on Solutions:

$$\mathcal{O} : \mathrm{Spaces}^{op} \longrightarrow \mathrm{CAlg}$$

$$X \longmapsto \mathcal{O}(X) = \text{alg of functions on } X$$

$\rightarrow \mathrm{Obs}^{cl} : \mathrm{Open}(M) \longrightarrow \mathrm{CAlg}$

$U \longmapsto \mathcal{O}(\mathrm{Sol}(U))$

Rmk: • This is a precosheaf

• U, V are disjoint

$\Rightarrow \mathrm{Sol}(U \sqcup V) \simeq \mathrm{Sol}(U) \times \mathrm{Sol}(V)$

$\Rightarrow \mathrm{Obs}^{cl}(U \sqcup V) \simeq \mathrm{Obs}^{cl}(U) \otimes \mathrm{Obs}^{cl}(V)$

factors!

• We might hope Obs^{cl} is a factorization alg.

the details depend on what Spaces, CAlg, and $\mathcal{O}$

In the BV formalism for classical theories, Obs^{cl} is a dg commutative alg with a 1-shifted Poisson bracket

For perturbative stuff, Obs^{cl} is a fact alg.

In BV formalization, quantization means

$$\underset{\substack{\circlearrowright\\ Obs^{cl}}}{d^{cl}} \rightsquigarrow \underset{\substack{\circlearrowright\\ Obs^{cl}[\![\hbar]\!]}}{d^{q}} = d^{cl} + \underbrace{\hbar\Delta + \hbar\{I^{q}, -\}}_{\text{quantum modifications}}$$

This deformation should respect the <u>support</u> of observables:

$$f \in Obs(U) \rightsquigarrow d^{q} f \in Obs(U)$$

<u>Rmk</u>: To construct deformation d^q in practice, requires <u>renormalization</u>

<u>Thm</u> (Costello – G.)

For classical BV theories where linearized EoM is elliptic, the classical observables Obs^{cl} form a commutative fact. alg on M

<u>If</u> a BV quantization exists, then the quantum observables Obs^q form a fact. algebra s.t.

$$Obs^{q} \bmod \hbar \cong Obs^{cl}.$$