Model Orbits and Unipotent Representations

Jing-Song Huang

CUHK, Shenzhen

Workshop

Minimal Representations and Theta Correspondence

ESI, Vienna, April 11-15, 2022

Outline

- Model orbits
- Model unipotent ideals
- Model unipotent representations
- Classification for split groups
  - Simply laced groups (type A, D, E) and G_2
  - Symplectic groups (type C)
  - Orthogonal groups (type B)
  - F_4

This is joint work with Lucas Mason-Brown.

Our results are inspired by the work of Gordan Savin and his friends.

Model nilpotent orbit

$\mathfrak{g}$: a simple Lie algebra/$\mathbb{C}$.

Simply connected Lie group $\mathbf{G}$ with $\mathrm{Lie}(\mathbf{G}) = \mathfrak{g}$.

$G = \mathbf{G}(\mathbb{R})$ the split real form. K a max'l compact subgroup of G.

There is a unique nilpotent (co)adjoint orbit O_{mod}, called the model orbit:

1) O_{mod} is spherical (Borel subgroup has an open orbit);
2) $\overline{O}_{\text{mod}}$ contains all spherical nilpotent orbit.

Partitions for classical and dimensions for exceptional of O_{mod}:

A_{2k-1}	A_{2k}	B_{2k}	B_{2k+1}	C_n	D_{2k}	D_{2k+1}
$[2^k]$	$[2^k,1]$	$[3,2^{2k-2},1^2]$	$[3,2^{2k}]$	$[2^n]$	$[3,2^{2k-2},1]$	$[3,2^{2k-2},1^3]$

G_2	F_4	E_6	E_7	E_8
8	28	40	70	128

Ref. McGovern, Comm. Algebra, 1994.

Maximal primitive ideal

$U(\mathfrak{g})$ the universal enveloping algebra with center $Z(\mathfrak{g})$.

Given $\chi\colon Z(\mathfrak{g}) \to \mathbb{C}$ an infinitesimal character.

Denote by J_χ the maximal ideal with infl. char. χ.

An alternative way to define the model orbit is

$$\overline{O}_{\mathrm{mod}} = AV(J_{\frac{1}{2}\rho}).$$

Set $Q = U(\mathfrak{g})/J_{\frac{1}{2}\rho}$. Then under adjoint action

$$Q = \bigoplus_{\mu \in \Lambda_r^d} V_\mu,$$

where Λ_r^d is set of dominant weights in root lattice s.t. $V_\mu \cong V_\mu^*$.

A Theorem of Loke-Savin and its extension

Theorem (Loke-Savin). The inclusion of $\mathfrak{k} \subset \mathfrak{g}$ induces an algebra isomorphism

$$t\colon U(\mathfrak{k})^K \to Q^K = U(\mathfrak{g})^K / J^K.$$

In particular, Q^K is commutative.

Ref. Loke-Savin, IMRN, 2012.

Extension. Let J_χ be a maximal ideal with infl. char. χ, s.t.

$$AV(J_\chi) = \overline{O}_{\mathrm{mod}}.$$

Set $Q_\chi = U(\mathfrak{g})/J_\chi$. Then we still have an algebra isomorphism

$$t\colon U(\mathfrak{k})^K \to Q_\chi^K.$$

Corollary. This implies that any two representations with the same annihilator J_χ and a same K-type are isomorphic.

The unitary dual of a reductive Lie group G

A central problem in representation theory is to classifying the equivalence classes of irreducible unitary representations of G.

The orbit method suggests a correspondence between irreducible unitary representations of G and orbits for G in $\mathfrak{g}_{\mathbb{R}}^*$

$$\{G - \text{orbits in } \mathfrak{g}_{\mathbb{R}}^*\} \rightsquigarrow \{\text{Irreducible unitary repns of } G\}$$

- One expects a finite set of irreducible unitary representations of G corresponding to the nilpotent co-adjoint G-orbits.
- They have a name—'*unipotent representations*'—but not yet a completely satisfactory definition.
- Properly defined unipotent representations form the building blocks of all irreducible unitary representations.

Ref. Vogan's 1986 Hermann Weyl Lectures notes.

Unipotent representations for $G(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be the finite field with q elements, let G be a connected reductive algebraic group defined over $\mathbb{F}_q$, and let $G(\mathbb{F}_q)$ be its $\mathbb{F}_q$-rational points.

In 1976, Deligne and Lusztig defined the notion of a unipotent representation of $G(\mathbb{F}_q)$ (geometric and case-free).

In 1984, Lusztig completed the classification of irreducible finite-dimensional representations of $G(\mathbb{F}_q)$, in particular,

1. The classification of all irreducible finite-dimensional representations of $G(\mathbb{F}_q)$ can be reduced to the classification of the unipotent representations, and
2. The unipotent representations are classified by certain geometric data related to the nilpotent co-adjoint orbits for the complex group associated to G.

Unipotent representations for real reductive G

The analogy between representations of finite groups of Lie type and reductive Lie groups suggests that the unitary dual is built over a finite set of building blocks parameterized by nilpotent co-adjoint orbits.

- The problem of correctly defining and classifying unipotent representations is one of central importance in the subject.
- Classifying the irreducible unitary representations of real reductive groups by construction from the unipotent representations.
- The solution would have major implications for representation theory and the Langlands program.

Unipotent ideals

Given a nilpotent orbit $\mathbb{O} \subset \mathfrak{g}^*$.

What are the primitive ideals J, s.t. $AV(J) = \overline{\mathbb{O}}$?

Beauville showed that Spec($\mathbb{C}[\mathbb{O}]$) has symplectic singularities.
Then the Namikawa space and Weyl group admit a Lie-theoretic description.
The canonical quantization $\mathcal{A}_0$ of Spec($\mathbb{C}[\mathbb{O}]$)
is **G**-equivariant and there is a uniquely defined co-moment map

$$\Phi_0 : U(\mathfrak{g}) \to \mathcal{A}_0.$$

Taking the kernel of Φ_0, we get a primitive ideal $J_0(\mathbb{O}) = \operatorname{Ker} \Phi_0$.

If we replace $\mathbb{O}$ with a **G**-equivariant cover $\widetilde{\mathbb{O}} \to \mathbb{O}$,
all of this remains true, and we get a primitive ideal

$$J_0(\widetilde{\mathbb{O}}) \subset U(\mathfrak{g}).$$

Ref. Losev, Transf. Groups 2021.

Losev, Mason-Brown, Matvieievskyi

Definition. Let $O \subset \mathfrak{g}^*$ be a nilpotent $\mathbf{G}$-orbit.
Let $\widetilde{O} \to O$ be a $\mathbf{G}$-equivariant cover.
The *unipotent ideal* attached to $\widetilde{O}$ is the primitive ideal

$$J_0(\widetilde{O}) \subset U(\mathfrak{g}).$$

The *unipotent infinitesimal character* attached to $\widetilde{O}$ is the infinitesimal character

$$\lambda_0(\widetilde{O}) \text{ for } J_0(\widetilde{O}).$$

They have determined all unipotent infinitesimal characters.

Harish-Chandra bimodules associated with the model orbits

Example. (Losev, Mason-Brown and Matvieievsky)
Let $\mathbf{G} = Sp(2n, \mathbb{C})$. *Then*

(i) *There is one unipotent Harish-Chandra bimodule attached to* O_{mod}. *It is parabolically induced from the trivial representation of the Segal parabolic.*

(ii) *There are two unipotent Harish-Chandra bimodules attached to* $\widetilde{O}_{mod}$. *One (the spherical) is the midpoint of the complementary series. The other (the anti-spherical) is unitarily induced from a nontrivial character of the Segal parabolic.*

Model unipotent infinitesimal characters

G	$\lambda_0(O_{\mathrm{mod}})$	$\lambda_0(\widetilde{O}_{\mathrm{mod}})$
A_{2n-1}	$\frac{1}{2}(n-1, n-1, n-3, n-3, \ldots, 1-n, 1-n)$	$\frac{1}{2}\rho$
A_{2n}	$\frac{1}{2}\rho$	no cover
B_{2n}	$\frac{1}{2}(2n-1, 2n-1, \ldots, 1, 1)$	$\frac{1}{2}(2n-1, 2n-1, \ldots, 1, 1)$
B_{2n+1}	$(n, n, n-1, n-1, \ldots, 1, 1, 0)$	$\frac{1}{2}(2n+1, 2n-1, 2n-1, \ldots, 1, 1)$
C_{2n}	$\frac{1}{2}(2n-1, 2n-1, 2n-3, 2n-3, \ldots, 1, 1)$	$(n, n-1, n-1, \ldots, 1, 1, 0)$
C_{2n+1}	$(n, n, n-1, n-1, \ldots, 1, 1, 0)$	$\frac{1}{2}(2n+1, 2n-1, 2n-1, \ldots, 1, 1)$
D_n	$\frac{1}{2}\rho$	$\frac{1}{2}\rho$

All exceptional groups: $\frac{1}{2}\rho$, except for F_4: $(1, 0, 1, 0)$.

Model unipotent representations

G: the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$.

$G = \mathbf{G}(\mathbb{R})$: the split real form of **G**.

$\widetilde{G}$: the two-fold nonlinear covering group of G.

Definition. *A unipotent representation of $\widetilde{G}$ attached to O_{mod} is an irreducible representation M of $\widetilde{G}$ such that*

(i) *M is unitary.*

(ii) *The annihilator of M is one of the unipotent ideals $J_0(\widetilde{O}_{mod})$.*

Remark. By a theorem of Vogan, Codim ≥ 2 condition implies that the associated variety of M is closure of a single $K_{\mathbb{C}}$-orbit.

Ref. Vogan, Associated Varieties and Unipotent Representations, 1991.

Model unipotent representations: type C

Note there are two unipotent ideals for these groups.
The interesting one comes from $\widetilde{O}_{\mathrm{mod}}$.

Theorem. (Huang and Mason-Brown) *The following are true:*

(i) *If n even, there are exactly $4n$ model unipotent representations of $Sp(2n,\mathbb{R})$ with annihilator $J_0(\widetilde{O}_{mod})$. All irreducible representations of $Sp(2n,\mathbb{R})$ with this annihilator are unitary and are obtained as theta-lifts of finite-dimensional unitary chbaracters of $O(p,q)$ with $p+q=n$.*

(ii) *If n is odd, there are no model unipotent representations of $Sp(2n,\mathbb{R})$ with annihilator $J_0(\widetilde{O}_{mod})$. There are exactly $4n$ model unipotent representations of $Mp(2n,\mathbb{R})$ with this annihilator. All irreducible representations of $Mp(2n,\mathbb{R})$ with this annihilator are unitary and are obtained as theta-lifts of unitary characters of $O(p,q)$ with $p+q=n$.*

Model unipotent representations: simply laced case and G_2

Suppose $\mathbf{G}$ is simply laced or G_2.
The model *genuine* unipotent representations have infl. char $\frac{1}{2}\rho$.

- A_{2n-1}: 4; A_{2n}: 1.
- G_2: 1
- E_6: 1
- E_7: 4
- E_8:1
- D_{2n}: 16, D_{2n+1}: 4.

They are lifted from the trivial of the linear group.

Ref. Tsai, IMRN, 2022.

Model unipotent representations: type B

Suppose **G** is of type B.

Let Spin(2n+1,2n) denote the connected and simply connected group.

The model *genuine* unipotent representations:

- B_{2n}, Spin(2n+1,2n): 8
- B_{2n+1}, Spin(2n+1,2n+2): 4

They are restrictions from the model unipotent representations of Spin(2n+1,2n+1) and Spin(2n+2,2n+2),

which are in turn obtained from restriction of the minimal representations of Spin(2n+1,2n+2) and Spin(2n+3,2n+2).

Ref. Loke-Savin, AJM 2008; Barbasch-Tsai, J. Lie Theory, 2021.

Model unipotent representations: F_4

Suppose **G** is of type F_4. The split real form is $G = F_{4(4)}$.

The model unipotent representations have infl. char $(1, 0, 1, 0)$.

The model orbits O_{mod} has 3 real forms.

The model unipotent representations:

- linear G: 3 (Atlas)
- nonlinear $\widetilde{G}$: 3

Remark By a theorem of Leung-Yu,
the codim ≥ 3 condition implies that $\#$ repns $= \#$ orbit data.

Ref. Leung-Yu, Duke, 2021.

Thank You!

Hi Gordan, Happy Birthday!