W-algebras and Bethe ansatz in 2d CFT

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July 25, 2024

ESI - Exactly Solvable Models

Co-funded by the European Union

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CZ.02.01.01/00/22_008/0004632

XXX $\mathfrak{su}(2)$ spin chain Bethe equations (Bethe, 1931)

$$1 = q \prod_{l=1}^{N} \frac{u_j + a_l - \epsilon}{u_j + a_l} \prod_{k \neq j} \frac{u_j - u_k + \epsilon}{u_j - u_k - \epsilon}$$

$$\Downarrow$$

CFT/$\mathcal{W}_{1+\infty}$ Bethe equations $\sim$ 2013

$$1 = q \prod_{l=1}^{N} \frac{u_j + a_l - \epsilon_3}{u_j + a_l} \prod_{k \neq j} \frac{(u_j - u_k + \epsilon_1)(u_j - u_k + \epsilon_2)(u_j - u_k + \epsilon_3)}{(u_j - u_k - \epsilon_1)(u_j - u_k - \epsilon_2)(u_j - u_k - \epsilon_3)}$$

(Nekrasov, Shatashvili 2009; Litvinov 2013; Bonelli, Sciarappa, Tanzini, Vaško 2014; Feigin, Jimbo, Miwa, Mukhin 2016; Kozlowski, Sklyanin, Torrielli 2016)

Bethe equations for gaussian matrix model (Hermite polynomials)

$$0 = x_j + \sum_{k \neq j} \frac{1}{x_j - x_k}$$

$$\Downarrow$$

Bethe equations for free boson (Wronskian Hermite polynomials)

$$0 = x_j + \frac{\ell(\ell+1)}{x_j^3} + \sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

$$\Downarrow$$

BLZ equations for Virasoro algebra ($\mathbb{Z}_{N+2}$ orbifold of ...)

$$0 = \frac{N}{2} x_j^{N-1} + \frac{\ell(\ell+1)}{x_j^3} + \sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

Overview

- $\mathcal{W}$-algebras and $\mathcal{W}_\infty$
- affine Yangian
- integrable structure - KdV and BLZ
- instanton R-matrix and ILW Bethe equations

W algebras - motivation

$\mathcal{W}$-algebas: extensions of the Virasoro algebra (2d CFT) by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV/KP) $\rightsquigarrow$ $\mathcal{W}$ is quant. KP
- (old) matrix models
- instanton partition functions and AGT
- holographic dual of 3d higher spin gravity
- quantum Hall effect
- topological strings (topological vertex..)
- $4d$ $\mathcal{N} = 4$ SYM at codimension 2 junction of three codimension 1 defects (Gaiotto, Rapčák)
- superconformal indices of 4d $\mathcal{N} = 2$ SCFTs
- geometric representation theory (equivariant cohomology of moduli spaces of instantons)

Zamolodchikov $\mathcal{W}_3$ algebra

$\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

together with spin 3 primary field $W(w)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + reg.$$

To close the algebra we need to find the OPE of W with itself consistent with associativity (Jacobi, crossing symmetry...).

The result:

$$\begin{aligned} W(z)W(w) \quad \sim \quad & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ & + \frac{1}{(z-w)^2}\left(\frac{32}{5c+22}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right) \\ & + \frac{1}{z-w}\left(\frac{16}{5c+22}\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right) + reg. \end{aligned}$$

Λ is a quasiprimary ‘composite’ (spin 4) field,

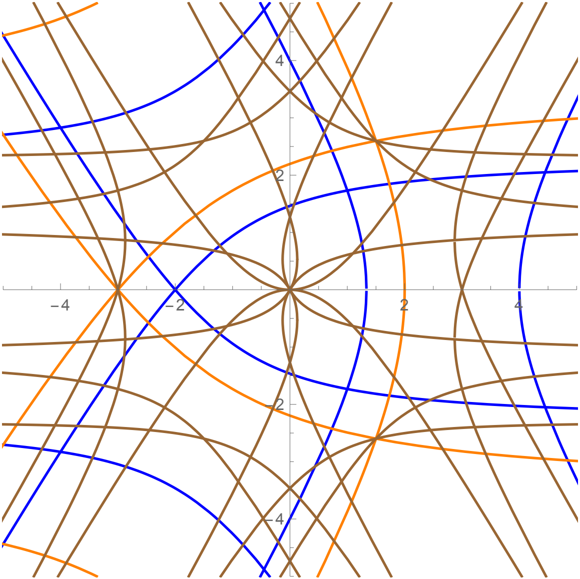
$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z).$$

The algebra is non-linear, not a Lie algebra in the usual sense

W_N series and $\mathcal{W}_\infty$ algebra

- W_N: an interesting family of W-algebras associated to $\mathfrak{sl}(N)$ Lie algebras (spins $2, 3, \ldots, N$, Virasoro $\leftrightarrow \mathfrak{sl}(2)$)
- $\mathcal{W}_\infty$: interpolating algebra for W_N series; spins $2, 3, \ldots$
- Gaberdiel-Gopakumar: solving associativity conditions for this field content $\rightsquigarrow$ two-parameter family: central charge c and rank parameter λ (proof cf. Andy Linshaw)
- choosing $\lambda = N \to$ truncation of $\mathcal{W}_\infty$ to $\mathcal{W}_N = \mathcal{W}[\mathfrak{sl}(N)]$, i.e. $\mathcal{W}_\infty$ is interpolating algebra for the whole $\mathcal{W}_N$ series
- adding spin 1 field, we have $\mathcal{W}_{1+\infty} \rightsquigarrow$ *many simplifications*
- **triality** symmetry of the algebra (Gaberdiel & Gopakumar) $\mathcal{W}_\infty[c, \lambda_1] \simeq \mathcal{W}_\infty[c, \lambda_2] \simeq \mathcal{W}_\infty[c, \lambda_3]$

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \qquad c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)$$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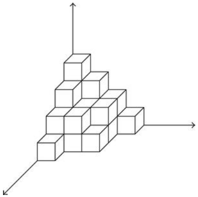
- MacMahon function as vacuum character of the algebra (enumerating all the local fields in the algebra)

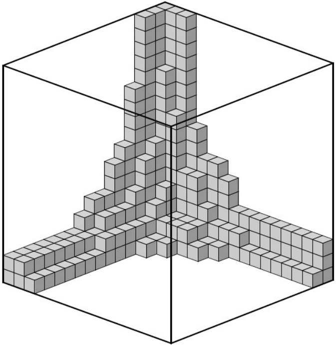
$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \cdots$$

- The same generating function is well-known to count the plane partitions (3d Young diagrams)

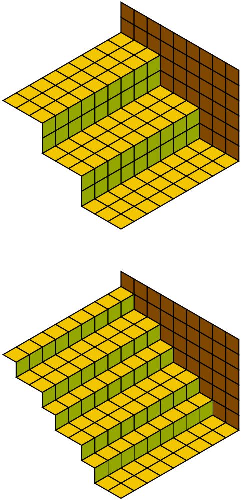


- triality acts by permuting the coordinate axes
- restriction to $\mathcal{W}_N$ corresponds to max N boxes in one of the directions

- this can be generalized to degenerate primaries (not only vacuum rep) by allowing 2d Young diagram asymptotics
- counting exactly as in topological vertex $\rightsquigarrow$ topological vertex can be interpreted as being a character of degenerate $\mathcal{W}_{1+\infty}$ representations



- box counting generalizes also to $\mathcal{W}_N$ minimal models (Ising...)
  $\rightsquigarrow$ periodic configurations - lozenge tilings on cylinder



The OPEs in $\mathcal{W}_N$ or $\mathcal{W}_{1+\infty}$ are very complicated, but can be written for all spins in terms of bilocal quantities (2014)

$$\begin{aligned}
U_3(z)U_5(w) \quad \sim \quad & \frac{1}{(z-w)^7}\Big(\frac{1}{2}\alpha(n-3)(n-2)(n-1)n\left(4\alpha^2\left(\alpha^2(n(5n-9)+1)-3n+4\right)+1\right)\mathbb{1}\Big) \\
& +\frac{1}{(z-w)^6}\Big(\frac{1}{6}(n-3)(n-2)(n-1)\left(6\alpha^4 n(2n-3)+\alpha^2(10-9n)+1\right)U_1(w)\Big) \\
& +\frac{1}{(z-w)^5}\Big(-\alpha(n-3)(n-2)(n-1)\left(-4\alpha^2+3\alpha^2 n-1\right)(U_1U_1)(w) \\
& +\alpha(n-3)(n-2)\left(4\alpha^2n^2-4\alpha^2 n-n-2\right)U_2(w) \\
& -\frac{1}{2}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1'(w)\Big) \\
& +\frac{1}{(z-w)^4}\Big(-\alpha(n-3)(n-2)(n-1)\left(\alpha^2(3n-4)-1\right)(U_1'U_1)(w) \\
& -\frac{1}{2}(n-3)(n-2)\left(2\alpha^2(n-1)-1\right)(U_1U_2)(w) \\
& +(n-3)\left(\alpha^2\left(n^2+2\right)-3\right)U_3(w) \\
& -\frac{1}{4}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1''(w) \\
& +\alpha(n-3)(n-2)\left(\alpha^2(n-1)n-1\right)U_2'(w)\Big) \\
& +\cdots
\end{aligned}$$

¿ Is there a better way to organize the algebra ?

Yangian of $\widehat{\mathfrak{gl}(1)}$

The Yangian of $\widehat{\mathfrak{gl}(1)}$ (Arbesfeld-Schiffmann-Tsymbaliuk) is an associative algebra with generators $\psi_j, e_j, f_j, j \geq 0$ and relations

$$\begin{aligned}
0 &= [e_{j+3}, e_k] - 3\,[e_{j+2}, e_{k+1}] + 3\,[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\
&\quad +\sigma_2\,[e_{j+1}, e_k] - \sigma_2\,[e_j, e_{k+1}] - \sigma_3\,\{e_j, e_k\} \\
0 &= [f_{j+3}, f_k] - 3\,[f_{j+2}, f_{k+1}] + 3\,[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}] \\
&\quad +\sigma_2\,[f_{j+1}, f_k] - \sigma_2\,[f_j, f_{k+1}] + \sigma_3\,\{f_j, f_k\} \\
0 &= [\psi_{j+3}, e_k] - 3\,[\psi_{j+2}, e_{k+1}] + 3\,[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\
&\quad +\sigma_2\,[\psi_{j+1}, e_k] - \sigma_2\,[\psi_j, e_{k+1}] - \sigma_3\,\{\psi_j, e_k\} \\
0 &= [\psi_{j+3}, f_k] - 3\,[\psi_{j+2}, f_{k+1}] + 3\,[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}] \\
&\quad +\sigma_2\,[\psi_{j+1}, f_k] - \sigma_2\,[\psi_j, f_{k+1}] + \sigma_3\,\{\psi_j, f_k\} \\
0 &= [\psi_j, \psi_k] \\
\psi_{j+k} &= [e_j, f_k]
\end{aligned}$$

'initial/boundary conditions'

$$[\psi_0, e_j] = 0, \qquad [\psi_1, e_j] = 0, \qquad [\psi_2, e_j] = 2e_j,$$
$$[\psi_0, f_j] = 0, \qquad [\psi_1, f_j] = 0, \qquad [\psi_2, f_j] = -2f_j$$

and finally the Serre-like relations

$$0 = \mathrm{Sym}_{(j_1,j_2,j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]], \quad 0 = \mathrm{Sym}_{(j_1,j_2,j_3)} [f_{j_1}, [f_{j_2}, f_{j_3+1}]].$$

Parameters $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{C}$ constrained by $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and

$$\sigma_2 = \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3$$
$$\sigma_3 = \epsilon_1\epsilon_2\epsilon_3.$$

We have both commutators and anticommutators in defining quadratic relations (but no $\mathbb{Z}_2$ grading) - for $\sigma_3 \neq 0$ not a Lie (super)-algebra.

Introducing generating functions (Drinfel'd currents)

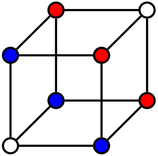
$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \quad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \quad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

the first set of formulas above (almost!) simplify to

$$e(u)e(v) \sim \varphi(u-v)e(v)e(u), \quad f(u)f(v) \sim \varphi(v-u)f(v)f(u),$$
$$\psi(u)e(v) \sim \varphi(u-v)e(v)\psi(u), \quad \psi(u)f(v) \sim \varphi(v-u)f(v)\psi(u)$$

with rational *structure function* (scattering phase in BAE)

$$\boxed{\varphi(u) = \frac{(u+\epsilon_1)(u+\epsilon_2)(u+\epsilon_3)}{(u-\epsilon_1)(u-\epsilon_2)(u-\epsilon_3)}}$$



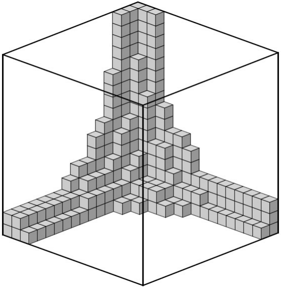
The representation theory of the algebra is much simpler in this Yangian formulation and φ controls basically everything

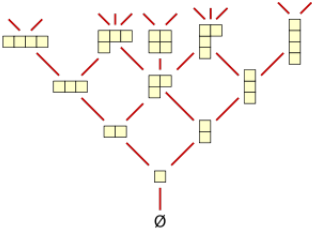
$\psi(u), e(u)$ and $f(u)$ in representations act like

$$\psi(u)\,|\Lambda\rangle = \psi_0(u) \prod_{\square\in\Lambda} \varphi(u-\epsilon_\square)\,|\Lambda\rangle$$

$$e(u)\,|\Lambda\rangle = \sum_{\square\in\Lambda^+} \frac{E(\Lambda\to\Lambda+\square)}{u-\epsilon_\square}\,|\Lambda+\square\rangle$$

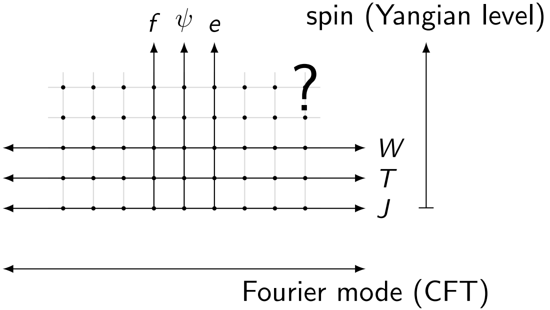
where the states $|\Lambda\rangle$ are associated to geometric configurations of boxes (plane partitions,...) and where $\epsilon_\square = \sum_j \epsilon_j x_j(\square)$ is the weighted geometric position of the box.





Two different descriptions of the algebra:

- *CFT/VOA* point of view with *horizontal* local fields $J(z), T(z), W(z), \ldots$ with increasingly complicated OPE as we go to higher spins
- *Yangian* point of view (Arbesfeld-Schiffmann-Tsymbaliuk) where all the spins are included in the *vertical* generating functions $\psi(u), e(u)$ and $f(u)$ but accessing higher Fourier modes is more difficult
- *Shuffle algebra* allows us to access both but only *half of the algebra*, some analogies to Moyal product



Integrable structures

- there are two natural distinct infinite families of commuting quantities/Hamiltonians/IOM (Cartan subalgebras):
  1. the Yangian (Bejamin-Ono) family of generators ψ_j
  2. the family of *local* conserved charges (BLZ, quantum KdV)
- the Yangian charges are very easy to diagonalize and their spectrum is determined by combinatorics of plane partitions
- the diagonalization of the local commuting quantities on the other hand is natural from CFT point of view, but is quite non-trivial and has a long history
- we can relate these two families by constructing a family of quantum ILW Hamiltonians that *interpolate* between these two families (Litvinov; Feigin,Jimbo,Miwa,Mukhin) - analogous to inequivalent choices of Cartan subalgebras

Classical KdV/KP

- in the classical limit, the theory reduces to the theory of integrable hierarchies of PDEs (KdV, KP)
- the classical object associated to Virasoro algebra is the one-dimensional Schrödinger operator

$$L^2 = \partial_x^2 + u(x)$$

- there exists an infinite dimensional family of continuous deformations of $u(x)$ which preserve the spectrum of L^2 and are organized into commuting flows
- the first such deformation is the trivial rigid translation of the potential

$$\partial_{t_1} u = \partial_x u$$

- the next one is already rather non-trivial and is captured by the Korteweg-de-Vries equation (Boussinesq 1877)

$$4\partial_{t_3} u = 6u\partial_x u + \partial_x^3 u.$$

- the space of Schödinger potentials is a Hamiltonian system if we equip it with Poisson bracket

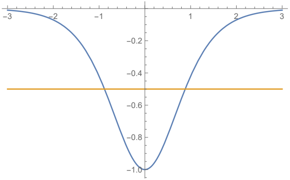
$$\{u(x), u(y)\} = -\delta'''(x-y) - 4u(x)\delta'(x-y) - 2u'(x)\delta(x-y)$$

(whose Fourier transform is just the classical Virasoro algebra)

- the deformations are generated by local Hamiltonians which are at the same time conserved quantities capturing the spectral data of the family of Schrödinger operators

$$I_1 = \int u(x)dx, \qquad I_3 = \int u^2(x)dx, \quad \ldots$$

- e.g. KdV soliton (Pöschl-Teller potential) with a single bound state



- the KdV conserved charges survive quantization in the form

$$
\begin{aligned}
I_1 &= \int T(x)dx = L_0 - \frac{c}{24} \\
I_3 &= \int (TT)(x)dx = L_0^2 + 2\sum_{m=1}^{\infty} L_{-m}L_m - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880} \\
I_5 &= \int \left[(T(TT))(x) - \frac{c-2}{12}(\partial T \partial T)(x) \right] dx = \ldots
\end{aligned}
$$

  so it makes sense to ask what their spectrum is
- since L_0 is part of the family, the problem is to diagonalize finite dimensional matrices level by level
- a surprising description of their spectrum was found by Bazhanov-Lukyanov-Zamolodchikov (in the context ODE/IM correspondence initiated by Dorey and Tateo)

- I_3 for $c = -22/5, \Delta = -1/5$ and Virasoro level 5:

$$I_3 = \begin{pmatrix} -18 & -\frac{48}{5} & -36 & -\frac{108}{5} & -\frac{144}{5} & -\frac{336}{5} & -288 \\ 30 & 12 & 54 & \frac{108}{5} & 24 & \frac{144}{5} & 0 \\ 30 & 0 & 48 & -\frac{12}{5} & -\frac{48}{5} & -\frac{48}{5} & 0 \\ 0 & 24 & 6 & \frac{192}{5} & 48 & \frac{288}{5} & 144 \\ 0 & 12 & 18 & 0 & \frac{216}{5} & \frac{24}{5} & 0 \\ 0 & 0 & 0 & 18 & 12 & \frac{258}{5} & 96 \\ 0 & 0 & 0 & 0 & 0 & 6 & 60 \end{pmatrix}$$

- I_5 for $c = -22/5, \Delta = -1/5$ and Virasoro level 5:

$$I_5 = \begin{pmatrix} -\frac{351760411}{756000} & -156 & -531 & -351 & -\frac{2412}{5} & -\frac{5436}{5} & -4536 \\ \frac{825}{2} & \frac{66685589}{756000} & \frac{1233}{2} & \frac{1053}{5} & 330 & \frac{1476}{5} & 0 \\ \frac{715}{2} & -48 & \frac{256441589}{756000} & -\frac{567}{5} & -\frac{1244}{5} & -\frac{1716}{5} & -864 \\ 75 & 334 & \frac{657}{2} & \frac{362583989}{756000} & 840 & \frac{5304}{5} & 2988 \\ 75 & 141 & \frac{783}{2} & 54 & \frac{271259189}{756000} & \frac{414}{5} & 0 \\ 0 & 60 & 15 & \frac{513}{2} & 327 & \frac{539563589}{756000} & 2016 \\ 0 & 0 & 0 & 15 & 0 & \frac{207}{2} & \frac{540243989}{756000} \end{pmatrix}$$

- not hermitian - expressed in non-orthogonal basis $L_{-\lambda} |0\rangle$
- null states not removed (laziness)

- consider a Schrödinger operator

$$\partial_z^2 + z^N + u + \frac{\ell(\ell+1)}{z^2}$$

associated to a CFT primary state (central charge c and conformal dimension Δ are encoded in N and ℓ) and dress it by allowing for additional collection of regular singular points

$$\sum_{j=1}^{M(N+2)} \left(\frac{2}{(z-z_j)^2} + \frac{\gamma_j}{z(z-z_j)} \right)$$

(M is the Virasoro level)

- the requirement of trivial monodromy around these singularities leads to a system of BLZ Bethe equations

$$0 = N x_j^{N-1} + \frac{2\ell(\ell+1)}{x_j^3} + \sum_{k \neq j} \frac{4}{(x_j - x_k)^3}$$

in coordinates convenient for AD spectral curve (Fioravanti)

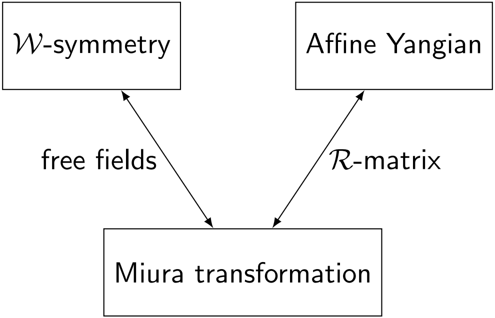
- given any solution of BLZ Bethe equations, the eigenvalues of I_j are determined, for instance

$$I_3 = \frac{(N-3)(N-1)(2N+3)}{1920(N+2)} - \frac{(-\ell(\ell+1)-2(N+2)M)^2}{8(N+2)} - \frac{(N+1)\left(-\ell(\ell+1)-2(N+2)M\right)}{16(N+2)} + \frac{N}{2}\sum_j x_j^{N+2}$$

- how to find these expressions for I_j in terms of Bethe roots? ODE/IM: using WKB! (Kudrna-Prochazka 2024?)
- generalization to higher ranks? (only $\mathcal{W}_3$ known explicitly)
- how are these local Hamiltonians related to Yangian conserved quantities?

Miura transformation and $\mathcal{R}$-matrix

the construction of interpolating family of Hamiltonians proceeds via algebraic Bethe ansatz starting from *instanton R-matrix* (Maulik-Okounkov 2012, Smirnov 2013, Matsuo-Zhu 2015)



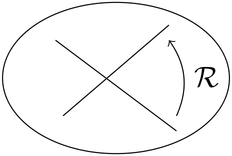
- consider the following factorization of N-th order differential operator

$$(\partial + \partial\phi_1(z)) \cdots (\partial + \partial\phi_N(z)) = \sum_{j=0}^{N} U_j(z) \partial^{N-j}$$

with N commuting free fields $\partial\phi_j(z)\partial\phi_k(w) \sim \delta_{jk}(z-w)^{-2}$

- OPEs of U_j generate $\mathcal{W}_N$ and furthermore are quadratic
- $\mathcal{W}_N \leftrightarrow$ quantization of N-th order differential operators
- the embedding of $\mathcal{W}_N$ in the bosonic Fock space depends on the way we order the fields on the LHS
- Maulik-Okounkov: $\mathcal{R}$-matrix as intertwiner between two embeddings, $\mathcal{R} : \mathcal{F}^{\otimes 2} \to \mathcal{F}^{\otimes 2}$

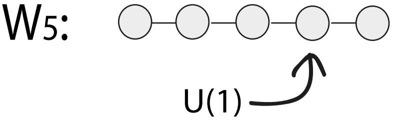
$$(\partial + \partial\phi_1)(\partial + \partial\phi_2) = \mathcal{R}^{-1}(\partial + \partial\phi_2)(\partial + \partial\phi_1)\mathcal{R}$$



- $\mathcal{R}$ defined in this way satisfies the Yang-Baxter equation (two ways of reordering $321 \to 123$)

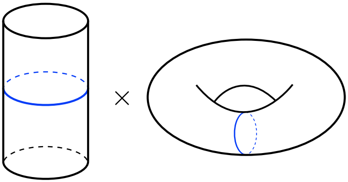
$$\mathcal{R}_{12}(u_1 - u_2)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{23}(u_2 - u_3) = \\ = \mathcal{R}_{23}(u_2 - u_3)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{12}(u_1 - u_2)$$

- the spectral parameter u - the global $GL(1)$ charge
- $\mathcal{R}$-matrix satisfies YBE $\rightsquigarrow$ apply the algebraic Bethe ansatz



- spin chain of length N $\rightsquigarrow$ associated symmetry algebra $\widehat{\mathfrak{gl}}(1) \times \mathcal{W}_N$

- a given $\mathcal{R}$-matrix describes a consistent coupling to a probe, in our case this is another CFT
- consider an *auxiliary* Fock space $\mathcal{F}_A$ and a *quantum* space $\mathcal{F}_Q \equiv \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_N$
- we associate to this the monodromy matrix $\mathcal{T}_{AQ} : \mathcal{F}_A \otimes \mathcal{F}_Q \to \mathcal{F}_A \otimes \mathcal{F}_Q$ defined as $\mathcal{T}_{AQ} = \mathcal{R}_{A1} \cdots \mathcal{R}_{AN}$
- in the usual algebraic Bethe ansatz the next step is to take the trace over the auxiliary space
- since our auxiliary spaces are infinite dimensional Fock spaces, we have to regularize the trace, $\mathcal{H}_q(u) = \mathrm{Tr}_A\, q^{L_{A,0}} \mathcal{T}_{AQ}(u)$



- this leads for every q to a different infinite family of commuting Hamiltonians (which can be identified with the Hamiltonians of quantum *intermediate long wave* equation), the first non-trivial being

$$H_2^q = (\Phi_3)_0 + \sum_{m>0} m \frac{1+q^m}{1-q^m} J_{-m} J_m$$

- interpolates between Yangian/BO Hamiltonians at $q \to 0$, local quantum KP/BLZ Hamiltonians at $q \to 1$ limit and to charge conjugate Yangian/BO Hamiltonians as $q \to \infty$

$$m \frac{1+q^m}{1-q^m} \to \begin{cases} |m| & q \to 0 \\ \frac{2}{1-q} - 1 + \ldots & q \to 1 \end{cases}$$

- these Hamiltonians can be diagonalized by Bethe ansatz equations (Litvinov 2013)

$$1 = q \prod_{l=1}^{N} \frac{u_j + a_l - \epsilon_3}{u_j + a_l} \prod_{k \neq j} \frac{(u_j - u_k + \epsilon_1)(u_j - u_k + \epsilon_2)(u_j - u_k + \epsilon_3)}{(u_j - u_k - \epsilon_1)(u_j - u_k - \epsilon_2)(u_j - u_k - \epsilon_3)}$$

- these equations are the same as in the simplest Heisenberg XXX $SU(2)$ spin chain, except for the fact that the interaction between Bethe roots is now a degree 3 rational function instead of degree 1!
- very rich structure of solutions: capture all the representation theory of Virasoro of $\mathcal{W}_N$ algebras (null states / characters / degenerate primaries / minimal models, ...)

- the parameter q is very natural from various points of view:
  1. the twist parameter from spin chain point of view
  2. encodes the shape (complex structure) of the auxiliary torus
  3. controls the non-locality of the Hamiltonians
  4. serves as a natural homotopy parameter for numerical solution of the equations
- once we solve Bethe ansatz equations, the spectrum of $\mathcal{H}_q(u)$ can be written as

$$\frac{\mathcal{H}_q(u)}{\mathcal{H}_{q=0}(u)} \to \frac{1}{\sum_\lambda q^{|\lambda|}} \sum_\lambda q^{|\lambda|} \prod_{\square \in \lambda} \psi_\Lambda(u - \epsilon_\square + \epsilon_3)$$

where

$$\psi_\Lambda(u) = A(u) \prod_j \varphi(u - x_j)$$

which is a Yangian limit of a formula proven by Feigin, Jimbo, Miwa, Mukhin (2016)

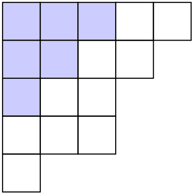
Local limit and free bosons

- the local limit $q \to 1$ is rather singular (actually any q a root of unity!), but in this limit the Heisenberg and $\mathcal{W}_\infty$ contributions to Hamiltonians decouple
- in particular, the Bethe roots associated to $\mathcal{W}_\infty$ remain finite in the $q \to 1$ limit while those associated to Heisenberg subalgebra diverge
- the singular behaviour of Bethe roots as $q \to 1$ must in a subtle way encode the associated Young diagram characterizing the Heisenberg excitation (Kudrna & TP 2024?)
- the Heisenberg Bethe roots have Puiseux expansion

$$u_j \sim \frac{\#}{1-q} + \frac{x_j}{(1-q)^{1/2}} + \frac{y_j}{(1-q)^{1/3}} + \mathcal{O}(1-q)^0$$

- some of the Heisenberg Bethe roots appear in pairs with order 2 monodromy around $q = 1$ while others appear in triples with order 3 monodromy (possibly also a single singlet)

- the distriution of these is determined by the Young diagram together with its two-core



- the two-core is always a triangular number

$$\frac{\ell(\ell+1)}{2}$$

  with is of a form $3k$ or $3k+1$ corresponding to k triples of Heisenberg roots plus possibly one singlet
- for the triplets and singlet we have $x_j = 0$ while for doublets (dominoes) we have Bethe equations

$$0 = x_j + \frac{\ell(\ell+1)}{x_j^3} + \sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

- how to solve these? for single row Young diagrams solved by *zeros* of Hermite polynomials $H_n(x)$ which famously satisfy

$$0 = x_j + \sum_{k \neq j} \frac{1}{x_j - x_k}$$

- solutions of our equations are instead given by zeros of *Wronskian* Hermite polynomials

$$H_\lambda(x) \equiv W[H_{\lambda_1+r-1}(x), H_{\lambda_2+r-2}(x), \ldots, H_{\lambda_{r-1}+1}(x), H_{\lambda_r}(x)]$$

- in our previous example

$$H_\lambda(x) \sim x^6 \left(32x^{10} + 16x^8 + 112x^6 - 168x^4 - 630x^2 + 2205\right)$$

  consistently with $\ell = 3$ and two triples of $x = 0$ and five dominoes

- what about the triple roots and y_j?

- the equations that we get for y_j are simply

$$\sum_{k\neq j} \frac{1}{(y_j - y_k)^3} = 0.$$

- this is missing the confining potential between the roots so there are continuously many moduli
- the space of solutions is the famous Airault–McKean–Moser locus and it exists only if the number of non-trivial y_j is

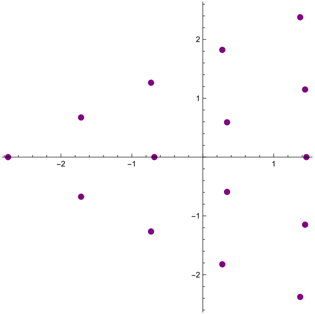
$$\frac{\ell(\ell+1)}{2}$$

- the points in this locus can be conveniently parametrized by using the KdV flows: starting with Schrödinger potential

$$\frac{\ell(\ell+1)}{2x^2},$$

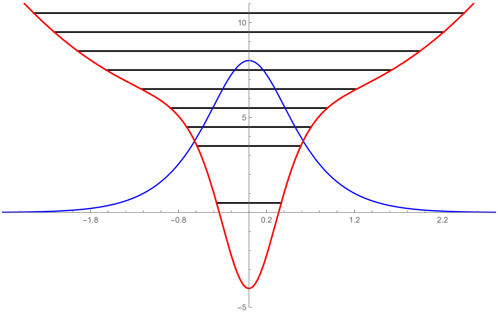
  the poles at the origin under j-th KdV flow expand into regular j-gons. In our case, due to symmetry, only triangles, hexagons, nonagons, ..., icosihenagons, ... are allowed

- unfortunately to straightforwardly decide what is happening the equations have to be solved to very high order... up to 15 roots everything is consistent with simply triangular flow,



- how to find y_j in general? new idea needed...

- to prove the solvability of domino Bethe equations in terms of Wronskian Hermite polynomials, the easiest way is via Bäcklund/Darboux transformations of the quantum harmonic oscillator
- idea: picking an arbitrary energy level, using the 'spectral shift' and 'conjugation by R-matrix' we can remove an arbitrary energy level, while the wave functions and the potential undergo a well-defined local transformation involving only the wave function of the state being removed
- applying sequence of these transformations produces rational deformations of harmonic oscillator labeled by Young diagrams (encoding which energy states we removed) which have no monodromy around the singularities in the potential so need to satisfy the domino Bethe equations



$$-\partial_x^2 + x^2 + \frac{8(2x^2-1)}{(2x^2+1)^2} + 4$$

Questions

- how to understand the Adler-Moser roots? meaning of the flows?
- how are the (solutions of?) ILW and BLZ Bethe ansatz equations related?
- another set of Bethe ansatz equations based on affine Gaudin model (nested BA structure)
- how can the ILW generating function be regularized to extract interesting information in $q \to 1$ limit? qq-characters?
- refined characters & modularity (Dijkgraaf, Maloney-Ng-Ross-Tsiares) & thermodynamic properties and geometrization
- quantum periods, TBA, mirror symmetry in topological string
- elliptic Calogero model (TBA eqns of Nekrasov-Shatashvili)

Thank you

... and HAPPY BIRTHDAY Harald!