Central extensions in infinite dimensions

-tutorial-

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Plan of the talk

1. Neeb's theorem on integrability of Lie algebra extensions

2. Integration via path groups

3. Integration via prequantization extension

Neeb's theorem on integrability of Lie algebra extensions

Lie algebra cohomology

The *continuous Lie algebra cohomology* $H^n_{\mathsf{CE}}(\mathfrak{g})$ of a Fréchet Lie algebra $\mathfrak{g}$ is the cohomology of the complex $(C^n(\mathfrak{g}), d)$ of continuous alternating n-linear maps $\psi \colon \mathfrak{g}^n \to \mathbb{R}$ with differential $d_{\mathsf{CE}} \colon C^n(\mathfrak{g}) \to C^{n+1}(\mathfrak{g})$

$$d_{\mathsf{CE}}\psi(x_0, \ldots, x_n) := \sum_{0 \le i < j \le n} (-1)^{i+j} \psi([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n)$$

$H^1_{\mathsf{CE}}(\mathfrak{g})$ classifies continuous characters of $\mathfrak{g}$

$H^2_{\mathsf{CE}}(\mathfrak{g})$ classifies continuous central extensions of $\mathfrak{g}$ by $\mathbb{R}$

$H^3_{\mathsf{CE}}(\mathfrak{g})$ classifies crossed modules over $\mathfrak{g}$.

Lie algebra 2-cocycles

A Lie algebra 2–cocycle $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ satisfies the cyclic identity

$$\psi([x_1, x_2], x_3) + \psi([x_2, x_3], x_1) + \psi([x_3, x_1], x_2) = 0.$$

It defines a Lie bracket on $\mathfrak{z} \times \mathfrak{g}$:

$$[(c, x), (c', x')] := \Big(\psi(x, x'), [x, x']\Big).$$

The coboundary $\psi = d_{\mathrm{CE}}\alpha$ of linear $\alpha : \mathfrak{g} \to \mathfrak{z}$,

$$\psi(x, x') = -\alpha([x, x']),$$

defines a central extension $\widehat{\mathfrak{g}}$ that is isomorphic to the trivial extension $\mathfrak{z} \times \mathfrak{g}$.

Theorem The continuous Lie algebra cohomology $H^2_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{z})$ parameterizes isomorphism classes of continuous central extensions of $\mathfrak{g}$ by $\mathfrak{z}$.

Central extensions

A *continuous central extension* of $\mathfrak{g}$ by $\mathfrak{z}$ is a (topologically split) exact sequence with $\mathfrak{z}$ is central in $\widehat{\mathfrak{g}}$:

$$0 \to \mathfrak{z} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \to 0$$

A continuous linear splitting $\sigma : \mathfrak{g} \to \widehat{\mathfrak{g}}$ gives a Lie algebra 2-cocycle

$$\psi(x, x') := [\sigma(x), \sigma(x')] - \sigma([x, x']) \in \mathfrak{z}$$

which defines a Lie algebra isomorphic to $\widehat{\mathfrak{g}}$.

A *smooth central extension* of G by Z is exact sequence with Z central in $\widehat{G}$

$$1 \to Z \longrightarrow \widehat{G} \longrightarrow G \to 1$$

that admits smooth local sections s (principal bundle). Group 2-cocycle

$$c(g, g') = s(g)s(g')s(gg')^{-1} \in Z$$

defines a Lie group isomorphic to $\widehat{G}$.

Lie group 2-cocycles

A normalized Lie group 2–cocycle $c : G \times G \to Z$ is smooth in a neighborhood of (e, e), satisfies $c(g, e) = c(e, g) = 1$, and the cocycle condition holds:

$$c(g_1, g_2)c(g_1g_2, g_3) = c(g_2, g_3)c(g_1, g_2g_3).$$

It defines a Lie group structure on $Z \times G$ by

$$(z, g)(z', g') = (zz'c(g, g'), gg'),$$

thus obtaining a central Lie group extension $\widehat{G}$ of G by Z.
The coboundary of $\lambda : G \to Z$

$$c(g, g') = \lambda(g)\lambda(g')\lambda(gg')^{-1},$$

defines a central extension $\widehat{G}$ that is isomorphic to the trivial extension $Z \times G$.

Theorem The Lie group cohomology $H^2(G, Z)$ parameterizes isomorphism classes of smooth central extensions of G by Z (i.e. extensions that admit smooth local sections).

Virasoro-Bott group

The Bott cocycle on $G = \mathrm{Diff}_+(S^1)$

$$c(\varphi, \psi) := \frac{1}{2} \int_{S^1} \log(\varphi \circ \psi)' d \log \psi',$$

with $\varphi' \in C^\infty(S^1)$ defined by $\varphi^*(dx) = \varphi' dx$.
The corresponding Lie algebra cocycle on $\mathfrak{g} = \mathfrak{X}(S^1)$ is the Virasoro cocycle

$$\psi(X, Y) = \int_{S^1} (X'Y'' - X''Y')dx$$

Short exact sequence for $Z = \mathfrak{z}/\Pi$ with $\Pi \subset \mathfrak{z}$ discrete subgroup:

$$H^2(G, Z) \xrightarrow{D} H^2_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{z}) \xrightarrow{P} \mathrm{Hom}(\pi_2(G), Z) \times \mathrm{Hom}(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z}))$$

with $D[c] = [\psi]$ for ψ the corresponding Lie algebra cocycle to c:

$$\psi(X, Y) = d^2_{(e,e)} c(X, Y) - d^2_{(e,e)} c(Y, X).$$

Integrability of Lie algebra extensions

There are two obstructions for the integration of a continuous Lie algebra 2-cocycle ψ on $\mathfrak{g}$ to a smooth Lie group extension of G.
Let ψ^ℓ be the closed left invariant 2-form on G determined by the cocycle ψ.
The *period group* Π_ψ is the image of the period homomorphism

$$\mathrm{per}_\psi : \pi_2(G) \to \mathfrak{z}, \quad \mathrm{per}_\psi([\sigma]) = \int_{S^2} \sigma^* \psi^\ell.$$

The *flux homomorphism* $F_\psi : \pi_1(G) \to \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})$ assigns to each piecewise smooth loop γ in G the linear map

$$X \in \mathfrak{g} \mapsto -\int_\gamma i_{X^r} \psi^\ell \in \mathfrak{z}.$$

Theorem [Neeb'04] The Lie algebra cocycle ψ with discrete period group Π_ψ and vanishing flux homomorphism F_ψ integrates to the Lie group extension

$$1 \to Z = \mathfrak{z}/\Pi_\psi \to \widehat{G} \to G \to 1.$$

Integration via path groups

Van Est method

Let ψ Lie algebra 2-cocycle with discrete period group $\Pi_\psi \subset \mathfrak{z}$ and $\{a_g\}_{g\in G}$ family of smooth paths in G from e to g.

By integrating the closed left invariant 2-form ψ^ℓ, we define

$$C : G \times G \to \mathfrak{z}, \quad C(f,g) := \int_{\Sigma_{f,g}} \psi^\ell,$$

with $\Sigma_{f,g}$ any piecewise smooth 2-simplex in G with boundary $fa_g - a_{fg} + a_f$.

Since $\Theta_{f,g,h} := f\Sigma_{g,h} - \Sigma_{fg,h} + \Sigma_{f,gh} - \Sigma_{f,g}$ is a spherical 2-cycle in G,

$$C(g,h) - C(fg,h) + C(f,gh) - C(f,g) = \int_{\Theta_{f,g,h}} \psi^\ell \in \Pi_\psi.$$

We get a group 2-cocycle $c : G \times G \to Z = \mathfrak{z}/\Pi_\psi$ that integrates ψ.

Central extensions of G/H

The 1-cochain $\lambda : H \to Z$ resolves the 2-cocycle $c : G \times G \to Z$ over the subgroup $H \subset G$ if the restriction of c to H is the coboundary of λ:

$$c(h_1, h_2) = \lambda(h_1)\lambda(h_2)\lambda(h_1 h_2)^{-1}, \quad h_1, h_2 \in H.$$

This is equivalent to $\mathsf{graph}(\lambda^{-1})$ is a Lie subgroup of $Z \times_c G$.
Assuming that H is a normal subgroup of G, the additional condition

$$c(g,h)c(ghg^{-1}, g)^{-1} = \lambda(h)\lambda(ghg^{-1})^{-1}, \quad g \in G, h \in H$$

ensures that the $\mathsf{graph}(\lambda^{-1})$ is a normal subgroup of $Z \times_c G$.
The rows and columns below are exact sequences of groups

$$\begin{array}{ccccc}
Z & \longrightarrow & (Z \times_c G)/\,\mathsf{graph}(\lambda^{-1}) & \longrightarrow & G/H \\
\uparrow & & \uparrow & & \uparrow \\
Z & \longrightarrow & Z \times_c G & \longrightarrow & G \\
\uparrow & & \uparrow & & \uparrow \\
Z & \longrightarrow & Z \times_c H & \longrightarrow & H
\end{array}$$

The path group

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and $\tilde{G}$ its universal covering group. The path group

$$PG = \{g \in C^\infty([0,1], G) | g(0) = e\}$$

is a smoothly contractible Lie group with path Lie algebra

$$P\mathfrak{g} = \{X \in C^\infty([0,1], \mathfrak{g}) | X(0) = 0\}.$$

Two Lie subgroups of PG integrating the loop Lie subalgebra $\Omega\mathfrak{g} \subset P\mathfrak{g}$ are the groups ΩG of loops and $\Omega_0 G$ of null-homotopic loops (both based at e). Exact sequences of Lie groups

$$\begin{array}{ccccc} \Omega G & \longrightarrow & PG & \longrightarrow & G \\ \uparrow & & \uparrow & & \uparrow \\ \Omega_0 G & \longrightarrow & PG & \longrightarrow & \tilde{G} \end{array}$$

with the same exact sequence of Lie algebras $\Omega\mathfrak{g} \to P\mathfrak{g} \to \mathfrak{g}$.

Path group cocycle

Each $\mathfrak{z}$-valued Lie algebra 2-cocycle ψ on $\mathfrak{g}$ defines a 2-cocycle on $P\mathfrak{g}$ that vanishes on the loop Lie algebra $\Omega\mathfrak{g}$:

$$P\psi : (X, Y) \mapsto \psi(X(1), Y(1)).$$

On PG there are natural system of paths $a_g(s)(t) = g(st)$ together with 2-simplexes $\Sigma_{f,g}(s,t) = a_f(s)a_g(st)$ with boundary $fa_g - a_{fg} + a_f$.

The Van Est method yields the following $\mathfrak{z}$-valued group 2-cocycle on PG integrating $P\psi$:

$$C(f, g) = \int_{\sigma_{f,g}} \psi^\ell = \int_0^1 \int_0^s \psi(\mathrm{Ad}_{g(t)^{-1}}\, \delta^\ell f(s), \delta^\ell g(t))dt\, ds,$$

where the 2-simplex $\sigma_{f,g}(s,t) = f(s)g(st)$ in G is $\Sigma_{f,g}$ evaluated at 1, and $\delta^\ell g = g^{-1}g' \in P\mathfrak{g}$ is the left logarithmic derivative of $g \in PG$.
A similar cocycle involving right log derivatives: $(f, g) \mapsto -C(g^{-1}, f^{-1})$.

Central extension of $\tilde{G}$

Assume the 2-cocycle ψ has discrete period group $\Pi_\psi \subset \mathfrak{z}$. The 2-cocycle $c(f,g) = C(f,g) \operatorname{mod} \Pi_\omega$ on PG is resolved over $\Omega_0 G$ by the 1-cochain

$$\lambda : \Omega_0 G \to Z, \quad \lambda(g) = \int_{\bar{g}} \psi^\ell \operatorname{mod} \Pi_\psi,$$

with $\bar{g}$ any homotopy from e to $g \in \Omega_0 G$. Thus

$$\begin{array}{ccccc}
Z & \longrightarrow & (Z \times_c PG)/\operatorname{graph}(\lambda^{-1}) & \longrightarrow & \tilde{G} \\
\uparrow & & \uparrow & & \uparrow \\
Z & \longrightarrow & Z \times_c PG & \longrightarrow & PG \\
\uparrow & & \uparrow & & \uparrow \\
Z & \longrightarrow & Z \times_c \Omega_0 G & \longrightarrow & \Omega_0 G
\end{array}$$

Theorem [Vizman'08] The cocycle ψ with discrete period group is integrated by the first line central Lie group extension.

Loop group [Mickelsson]

Let H be a compact simple Lie group with Lie algebra $\mathfrak{h}$. We consider the Killing form κ on $\mathfrak{h}$ and the Cartan 3-form $\eta = \frac{1}{6}\kappa(\theta^l, [\theta^l, \theta^l]) \in \Omega^3(H)$, where $\theta^l \in \Omega^1(H, \mathfrak{h})$ is the left Maurer-Cartan 1-form on H.
If H simply connected, then the loop group $G = C^\infty(S^1, H)$ is simply connected. The cocycle on its loop algebra $\mathfrak{g} = C^\infty(S^1, \mathfrak{h})$,

$$\psi(X, Y) = \int_{S^1} \kappa(X, dY) \in \mathbb{R},$$

has discrete period group. The circle-valued cocycle on PG integrating $P\psi$,

$$c(f, g) = \int_{I\times S^1} \delta^l f \wedge_\kappa \delta^r g \bmod \mathbb{Z},$$

when restricted to $\Omega_0 G$, is the coboundary of

$$\lambda(g) = \int_{I\times I\times S^1} \bar{g}^* \eta \bmod \mathbb{Z}.$$

This follows also via the Polyakov-Wiegmann formula

$$m^*\eta = \mathrm{pr}_1^*\,\eta + \mathrm{pr}_2^*\,\eta - d(\kappa(\mathrm{pr}_1^*\,\theta^l, \mathrm{pr}_2^*\,\theta^r)).$$

Current group [Losev-Moore-Nekrasov-Shatashvili]

Let $G = C^\infty(M, H)_0$ be the identity component of the current group, with current Lie algebra $\mathfrak{g} = C^\infty(M, \mathfrak{h})$, for M a compact manifold.
Continuous Lie algebra 2-cocycle with values in $\mathfrak{z} = \Omega^1(M)/dC^\infty(M)$:

$$\psi(X, Y) = [\kappa(X, dY) - \kappa(Y, dX)] = 2[\kappa(X, dY)].$$

The period group $\Pi_\psi = \Omega^1_{\mathbb{Z}}(M)/dC^\infty(M)$ (de Rham cohomology with integral periods) of the continuous Lie algebra 2-cocycle ψ is discrete.

The Z-valued group cocycle c on PG,

$$c(f, g) = \left[\int_I \delta^l f \wedge_\kappa \delta^r g\right] \bmod \Pi_\psi,$$

that integrates the Lie algebra cocycle $P\psi$ is resolved over $\Omega_0 G$ by

$$\lambda : \Omega_0 G \to Z, \quad \lambda(g) = \left[\int_{I\times I} \bar{g}^*\eta\right] \bmod \Pi_\psi.$$

Lichnerowicz cocycles

On the compact manifold M we consider a volume form μ normalized by $\int_M \mu = 1$. Let $G = \mathrm{Diff}(M,\mu)_0$. Given η a closed 2-form on M, the Lichnerowicz cocycle

$$\psi_\eta(X,Y) = \int_M \eta(X,Y)\mu$$

is a 2-cocycle on the Lie algebra $\mathfrak{g} = \mathfrak{X}(M,\mu)$ of divergence free vector fields.

The period group of ψ_η is contained in the period group of η, thus $\Pi_{\psi_\eta} \subseteq \mathbb{Z}$ if η is a closed integral 2-form. The path group construction gives a construction of the central extension of $\tilde{G}$ integrating the Lichnerowicz cocycle.

To write the cocycle c on PG one uses a connection on a principal circle bundle over M with curvature η. It is resolved over $\Omega_0 G$ by

$$\lambda(g) = \int_{I\times I\times M} p^*\eta \wedge \bar{g}^*\mu \mod \mathbb{Z},$$

with projection $p : I\times I\times M \to M$ and a path $\bar{g}$ from id_M to g in $\Omega_0 G$ viewed as $(s,t,x) \in I\times I\times M \mapsto \bar{g}(s,t)(x) \in M$.

Integration via prequantization extension

Prequantization extension

$(\mathcal{M},\Omega)$ prequantizable symplectic manifold, i.e. there exists $\mathcal{P}\to\mathcal{M}$ principal circle bundle with principal connection $\Theta\in\Omega^1(\mathcal{P})$ with curvature Ω.

The Lie algebra extension

$$0\to\mathbb{R}\to C^\infty(\mathcal{M})\to\mathfrak{X}_{\mathrm{ham}}(\mathcal{M})\to 0,$$

with Poisson bracket $\{f,g\}=\Omega(X_g,X_f)$ on $C^\infty(\mathcal{M})$, integrates to the prequantization central extension [Kostant'70, Souriau'70]

$$1\to S^1\to \mathrm{Quant}(\mathcal{P})\to \mathrm{Ham}(\mathcal{M})\to 1,$$

with $\mathrm{Quant}(\mathcal{P})=\{\psi\in\mathrm{Aut}(\mathcal{P}):\psi^*\Theta=\Theta\}$ *the quantomorphism group*. The Lie algebra extension splits for compact $\mathcal{M}$ (the Kostant-Souriau 2-cocycle is a coboundary).

Theorem [Neeb-V.'03] The pull-back of the prequantization central extension for infinite dimensional $\mathcal{M}$ yields smooth Lie group extensions integrating the Lie algebra cocycle $(X,Y)\mapsto\Omega(\zeta_X,\zeta_Y)(m_0)$.

Universal central extension

Theorem [Conjectured by Roger'95, Janssens-Ryvkin-V.'24]
Let (M^m, μ) be compact with $m > 2$ and volume form μ.
Then the Lie algebra cohomology is $H^2_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{ex}}(M)) = H^2_{\mathrm{dR}}(M)$ and the universal central extension is

$$0 \to H^{m-2}_{\mathrm{dR}}(M) \to \Omega^{m-2}(M)/d\Omega^{m-3}(M) \to \mathfrak{X}_{\mathrm{ex}}(M, \mu) \to 0$$

with Lie bracket $\{[\alpha_1], [\alpha_2]\} = [i_{X_{\alpha_1}} i_{X_{\alpha_2}} \mu]$.

Integrability of Lichnerowicz cocycles

The Lichnerowicz cocycle

$$\psi_\eta(X,Y)=\int_M \eta(X,Y)\mu,$$

for integral $[\eta]\in H^2_{\mathrm{dR}}(M)$ Poincaré dual to $[N]\in H_{m-2}(M,\mathbb{R})$ with codimension 2 submanifold $N\subset M$, is cohomologous to

$$\psi_N(X,Y)=\int_N i_Y i_X \mu$$

Theorem [Ismagilov'96, Haller-V.'04] The above cocycle integrates to a central Lie group extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$.

One considers the Marsden-Weinstein symplectic form $\Omega=\tilde{\mu}$ on connected component $\mathcal{M}\subset \mathrm{Gr}_{m-2}(M)$ of N, and the pullback of the prequantization extension by Hamiltonian action

$$\mathrm{Diff}_{\mathrm{ex}}(M)\to \mathrm{Ham}(\mathcal{M}).$$

MW symplectic form is prequantizable

Lemma Let $\alpha \in \Omega^{k+2}(M)$ be a closed differential form with integral cohomology class. Then the non-linear Grassmannian $\mathrm{Gr}_k(M)$ endowed with the closed 2-form $\tilde{\alpha}$ is prequantizable.

In [Haller-V.'04] a principal circle bundle $(\mathcal{P}, \theta)$ over $\mathrm{Gr}_k(M)$ with curvature $\tilde{\alpha}$ has been constructed through its Čech 1-cocycle.

In [Diez-Janssens-Neeb-V.'24] we get such a prequantum bundle using the transgression $\tilde{h}$ to $\mathrm{Gr}_k(M)$ of a differential character h with curvature α. The degree one differential character $\tilde{h}$ represents the holonomy of a principal bundle with curvature $\tilde{\alpha}$.

The diff. characters $h_{(\mathcal{P},\theta)}$ and $\tilde{h}$ may differ by an element in $H^1(\mathrm{Gr}_k(M), \mathbb{T})$.

Universal central extension

Theorem [Conjectured by Roger'95] [Janssens-V.'16]
Let (M, ω) be compact symplectic. The Lie algebra cohomology is $H^2_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{ham}}(M)) = H^1_{\mathrm{dR}}(M)$ and the universal central extension is

$$H^{2n-1}_{\mathrm{dR}}(M) \to \Omega^{2n-1}(M)/d\Omega^{2n-2}(M) \to \mathfrak{X}_{\mathrm{ham}}(M, \omega),$$

with Lie bracket

$$[[\gamma_1], [\gamma_2]] = [f_1 df_2 \wedge \omega^{n-1}/(n-1)!], \text{ where } f\omega^n/n! = d\gamma.$$

Each closed $\alpha \in \Omega^1(M)$ determines Roger cocycle on $\mathfrak{X}_{\mathrm{ham}}(M)$:

$$\sigma_\alpha(X_f, X_g) = \int_M f\alpha(X_g)\omega^n/n!.$$

If $[\alpha] \in H^1_{\mathrm{dR}}(M)$ is Poincaré dual to $[C] \in H_{2n-1}(M, \mathbb{R})$, it is cohomologous to

$$\sigma_C(X_f, X_g) = \int_C f dg \wedge \omega^{n-1}/(n-1)!.$$

Integrability of Roger cocycles

Let $\pi : (P^{2n+1}, \theta) \to (M^{2n}, \omega)$ be a prequantum bundle over the symplectic manifold. The volume form $\mu = \theta \wedge (d\theta)^n/(n+1)!$ on the contact manifold (P, θ) has total volume $V = \frac{2\pi}{n+1} \mathrm{vol}_\omega(M)$.

The identity component $\mathtt{Quant}(P)_0$ is a subgroup of $\mathtt{Diff}_{\mathrm{ex}}(P, \mu)$.

The pullback of the integrable Lichnerowicz cocycle $\frac{1}{V}\psi_N$ on $\mathfrak{X}_{\mathrm{ex}}(P, \mu)$ to the Lie algebra $C^\infty(M)$ is cohomologous to $\frac{1}{V}\sigma_{\pi_* N}$.

Integrability of Roger cocycles

Theorem [Janssens-V.'16] The classes $[\sigma_C] \in H^2_{\mathrm{CE}}(C^\infty(M))$ corresponding to the lattice

$$\frac{1}{V}\pi_* H_{2n-1}(P,\mathbb{Z}) \subseteq H_{2n-1}(M,\mathbb{R})$$

give rise to integrable cocycles of the Poisson Lie algebra $C^\infty(M)$.

The same holds for Roger cocycles $[\sigma_\alpha]$ with $[\alpha]$ in the lattice

$$\frac{1}{V}\pi_! H^2_{\mathrm{dR}}(P)_{\mathbb{Z}} \subseteq H^1_{\mathrm{dR}}(M)\,,$$

where the map $\pi_!$ is given by fibre integration and $H^2_{\mathrm{dR}}(P)_{\mathbb{Z}}$ denotes the space of integral cohomology classes.

Hamiltonian group

By Lie's Second Theorem for regular Lie groups [Kriegl-Michor'97], the Lie algebra homomorphism that splits the prequantization extension,

$$\kappa : \mathfrak{X}_{\mathsf{ham}}(M) \to C^\infty(M), \quad \kappa(X_f) := f - \frac{1}{\mathsf{vol}_\omega(M)} \int_M f\omega^n/n!\,,$$

integrates to a group homomorphism on the universal covering group of the group of Hamiltonian diffeomorphisms:

$$K : \widetilde{\mathsf{Ham}}(M) \to \mathsf{Quant}(P)_0\,.$$

The above extensions of the quantomorphism group can be pulled back by K to obtain extensions of the universal covering group $\widetilde{\mathsf{Ham}}(M)$.