Higher symmetry enriched topological phases

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Topological Orders and Higher Structures
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arXiv:2003.08898, 2005.14178

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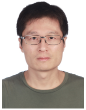

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Introduction

- A blueprint for the framework of topological phases with higher symmetry. Mathematically, many contents should be regarded as conjectures.
- Up to invertibles, describe topological phases by fusion n-categories with "trivial" center.
- "Trivial" with respect to a given higher symmetry $\Rightarrow$ non-trivial structures arise.

Warm up: Symmetry in a quantum system

Let H, V be the Hamiltonian and Hilbert space of a quantum system. G a group, represented by $\rho_g \in \mathrm{GL}(V)$, is the symmetry group of the system if

$$\rho_g H = H\rho_g,\ \forall g \in G.$$

- H, V are meant to be generic, not specific.

  Allow symmetric perturbations, for example.
- More precisely, one should consider a class of Hilbert spaces where G acts on, and all possible Hamiltonians that commute with G action.
- For completeness, just take $\mathrm{Rep}(G)$
  - objects: (V, ρ), vector space V with a G action $\rho : G \to \mathrm{GL}(V)$
  - morphisms: symmetric operators (those that commute with G action)

Tannaka duality

Duality between invariants and transformations

$$\mathrm{Rep}(G) \xrightarrow{\mathrm{Fgt}} \mathrm{Vec}, \quad G \cong \mathrm{Aut}(\mathrm{Fgt}).$$

- $\mathrm{Fgt} : (V, \rho) \mapsto V$ is the forgetful functor.
- It is faithful and picks the symmetric operators.
- The commutation relation is encoded in the definition of natural transformation, $\forall g \in \mathrm{Aut}(\mathrm{Fgt}),\ f : V \to V'$,

$$\begin{array}{ccc} V = \mathrm{Fgt}(V, \rho) & \xrightarrow{g_V = \rho_g} & \mathrm{Fgt}(V, \rho) = V \\ \Big\downarrow f & & \Big\downarrow f \\ V' = \mathrm{Fgt}(V', \rho') & \xrightarrow{g_{V'} = \rho'_g} & \mathrm{Fgt}(V', \rho') = V' \end{array}$$

Tannaka formalism as symmetry breaking

We can also interpret the data of Tannaka formalism

$$\mathrm{Rep}(G) \xrightarrow{\mathrm{Fgt}} \mathrm{Vec}$$

as

- $\mathrm{Rep}(G)$: the systems with symmetry
- Vec: the systems with no symmetry
- Fgt: the process of symmetry breaking

Generalizing symmetry

Thus the most general form of "symmetry" is, in an appropriate category e.g. the category of physical theories of your interest

$$\mathcal{R} \xrightarrow{\beta} \mathcal{V},$$

where

- $\mathcal{V}$ is the trivial theory without symmetry "ground field";
- $\mathcal{R}$ is the theory with symmetry;
- β is the symmetry breaking morphism in this appropriate category.
- The symmetry "algebra" is, by Tannaka duality, $\mathrm{End}(\beta)$.

Which category is for topological phases?
The category of fusion n-categories.

Higher symmetry

Consider symmetry in the category of fusion n-categories. Let $\mathcal{V}$ be the fusion n-category for the "elementary local" excitations.

Definition

A fusion n-category $\mathcal{R}$ equipped with monoidal n-functor

$$\beta : \mathcal{R} \to \mathcal{V},$$

which is

- surjective: the image of β generates $\mathcal{V}$;
- top-faithful: injective on n-morphisms (i.e., operators);

is called a $\mathcal{V}$-local fusion n-category.

We refer to $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$ as a higher symmetry.

Higher symmetry

$\mathcal{V}$ is the fusion n-category for the "elementary local" excitations:

- Boson system: $\mathcal{V} = n\mathrm{Vec} \equiv \Sigma^{n-1}\mathrm{Vec}$.

  Σ: delooping and condensation completion — D. Gaiotto, T. Johnson-Freyd, arXiv:1905.09566.

  T. Johnson-Freyd, arXiv:2003.06663.

  - $\mathcal{R} = n\mathrm{Rep}(G) \equiv \Sigma^{n-1}\mathrm{Rep}(G)$: symmetry charges and higher dimensional defects from condensation of symmetry charges; β forgets group action. 0-form or global symmetry.
  - $\mathcal{R} = n\mathrm{Vec}_G$: G graded n-vector spaces, the symmetry domain walls in the spontaneous G-symmetry breaking phase; β forgets grading. Algebraic higher symmetry in the sense of X.-G. Wen.
  - ...
- Fermion system: $\mathcal{V} = n\mathrm{sVec} \equiv \Sigma^{n-1}\mathrm{sVec}$.
  - $\mathcal{R} = n\mathrm{Rep}(G, z)$.
  - ...
- Anyon system, string system, ... with more exotic $\mathcal{V}$.

Partial characterization of higher SET

The macroscopic observables topological defects or extended operators of an $(n+1)$D topological phase, whether with symmetry or not, always form a fusion n-category.

"With symmetry" $\Leftrightarrow$ contains the macroscopic observables of a higher symmetry, which are nothing but $\mathcal{R}$

Definition (Partial)

A (potentially anomalous) $(n+1)$D topological phase with higher symmetry $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$, also called a higher symmetry enriched topological (SET) phase, is partially characterized by a fusion n-category $\mathcal{C}$ with embedding

$$\iota : \mathcal{R} \to \mathcal{C}.$$

By embedding, we mean that $\mathcal{R}$ is equivalent to the image of ι.

Anomaly detection: compute bulk

- Anomaly-free $\Leftrightarrow$ trivial bulk
- Bulk $\Leftrightarrow$ center, here we need to formulate the proper notion of center relative to the higher symmetry.
- Conjecture: the bulk of $\mathcal{C}$ with symmetry is given by the E_1 center $Z_1(\mathcal{C}) \equiv \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C},\mathcal{C})$ with additional structures.
- Trivial phase must have $\mathcal{C} = \mathcal{R}$; the bulk of $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$ is trivial.
- However, the trivial bulk with symmetry is not completely trivial. We need extra data to characterize how the bulk of $\mathcal{C}$ is trivial.
- More precisely, the extra data is at least an equivalence $Z_1(\mathcal{R}) \simeq Z_1(\mathcal{C})$ higher structures?; we also need to formulate how the additional structure of symmetry is preserved by such equivalence.

Special case: $\mathcal{R}$ is symmetric or braided

Begin with the simpler case that $\mathcal{R}$ is symmetric (or braided).

$\mathcal{R}$ can be canonically embedded into $Z_1(\mathcal{R})$

$$\iota_{\mathcal{R}} : \mathcal{R} \to Z_1(\mathcal{R}).$$

$\mathcal{R} \xrightarrow{\iota_{\mathcal{R}}} Z_1(\mathcal{R})$ describes the trivial phase with symmetry in one higher dimension. $Z_1(\mathcal{R})$ is the "trivial" minimal modular extension of symmetric $\mathcal{R}$.

Let

$$\begin{aligned} \mathrm{Fgt}_{\mathcal{C}} : Z_1(\mathcal{C}) \equiv \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C}) &\to \mathcal{C} \\ f &\mapsto f(\mathbf{1}_{\mathcal{C}}) \end{aligned}$$

be the forgetful functor. We have a natural formulation for an equivalence between trivial bulk with symmetry.

Special case: $\mathcal{R}$ is symmetric or braided

Definition

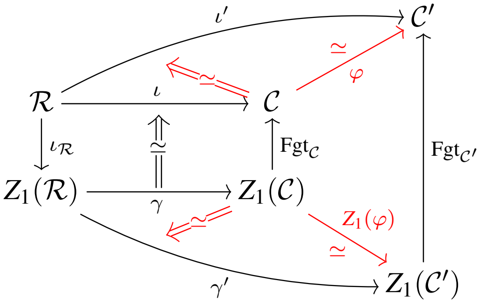
When $\mathcal{R}$ is symmetric (or braided), an anomaly-free $n+1$D higher SET phase is charaterized by up to invertible ones without symmetry

1. A fusion n-category $\mathcal{C}$;
2. An embedding $\iota : \mathcal{R} \to \mathcal{C}$;
3. A braided equivalence $\gamma : Z_1(\mathcal{R}) \simeq Z_1(\mathcal{C})$ such that

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\iota} & \mathcal{C} \\ \downarrow{\scriptstyle \iota_{\mathcal{R}}} & & \uparrow{\scriptstyle \mathrm{Fgt}_{\mathcal{C}}} \\ Z_1(\mathcal{R}) & \xrightarrow[\gamma]{} & Z_1(\mathcal{C}) \end{array}$$

L. Kong, TL, X.-G. Wen, Z.-H. Zhang, and H. Zheng, arXiv:2003.08898.

Open problem: Higher structures



- Equivalences $\varphi : \mathcal{C} \simeq \mathcal{C}'$, natural isomorphisms, higher homotopies, . . . ;
- Even for $\mathcal{C}' = \mathcal{C}$, the data γ is up to $\mathrm{Aut}(\mathcal{C})$; for $\varphi = \mathrm{id}_{\mathcal{C}}$, $\mathrm{Fgt}_{\mathcal{C}} \circ \gamma \circ \iota_{\mathcal{R}} \Rightarrow \iota$ is up to $\mathrm{Aut}(\iota)$ and $\mathrm{Aut}(\gamma)$.

Higher SPT

When $\mathcal{C} = \mathcal{R}$, the autoequivalences of $Z_1(\mathcal{R})$ preserving the embedding $\mathcal{R} \xrightarrow{\iota_\mathcal{R}} Z_1(\mathcal{R})$, denoted by $\mathrm{Aut}(Z_1(\mathcal{R}), \iota_\mathcal{R})$, classify the higher symmetry protected topological (SPT) phases.

$$\begin{array}{ccc} & \mathcal{R} & \\ {}^{\iota_\mathcal{R}}\swarrow & & \searrow^{\iota_\mathcal{R}} \\ Z_1(\mathcal{R}) & \xrightarrow[\gamma]{\qquad\qquad} & Z_1(\mathcal{R}) \end{array}$$

Example

- $\mathcal{R} = \mathrm{Rep}G$ 1+1D bosonic: $\mathrm{Aut}(Z_1(\mathcal{R}), \iota_\mathcal{R}) = \mathrm{Pic}(\mathrm{Rep}(G)) = H^2(G, U(1))$.
- $\mathcal{R} = \mathrm{Rep}(G, z)$ 1+1D fermionic: $\mathrm{Aut}(Z_1(\mathcal{R}), \iota_\mathcal{R}) = \mathrm{Pic}(\mathrm{Rep}(G, z)) =$ $\begin{cases} H^2(G, U(1)) \times \mathbb{Z}_2 & \text{if } G = G_b \times \langle z \rangle \\ H^2(G, U(1)) & \text{otherwise} \end{cases}$.
- $\mathcal{R} = 2\mathrm{Rep}G$ 2+1D bosonic: conjecture $\mathrm{Pic}(2\mathrm{Rep}(G)) = H^3(G, U(1))$.
- $\mathcal{R} = 2\mathrm{sVec}$: conjecture $\mathrm{Aut}(Z_1(\mathcal{R}), \iota_\mathcal{R}) = \mathbb{Z}_{16}$.

L. Kong, TL, X.-G. Wen, Z.-H. Zhang, and H. Zheng, arXiv:2003.08898.

General case

To be more general, we no longer assume that $\mathcal{R}$ is braided.

For example, $\mathcal{R} = n\mathrm{Vec}_G$ with non-abelian G.

We can no longer embed $\mathcal{R}$ into $Z_1(\mathcal{R})$.
How to consider the symmetry in the bulk?

Lemma (Bruguières, Natale, Proposition 5.1, arXiv:1006.0569)

Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories, $F : \mathcal{C} \to \mathcal{D}$ a monoidal functor and R the right adjoint of F. Then $A = R(\mathbf{1}_\mathcal{D})$, with a natural half braiding, has a canonical structure of commutative algebra in $Z_1(\mathcal{C})$ and $\mathcal{D} \simeq A\text{-}\mathrm{Mod}_\mathcal{C}$, F coincides with the free module functor $\mathcal{C} \to A\text{-}\mathrm{Mod}_\mathcal{C}$.

Dual algebra

We believe the similar for higher categories:
$\beta : \mathcal{R} \to \mathcal{V}$ canonically determines a commutative (higher) algebra A_β in $Z_1(\mathcal{R})$, such that $\mathcal{V} \simeq A_\beta\text{-Mod}_\mathcal{R}$ and β is reconstructed as the free module functor.
Physically, condensing A_β breaks the symmetry $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$.

Example

Take $\mathrm{Rep}(G) \xrightarrow{\mathrm{Fgt}} \mathrm{Vec}$. $\mathrm{End}(\mathrm{Fgt}) = \mathbb{C}[G]$.
$A_{\mathrm{Fgt}} = \mathrm{Fun}(G) = \mathrm{Hom}(\mathbb{C}[G], \mathbb{C}) = \mathbb{C}[G]^*$. $\mathrm{Fun}(G)$ contains every irreducable representation as a direct summand.

In general we may think A_β as the algebra dual to the symmetry algebra $\mathrm{End}(\beta)$.
When $\mathcal{R}$ is braided, $A_\beta = \iota_\mathcal{R} \circ \mathrm{Fgt}_\mathcal{R}(A_\beta)$. We like to replace $\mathcal{R} \xrightarrow{\iota_\mathcal{R}} Z_1(\mathcal{R})$ for A_β.

General case

Definition

An anomaly-free $n+1$D higher SET phase with symmetry $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$ is characterized by up to invertible ones without symmetry

1. A fusion n-category $\mathcal{C}$;
2. An embedding $\iota : \mathcal{R} \to \mathcal{C}$;
3. A braided equivalence $\gamma : Z_1(\mathcal{R}) \simeq Z_1(\mathcal{C})$ such that

$$\mathrm{Fgt}_{\mathcal{C}} \circ \gamma(A_\beta) = \iota \circ \mathrm{Fgt}_{\mathcal{R}}(A_\beta).$$

$$\begin{array}{ccc}
\mathrm{Fgt}_{\mathcal{R}}(A_\beta) \in \mathcal{R} & \xmapsto{\ \iota\ } & \mathrm{Fgt}_{\mathcal{C}} \circ \gamma(A_\beta) = \iota \circ \mathrm{Fgt}_{\mathcal{R}}(A_\beta) \in \mathcal{C} \\
\uparrow {\scriptstyle \mathrm{Fgt}_{\mathcal{R}}} & & \uparrow {\scriptstyle \mathrm{Fgt}_{\mathcal{C}}} \\
A_\beta \in Z_1(\mathcal{R}) & \xmapsto[\ \gamma\ ]{} & \gamma(A_\beta) \in Z_1(\mathcal{C})
\end{array}$$

L. Kong, TL, X.-G. Wen, Z.-H. Zhang, and H. Zheng, arXiv:2005.14178.

Open problem: Higher structures

- Allow algebra isomorphism $\mathrm{Fgt}_{\mathcal{C}} \circ \gamma(A_\beta) \overset{?}{\simeq} \iota \circ \mathrm{Fgt}_{\mathcal{R}}(A_\beta)$?
- $\iota \circ \mathrm{Fgt}_{\mathcal{R}}(A_\beta)$ is the macroscopic observable of symmetry in $\mathcal{C}$. If the difference is too large, we would regard it as a different symmetry.
- Indeed, higher structures such as natural isomorphisms $\alpha \in \mathrm{Aut}(\iota)$ may introduce algebra isomorphisms $\alpha_{\mathrm{Fgt}_{\mathcal{R}}(A_\beta)} \in \mathrm{Aut}(\iota \circ \mathrm{Fgt}_{\mathcal{R}}(A_\beta))$. We expect $\mathrm{Fgt}_{\mathcal{C}} \circ \gamma(A_\beta) = \iota \circ \mathrm{Fgt}_{\mathcal{R}}(A_\beta)$ up to such higher structures.

Pushout and symmetry breaking in $\mathcal{C}$

It is natural to consider the pushout, in the category of fusion n-categories,

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\iota} & \mathcal{C} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\iota_*\beta} \\ \mathcal{V} & \longrightarrow & \mathcal{C}_0 \end{array}$$

- $\iota_*\beta$ describes symmetry breaking in the higher SET;
- $\mathcal{C}_0$ is the resulting topological phase without $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$ symmetry.

Similarly $\iota_*\beta$ canonically determines a commutative algebra $A_{\iota_*\beta}$ in $Z_1(\mathcal{C})$, $\mathrm{Fgt}_{\mathcal{C}}(A_{\iota_*\beta}) = \iota \circ \mathrm{Fgt}_{\mathcal{R}}(A_\beta)$. So we may instead write the condition

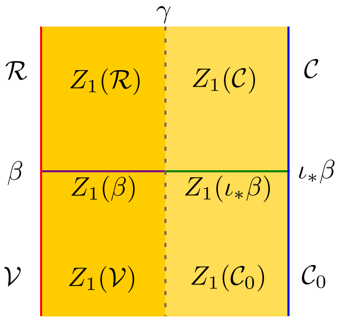
$$\gamma(A_\beta) \simeq A_{\iota_*\beta}.$$

The bulk with symmetry breaking domain wall

The bulk, or center Z_1, is functorial. L. Kong, X.-G. Wen, H. Zheng, arXiv:1502.01690

L. Kong, H. Zheng, arXiv:1507.00503

Compute the bulk of $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$ as a whole: β gives a symmetry breaking domain wall in the bulk.



γ naturally gives rise to an equivalence of the whole bulk.

The bulk with symmetry breaking domain wall

- $Z_1(\mathcal{R}) \equiv \mathrm{Fun}_{\mathcal{R}|\mathcal{R}}(\mathcal{R}, \mathcal{R})$, $Z_1(\mathcal{V}) \equiv \mathrm{Fun}_{\mathcal{V}|\mathcal{V}}(\mathcal{V}, \mathcal{V})$;
- β naturally makes $\mathcal{V}$ a $\mathcal{R}$-$\mathcal{V}$-bimodule ${}_\beta\mathcal{V}$, we take $Z_1(\beta) \equiv \mathrm{Fun}_{\mathcal{R}|\mathcal{V}}({}_\beta\mathcal{V}, {}_\beta\mathcal{V})$;
- The following two functors are monoidal and central

$$F_\beta^{\mathcal{R}} : Z_1(\mathcal{R}) = \mathrm{Fun}_{\mathcal{R}|\mathcal{R}}(\mathcal{R}, \mathcal{R}) \to \mathrm{Fun}_{\mathcal{R}|\mathcal{V}}({}_\beta\mathcal{V}, {}_\beta\mathcal{V}) = Z_1(\beta)$$
$$f \mapsto \beta(f(\mathbf{1}_{\mathcal{R}})) \otimes -,$$
$$F_\beta^{\mathcal{V}} : Z_1(\mathcal{V}) = \mathrm{Fun}_{\mathcal{V}|\mathcal{V}}(\mathcal{V}, \mathcal{V}) \to \mathrm{Fun}_{\mathcal{R}|\mathcal{V}}({}_\beta\mathcal{V}, {}_\beta\mathcal{V}) = Z_1(\beta)$$
$$f \mapsto - \otimes f(\mathbf{1}_{\mathcal{V}}) = f,$$

and makes $Z_1(\beta)$ a monoidal $Z_1(\mathcal{R})$-$Z_1(\mathcal{V})$-bimodule.

To conclude, the bulk of $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$ is $Z_1(\mathcal{R}) \xrightarrow{F_\beta^{\mathcal{R}}} Z_1(\beta) \xleftarrow{F_\beta^{\mathcal{V}}} Z_1(\mathcal{V})$.

Similarly the bulk of $\mathcal{C} \xrightarrow{\iota_*\beta} \mathcal{C}_0$ is $Z_1(\mathcal{C}) \xrightarrow{F_{\iota_*\beta}^{\mathcal{C}}} Z_1(\iota_*\beta) \xleftarrow{F_{\iota_*\beta}^{\mathcal{C}_0}} Z_1(\mathcal{C}_0)$.

The bulk with symmetry breaking domain wall

The bulk of $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$ is $Z_1(\mathcal{R}) \xrightarrow{F_\beta^{\mathcal{R}}} Z_1(\beta) \xleftarrow{F_\beta^{\mathcal{V}}} Z_1(\mathcal{V})$.

The bulk of $\mathcal{C} \xrightarrow{\iota_*\beta} \mathcal{C}_0$ is $Z_1(\mathcal{C}) \xrightarrow{F_{\iota_*\beta}^{\mathcal{C}}} Z_1(\iota_*\beta) \xleftarrow{F_{\iota_*\beta}^{\mathcal{C}_0}} Z_1(\mathcal{C}_0)$.

$\gamma : Z_1(\mathcal{R}) \simeq Z_1(\mathcal{C})$ induces equivalences $\tilde{\gamma} : Z_1(\beta) \simeq Z_1(\iota_*\beta)$ and $\gamma_0 : Z_1(\mathcal{V}) \simeq Z_1(\mathcal{C}_0)$, which satisfy

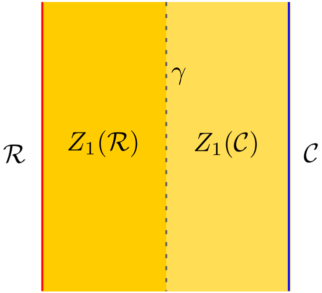
$$\begin{array}{ccccc}
Z_1(\mathcal{R}) & \xrightarrow{F_\beta^{\mathcal{R}}} & Z_1(\beta) & \xleftarrow{F_\beta^{\mathcal{V}}} & Z_1(\mathcal{V}) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \tilde{\gamma}} & & \downarrow{\scriptstyle \gamma_0} \\
Z_1(\mathcal{C}) & \xrightarrow{F_{\iota_*\beta}^{\mathcal{C}}} & Z_1(\iota_*\beta) & \xleftarrow{F_{\iota_*\beta}^{\mathcal{C}_0}} & Z_1(\mathcal{C}_0)
\end{array}$$

- $Z_1(\beta) \simeq A_\beta\text{-Mod}_{Z_1(\mathcal{R})}$ and $Z_1(\iota_*\beta) \simeq A_{\iota_*\beta}\text{-Mod}_{Z_1(\mathcal{C})}$. Thus, $\gamma(A_\beta) \simeq A_{\iota_*\beta}$ induces $\tilde{\gamma} : Z_1(\beta) \simeq Z_1(\iota_*\beta)$.
- $F_\beta^{\mathcal{V}}$ and $F_{\iota_*\beta}^{\mathcal{C}_0}$ are embeddings; γ_0 is the restriction of $\tilde{\gamma}$.

Categorical gauging

With $\gamma : Z_1(\mathcal{R}) \simeq Z_1(\mathcal{C})$, we can categorically gauge the higher symmetry $\mathcal{R} \xrightarrow{\beta} \mathcal{V}$, and obtain the gauged theory

$$\mathcal{R}^{\mathrm{rev}} \underset{Z_1(\mathcal{R})}{\otimes} \gamma \underset{Z_1(\mathcal{C})}{\otimes} \mathcal{C},$$



a fusion n-category multifusion for $n = 1$ which describes a topological phase without symmetry (its bulk is $n\mathrm{Vec}$).

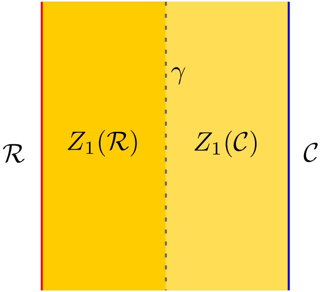
Categorical gauging

The old way of categorical gauging: G-crossed extension and minimal modular extension.

M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, PRB 100, 115147 (2019), arXiv:1410.4540.

TL, L. Kong, and X.-G. Wen, Phys. Rev. B 95, 235140 (2017), arXiv:1602.05946.

TL, L. Kong, and X.-G. Wen, Commun. Math. Phys. 351, 709 (2016), arXiv:1602.05936.



Conjecture

For symmetric $\mathcal{R}$, $\Omega(\mathcal{R} \underset{Z_1(\mathcal{R})}{\otimes} \gamma \underset{Z_1(\mathcal{C})}{\otimes} \mathcal{C})$ is a minimal modular extension of $\Omega\mathcal{C}$. Moreover, there is a bijection bewteen minimal modular extensions and equivalence functors γ in the bulk.

L. Kong, TL, X.-G. Wen, Z.-H. Zhang, and H. Zheng, arXiv:2003.08898.

Outlook

- Higher structures: Study the higher category of higher SETs.
- Boundary theory, anomalous higher SETs.
- ...

Thanks for attention!