Noncrossing Arc Diagrams and Dynamics

Emily Barnard

DePaul University

February 20, 2025

Overview

Goals for the week

- Survey the many lattice-posets which appear in “nature”.
- Practice techniques for proving when a poset is a lattice.
- Recognize semidistributive lattices via and edge labeling.
- Practice using the FTFDL.
- Practice finding and checking for lattice quotients.
- **Practice dynamics related to Tamari lattices.**
- **Survey some of the mathematicians who study lattice-posets in combinatorics, representation theory and dynamics.**

Overview

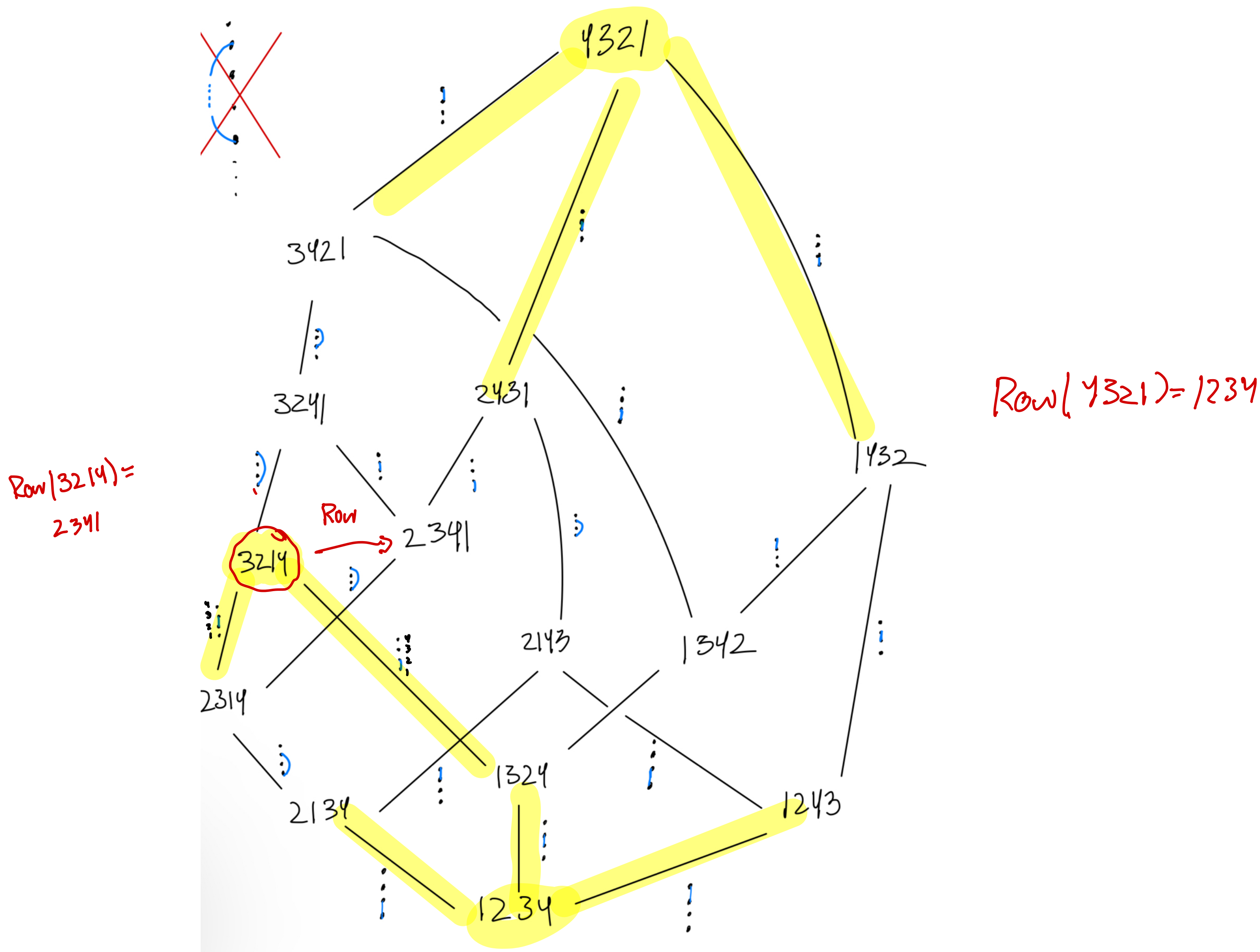
Goals for the day

- Define pop-stack sorting
- Describe c-Cambrian lattices, a generalization of the Tamari lattice
- Make a brief connection to representation theory
- Meet one more lattice theory extraordinaire!

Dynamical

- Jessica Striker, professor at North Dakota State University is one of founders of dynamical algebraic combinatorics.
- With Nathan Williams she has explored the connection between the Kreweras complement on noncrossing partitions and rowmotion on order ideals of the root poset.

Rowmotion on Semidistributive lattices



Pop-stack Sorting

Definition

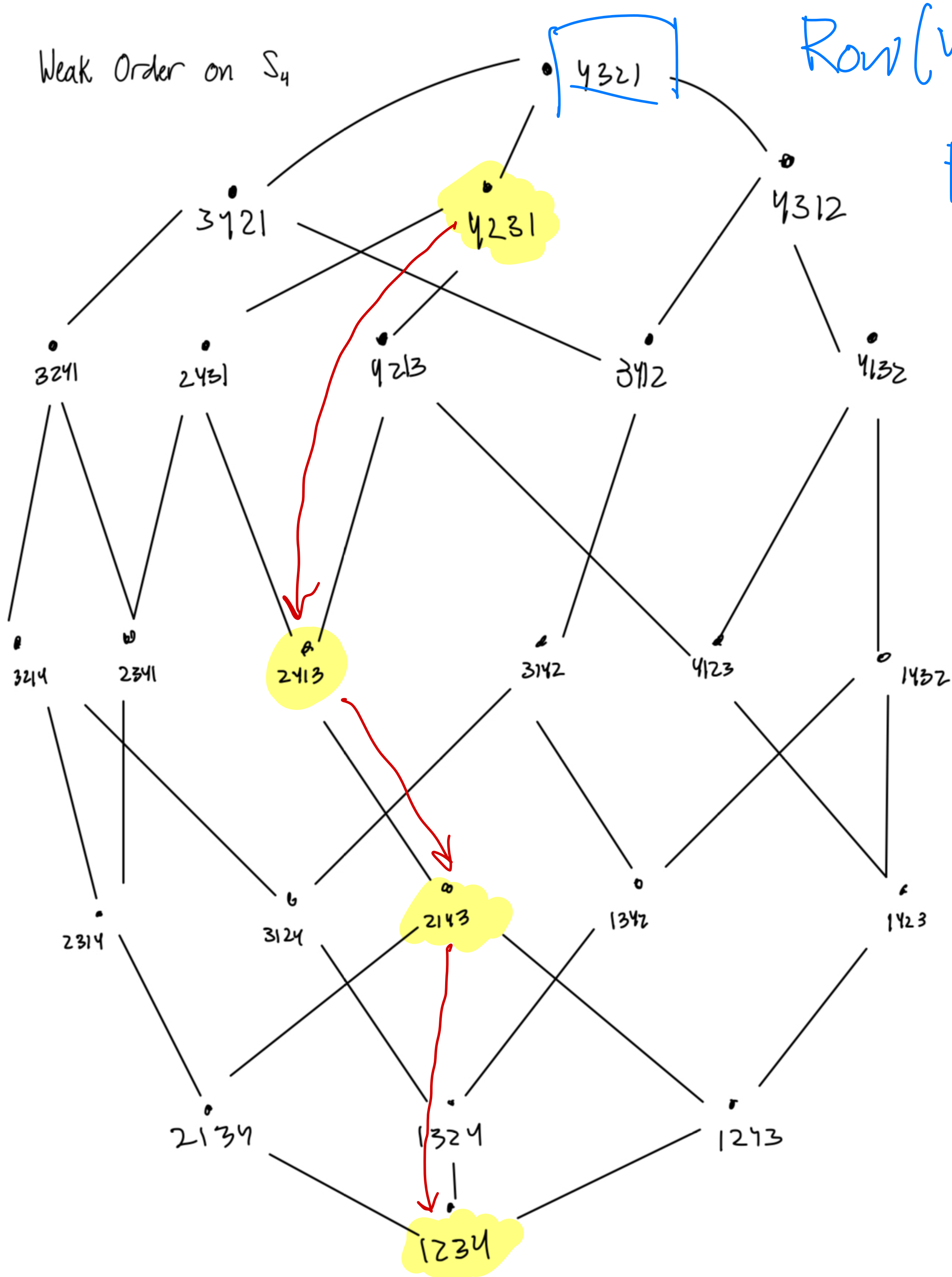
Let $w = w_1 w_2 \ldots w_n$ be a permutation. $\mathrm{pop}^{\downarrow}(w)$ reverses each of the (maximal) descending runs of w without changing their relative positions.

Example

Sorted

$$4231 \mapsto 2413 \mapsto 2143 \mapsto 1234$$

Weak Order on S_4

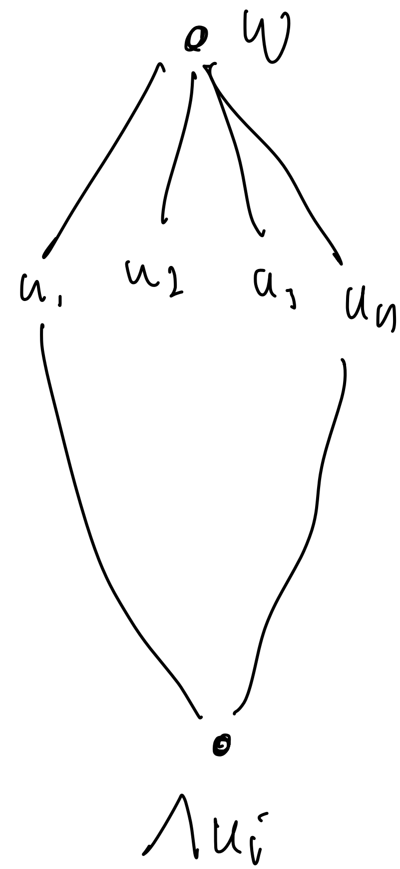


$\text{Row}(4321) = 1234$

$\text{pop}_j(4321) =$

1234

Lattice-Theoretic Definition



Pop-stack via Meets

Given a permutation w in the weak order on S_n

$$\mathrm{pop}^{\downarrow}(w) = \bigwedge \{u : w \gtrdot u\} \cup \{w\}$$

pop↓ takes us monotonically down in L.

when $w = \hat{0}$

$$\mathrm{pop}\downarrow(\hat{0}) = \left(\bigwedge \emptyset\right) \wedge \hat{0} = \hat{0}$$

Lattice-Theoretic Definition

$\mathrm{pop}_{\downarrow} : L \to L$

Pop-stack via Meets

Given a permutation w in the weak order on S_n

$$\mathrm{pop}^{\downarrow}(w) = \bigwedge \{u : w \gtrdot u\} \cup \{w\}$$

Theorem [Defant and Williams]

If L is semidistributive (more generally, if L is semidistrim), then for any $w \in L$

- $\mathrm{pop}^{\downarrow}(w) = w \wedge \mathrm{Row}(w)$
- $|\{w \in L :\ w \geq \mathrm{Row}(w)\}| = |\mathrm{pop}^{\downarrow}(w)|$

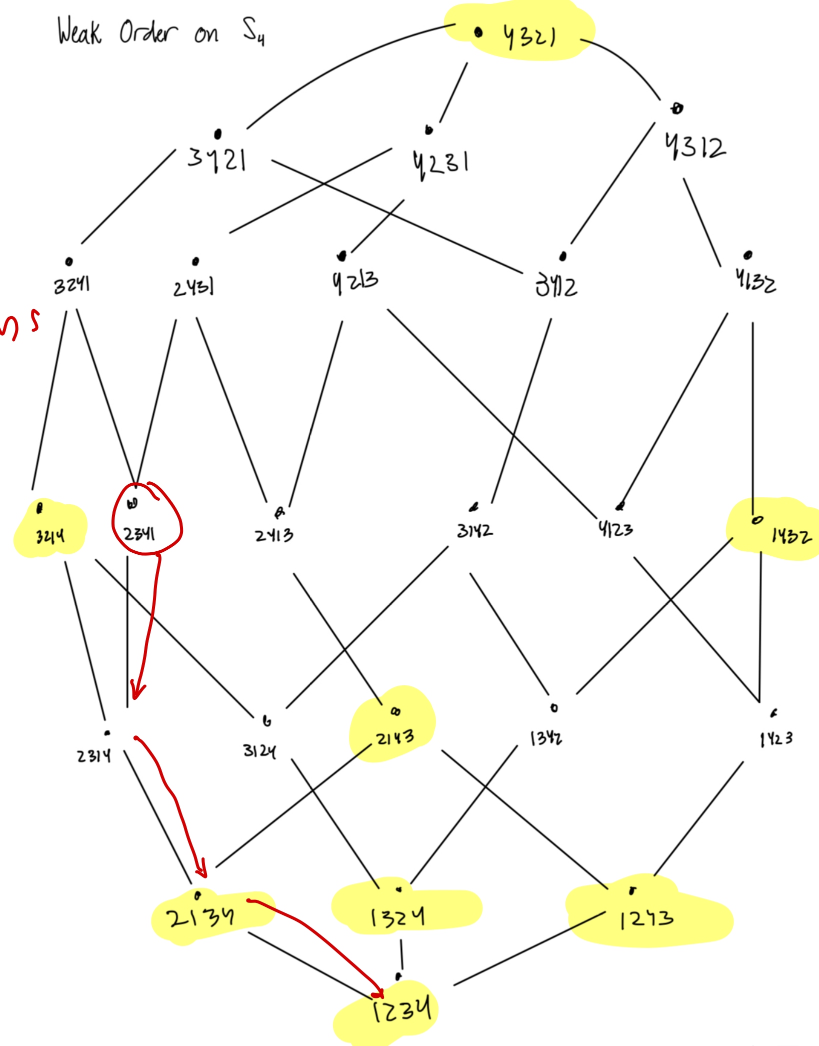
P-pop stack sortable

Definition

A permutation is p-pop stack sortable provided that it is sorted after at most p iterations of $\mathrm{pop}^{\downarrow}$.

S_4

- What's the longest it can take to be sorted?
  - 4231 takes 3 iterations



- How many elements take at most 1 iteration of pop to be sorted.

2^{n-1}

History

- Pudwell-Smith found generating functions for 2-stack sortable permutations.
- Claesson and Gundmundsson showed that the generating function for p-pop stack sortable permutations is rational for all p.
- Ungar showed that each permutation in S_n can be sorted in at most $n - 1$ iterations.
- More recently, Defant and Williams extended the definition of $\mathrm{pop}^{\downarrow}$ to a larger family of posets, including the weak order for reflection groups called Coxeter groups.

Set up

We want to study the pop-stack sorting method on a famous family of posets called c-Cambrian lattices.

- Each c-Cambrian lattice is defined by a choice of Coxeter system (W, S) and a Coxeter element c.
- The number of elements in any c-Cambrian is the W-Catalan number.
- Important connections to noncrossing partitions, cluster algebras and representation theory.
- Like the Tamari lattice, each c-Cambrian lattice is semidistributive. + trim

Popping in a c-Cambrian lattice

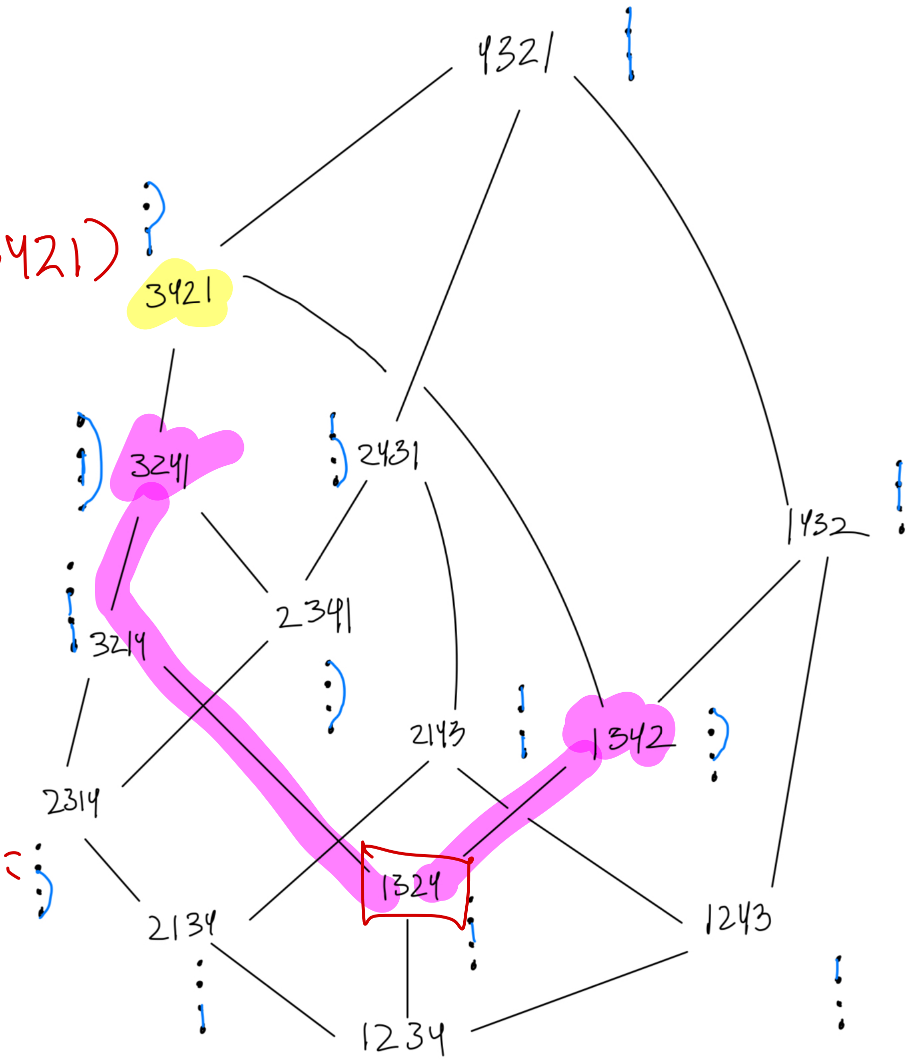
Tamari lattice

$\mathrm{pop}_{\downarrow}(3421)$

$= 3124$

missing!

312 avoiding



Actual $\mathrm{pop}_{\downarrow}(3421) =$

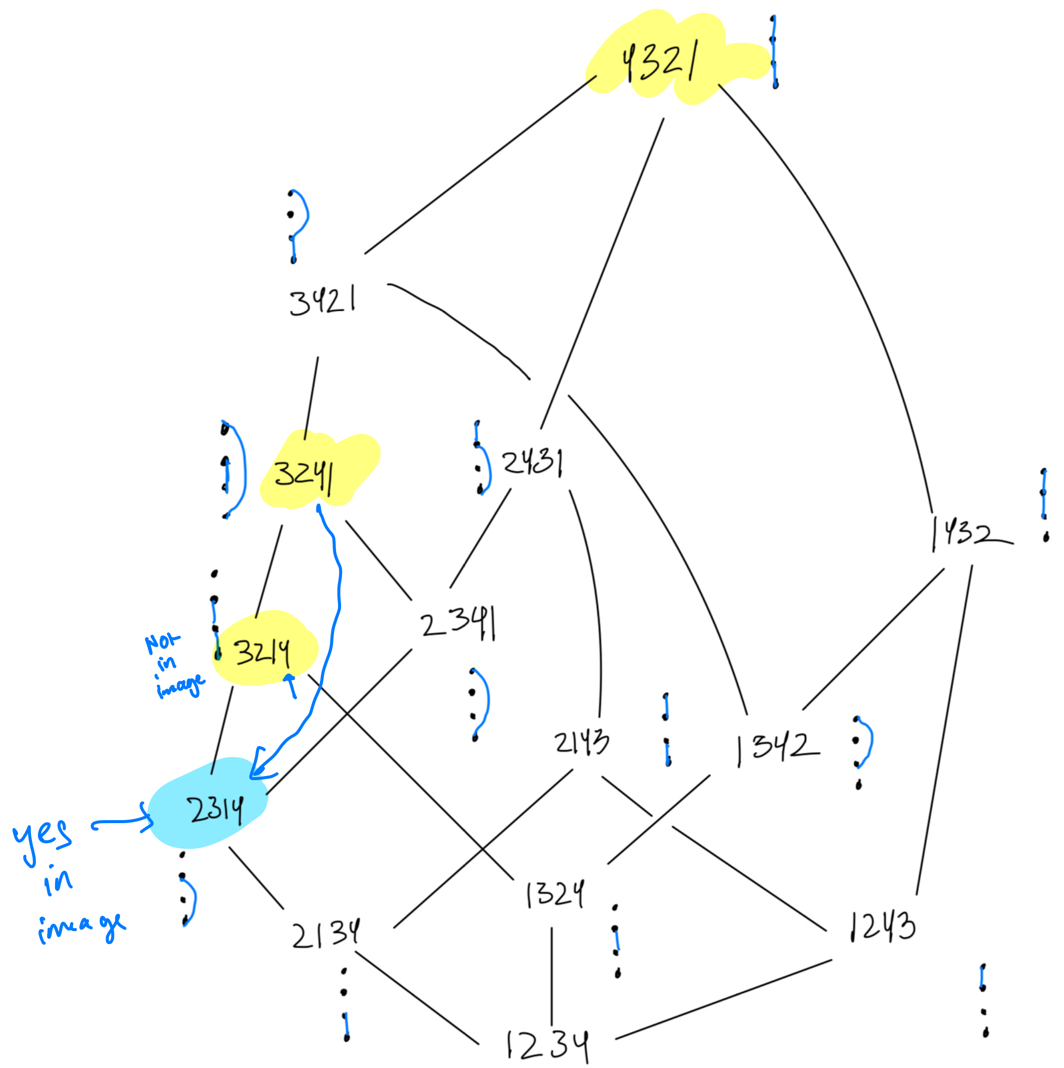
1324

Obs:

$\mathrm{pop}_{\downarrow}$ in Tamari lattice $\neq$

$\mathrm{pop}_{\downarrow}$ in weak order.

Can you find two elements that are **not** in the image of $\mathrm{pop}^{\downarrow}$?

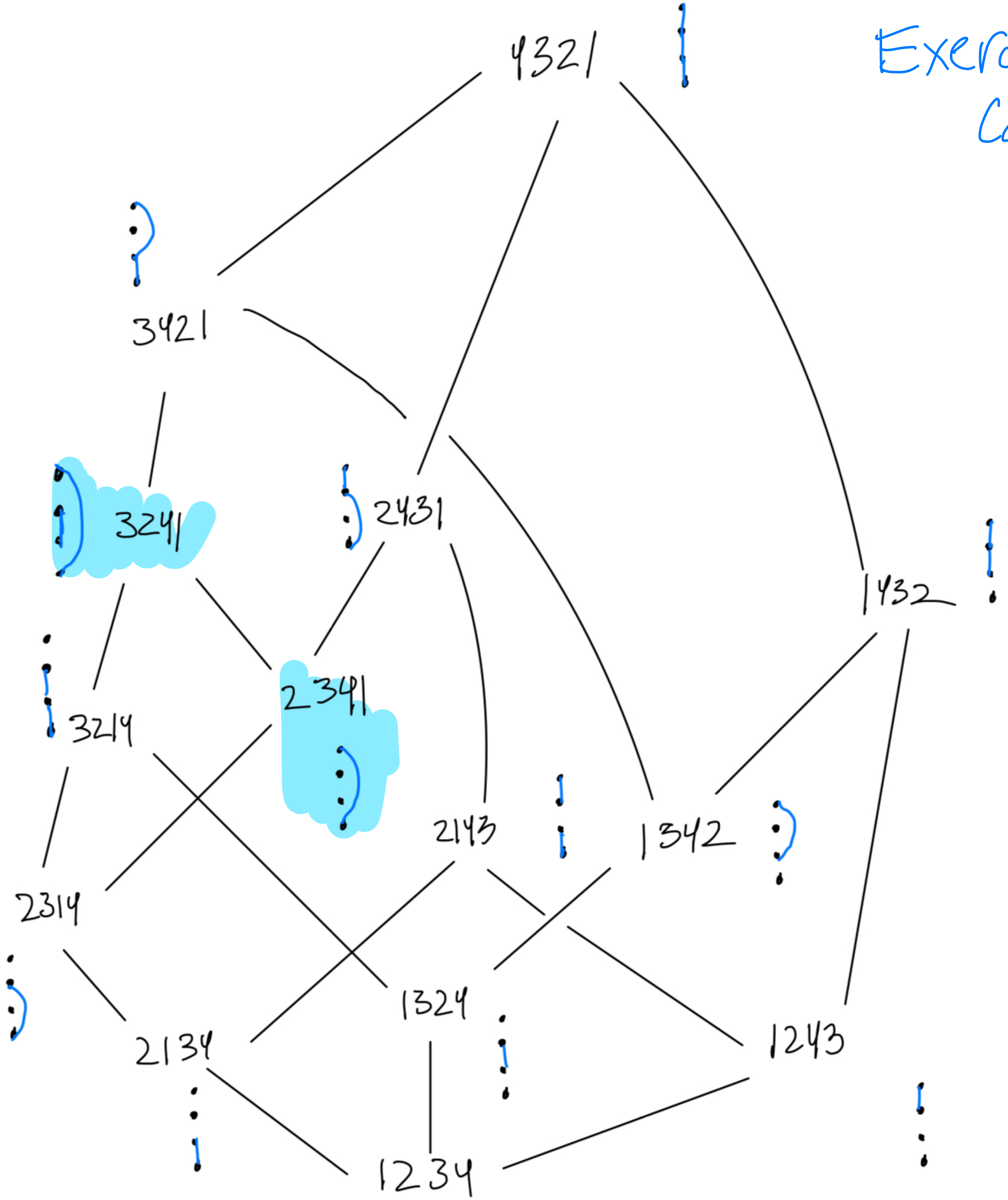


Proposition

Let 31-2 avoiding permutation $w = w_1 w_2 \cdots w_n$ in the Tamari lattice. Then w is in the image of $\mathrm{pop}^{\downarrow}$ if and only if $w_n = n$ and it has no descending run with more than two entries.

Theorem

Let 31-2 avoiding permutation $w = w_1 w_2 \cdots w_n$ be in the Tamari lattice. Then w is $(n-1)$-pop stack sortable if and only if $(1, n)$ is a descent in w.



Take longest to sort

Exercise: Count them!

Type-A Cambrian Lattices

possible
middle

Let us extend our pattern-avoidance condition.

Observations

- In the Tamari lattice, think of each number $2, 3, \ldots n-1$ is a possible MIDDLE value for a our **BIG SMALL-MIDDLE** pattern.

- You can also realize the Tamari lattice by avoiding the pattern **MIDDLE-BIG SMALL**. This is the pattern 2-31.

- Each possible middle value is disallowed from a side.

- For example, the middle value is never on the right side for the 31-2 pattern.

Type-A Cambrian Lattices

Let us extend our pattern-avoidance condition.

Expanding our pattern-avoiding condition

- For each possible middle value $2, 3, \ldots, (n-1)$ choose a side to disallow.
- Example:
  - For **2** disallow $31 -$ **2**.
  - For **3** disallow **3** $- 42$.
  - For **4** disallow $53 -$ **4**.

  - In general, **odd**-BIG SMALL and BIG SMALL - **even**.

We can do any variation of the above!

c-Cambrian lattices

We encode the left-right choice that we make for each 'middle' number as a $(0-1)$-vector, which we denote as c.

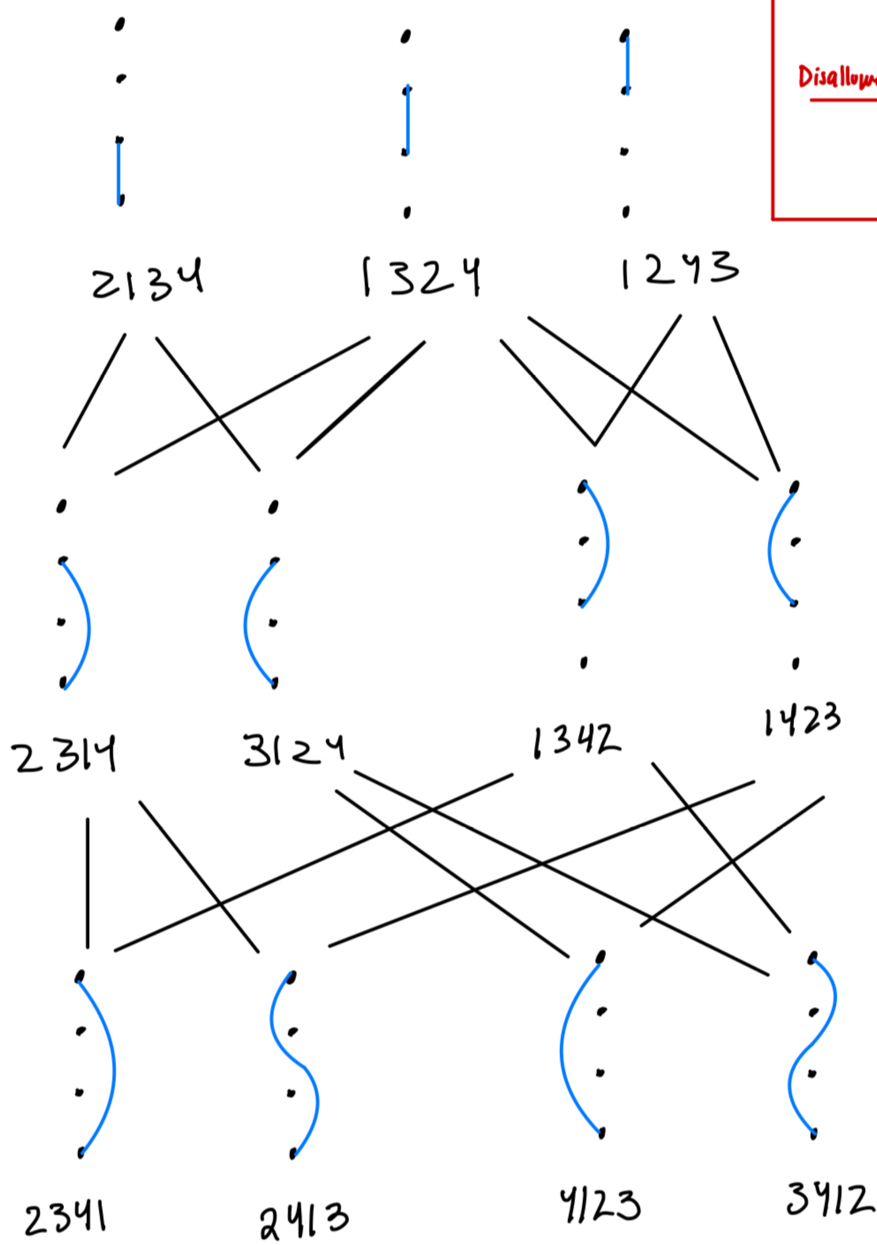
Definition

The subposet induced by the set of pattern-avoiding permutations determined by c is called a c-Cambrian lattice (in type A).

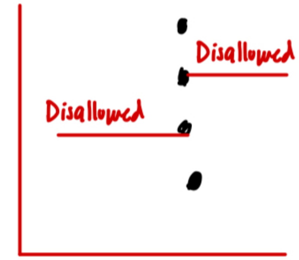
Examples

- The Tamari lattice is encoded with $c = (0, 0, \cdots, 0)$ or $c = (1, 1, \ldots, 1)$.
- The pattern avoidance condition in which even middle numbers cannot be right, and odd middle numbers cannot be left is encoded by $c = (0, 1, 0, 1, \ldots)$. This c-Cambrian lattice is also called **bipartite**.

Example

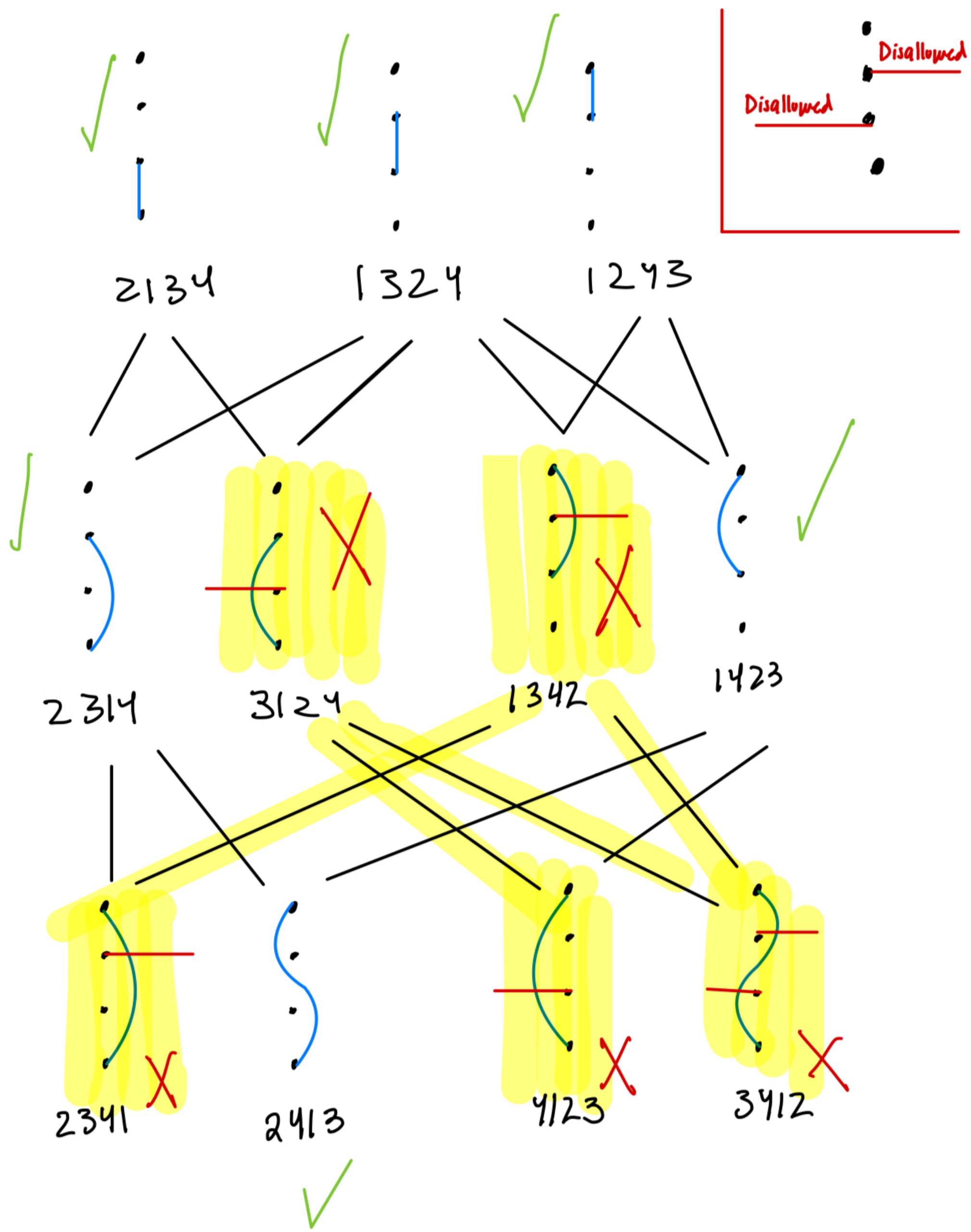


2134
1324
1243
2314
3124
1342
1423
2341
2413
4123
3412



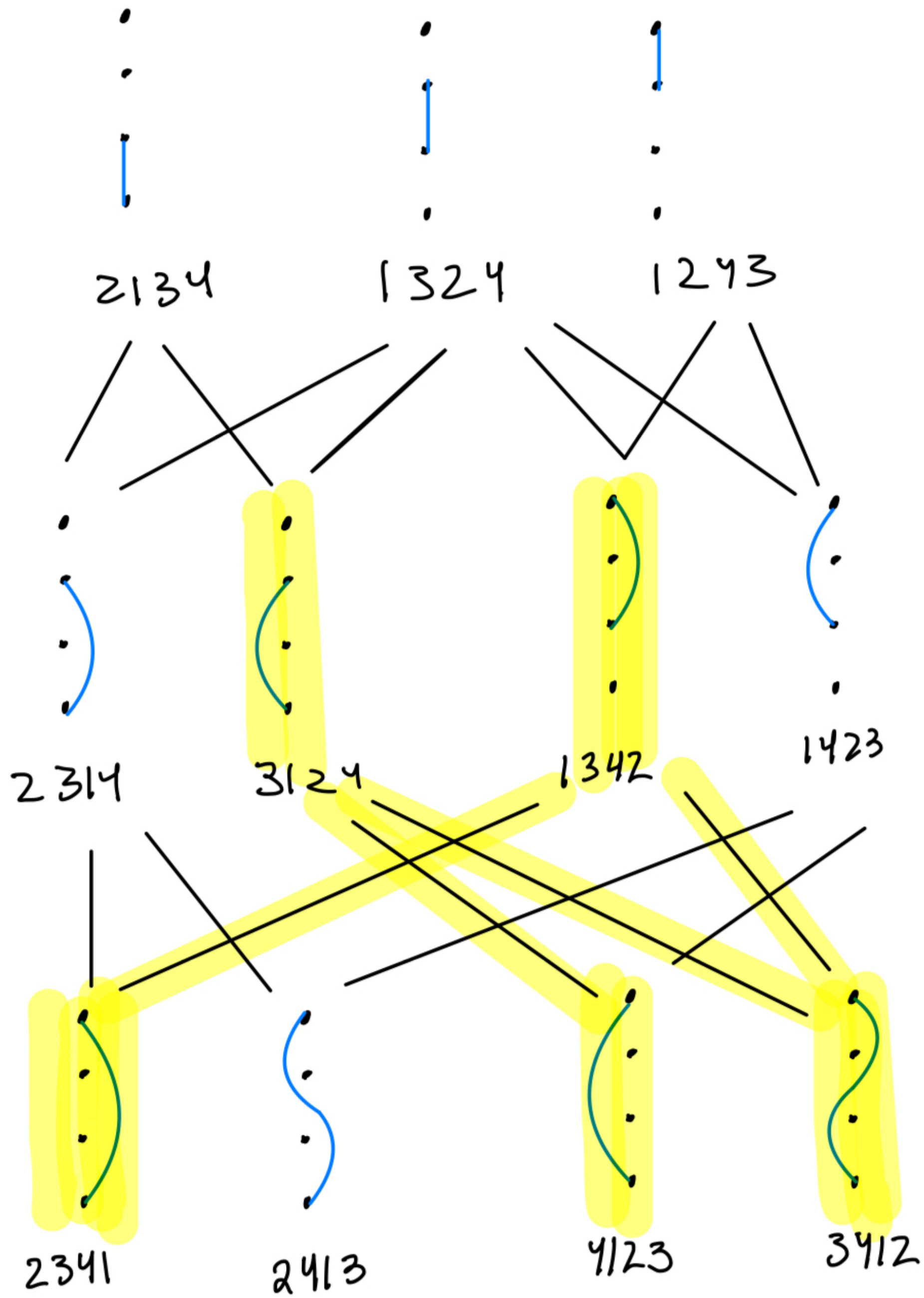
Disallowed
Disallowed

Example



Disallowed
Disallowed
2134
1324
1243
2314
3124
1342
1423
2341
2413
4123
3412

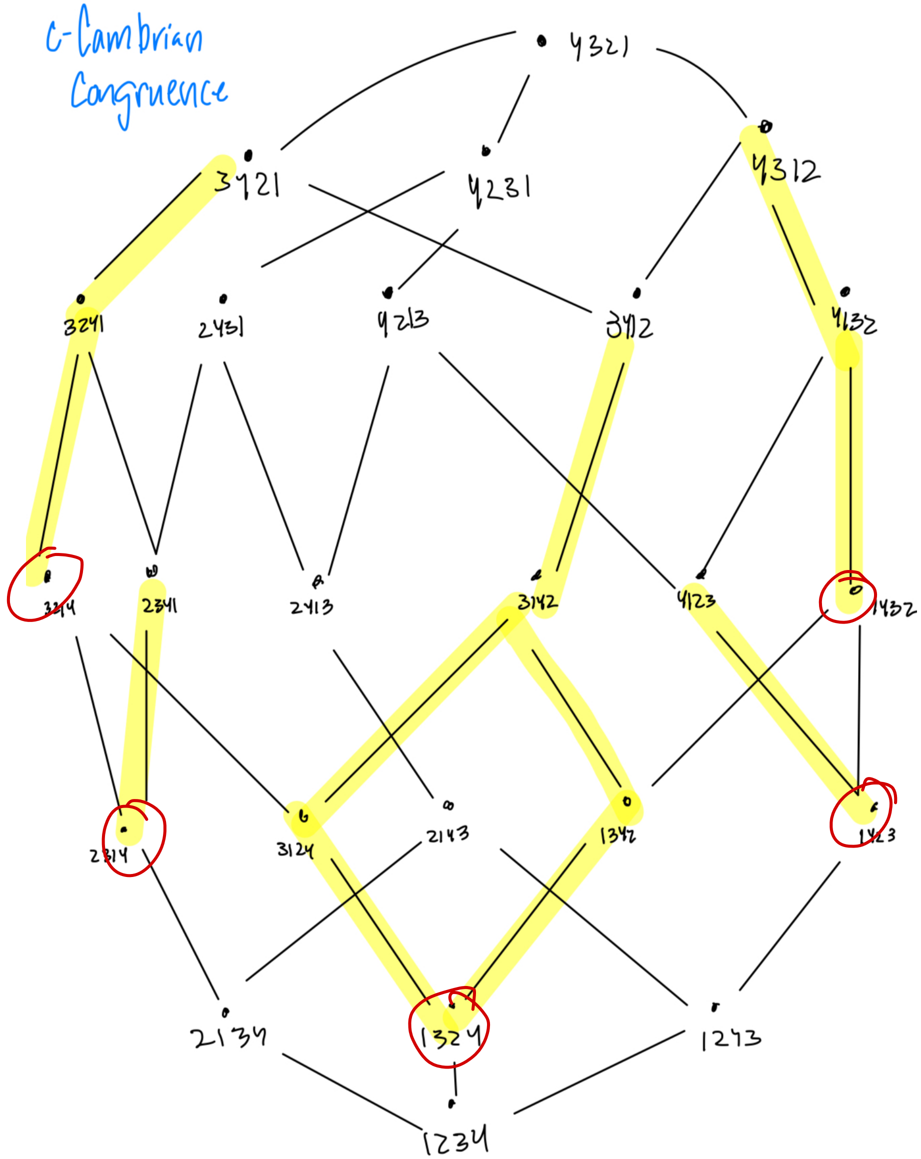
Example



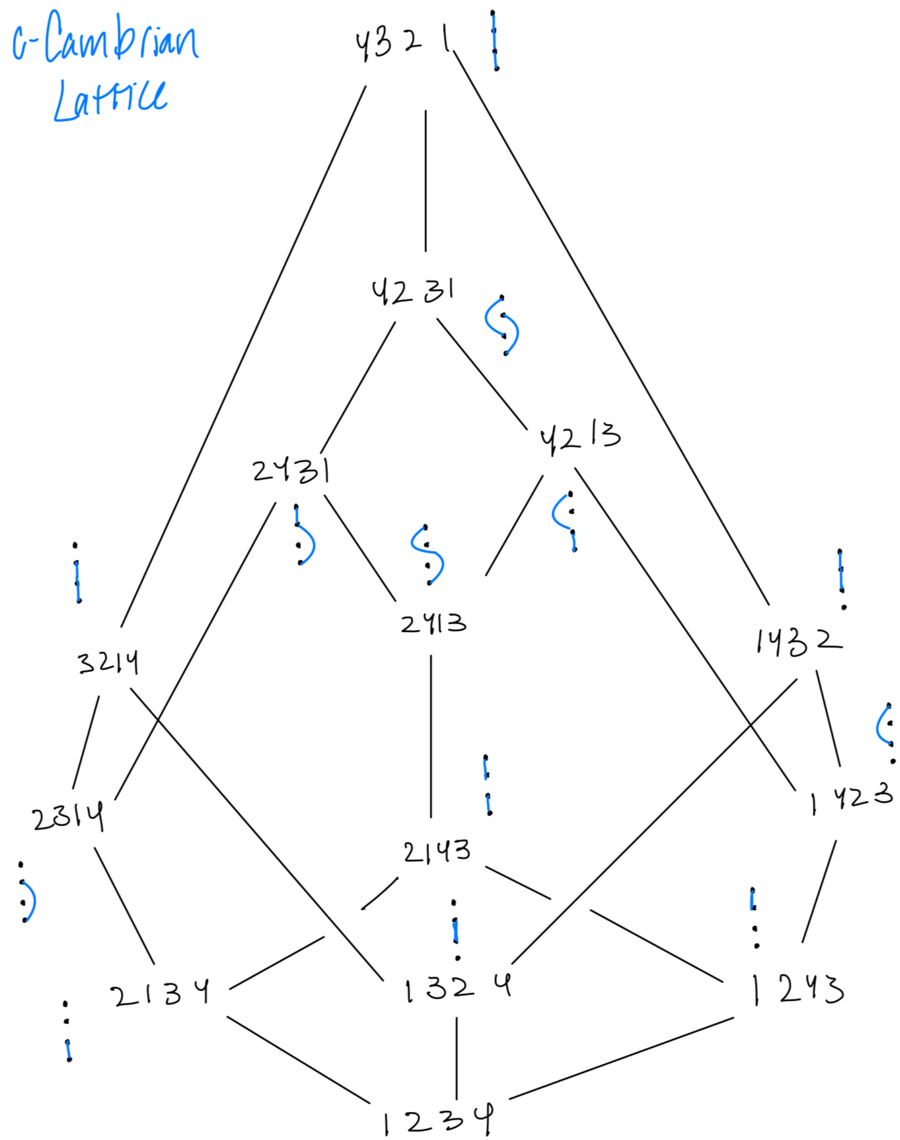
Example

Resulting lattice quotient

c-Cambrian congruence



Example


c-Cambrian Lattice
4321
4231
2431
4213
3214
2413
1432
2314
2143
1423
2134
1324
1243
1234

Main Results

w

$\mathrm{pop}_\downarrow(w)$

Definition

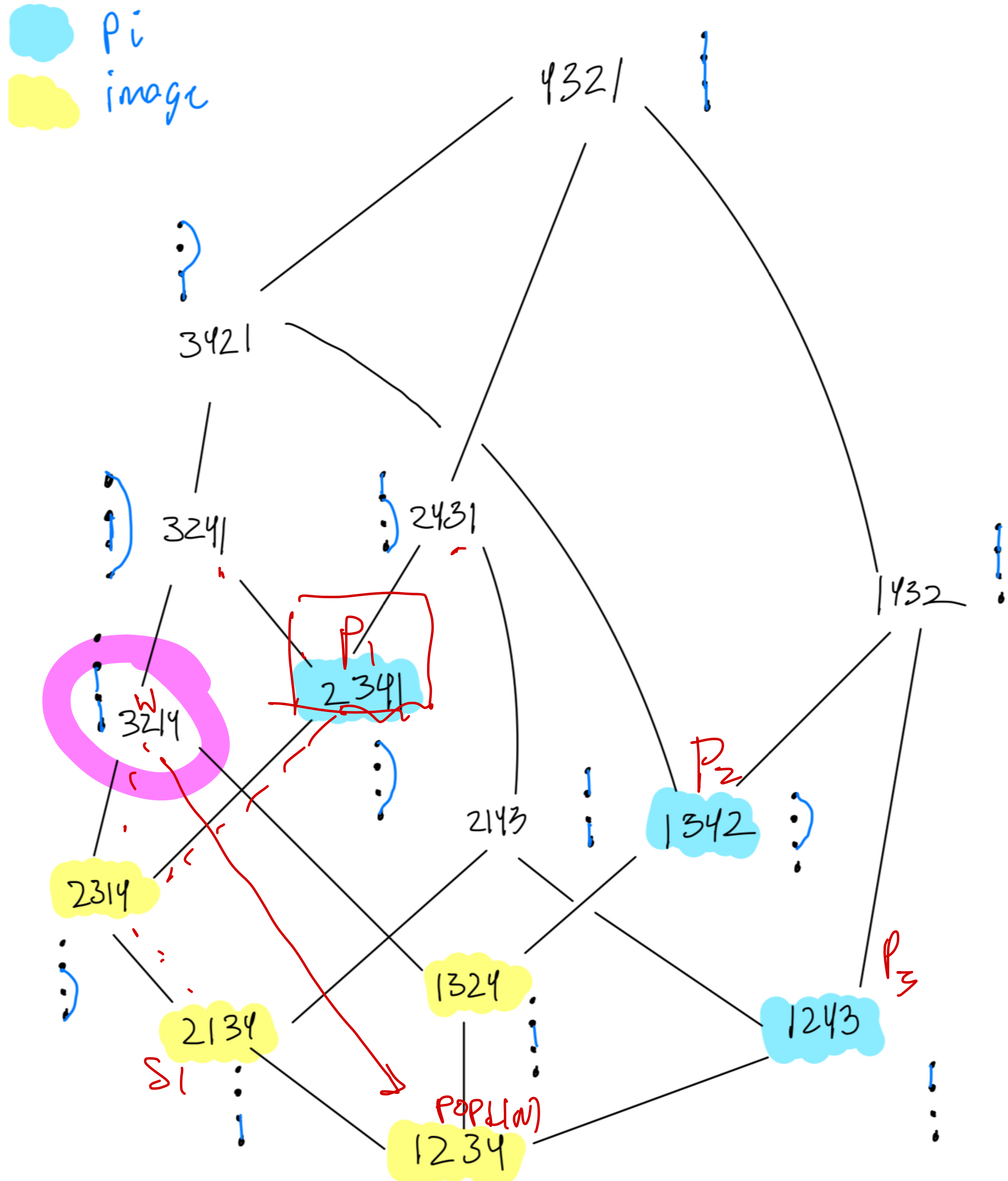
Let W be a Coxeter group of rank n with simple generators $s_1, s_2, \ldots, s_n$. For each $i \in [n]$ **define** p_i to be the unique greatest element above the simple s_i and not above any other simple s_j for $j \neq i$.

Theorem [B., Defant, Hanson]

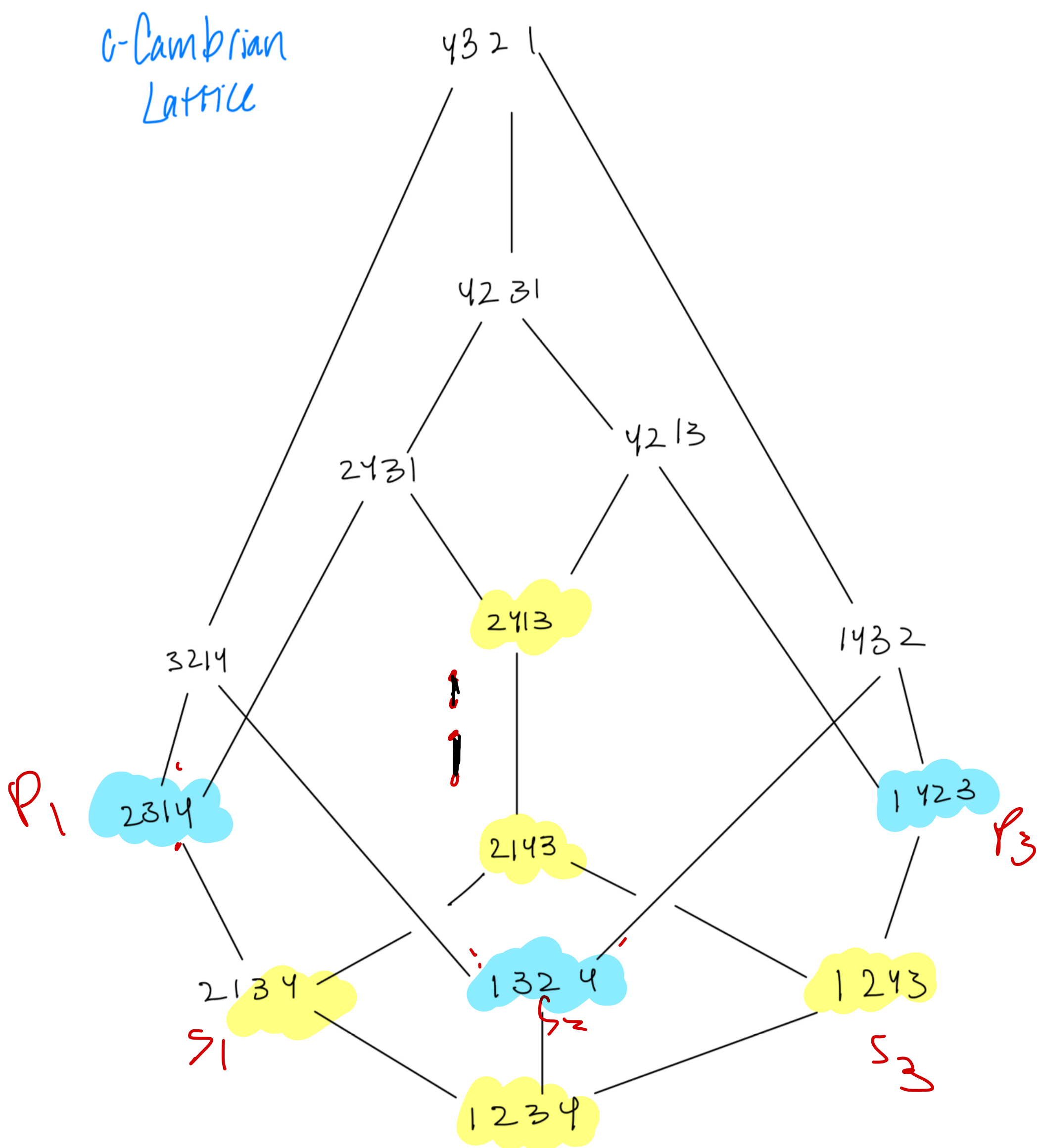
Let W be a finite irreducible Coxeter group, and let $c \in W$ be a Coxeter element. For $w \in \mathrm{Camb}_c$, the following are equivalent.

1. w is in the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}$.
2. The descents of w all commute, and w has no left inversions in common with c^{-1}.
3. The interval $\underbrace{[\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}(w), w]}_{\text{descents of } w \text{ commute;}}$ is Boolean, and $p_i \not\leq w$ for all $i \in [n]$.

No descending runs w/ more than 2 elmts.


pi
image
4321
3421
3241
2431
1432
3214
2341
2143
1342
2314
1324
1243
2134
1234
P1
P2
P3
w
s1
pop_L(w)

P_1, P_2, P_3



c-Cambrian Lattice
4321
4231
2431
4213
2413
3214
1432
2314
P_1
2143
1423
P_3
2134
1324
1243
s_1
s_2
s_3
1234

Theorem [B., Defant, Hanson]

Let c be a Coxeter element of a finite Coxeter group W. For every c-sortable element w, the following are equivalent.

1. The descents of w commute.
2. The interval $[\mathrm{pop}^{\downarrow}_{\mathrm{Weak}(W)}(w), w]$ of $\mathrm{Weak}(W)$ is a distributive lattice.
3. The interval $[\mathrm{pop}^{\downarrow}_{\mathrm{Weak}(W)}(w), w]$ of $\mathrm{Weak}(W)$ is a Boolean lattice.
4. The interval $[\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}(w), w]$ of Camb_c is a distributive lattice.
5. The interval $[\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}(w), w]$ of Camb_c is a Boolean lattice.
6. The interval $[\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}(w), w]$ of Camb_c equals the interval $[\mathrm{pop}^{\downarrow}_{\mathrm{Weak}(W)}(w), w]$ of $\mathrm{Weak}(W)$.

Mini-proof

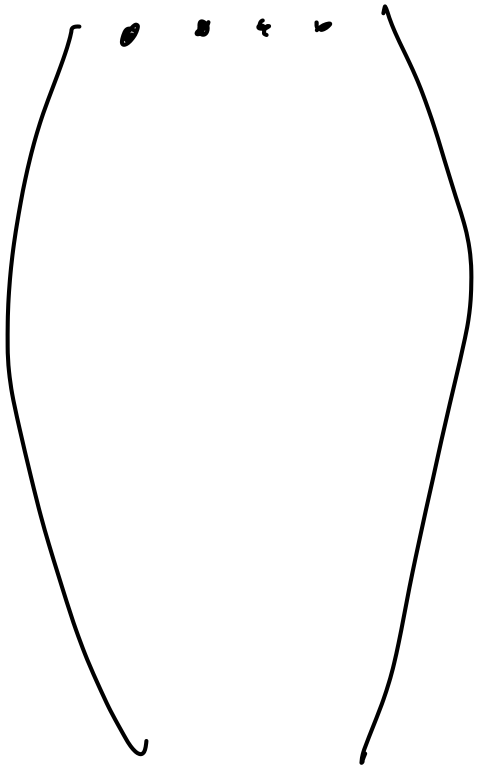
Lemma

Let L be a finite lattice, and let $x \in L$. If the interval $[\mathrm{pop}_L^{\downarrow}(x), x]$ is distributive, then it is Boolean.

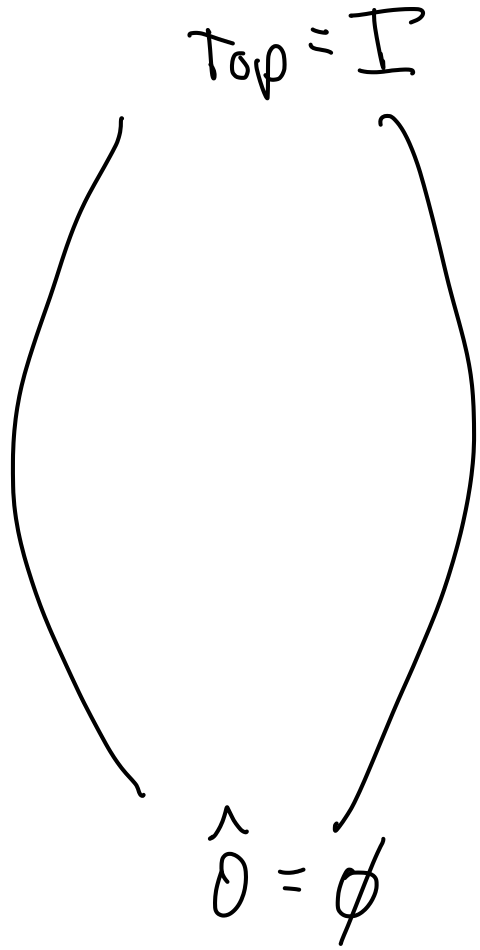
Proof

- Assume that $[\mathrm{pop}_L^{\downarrow}(x), x]$ is distributive.
- By the fundamental theorem of finite distributive lattices ...

P



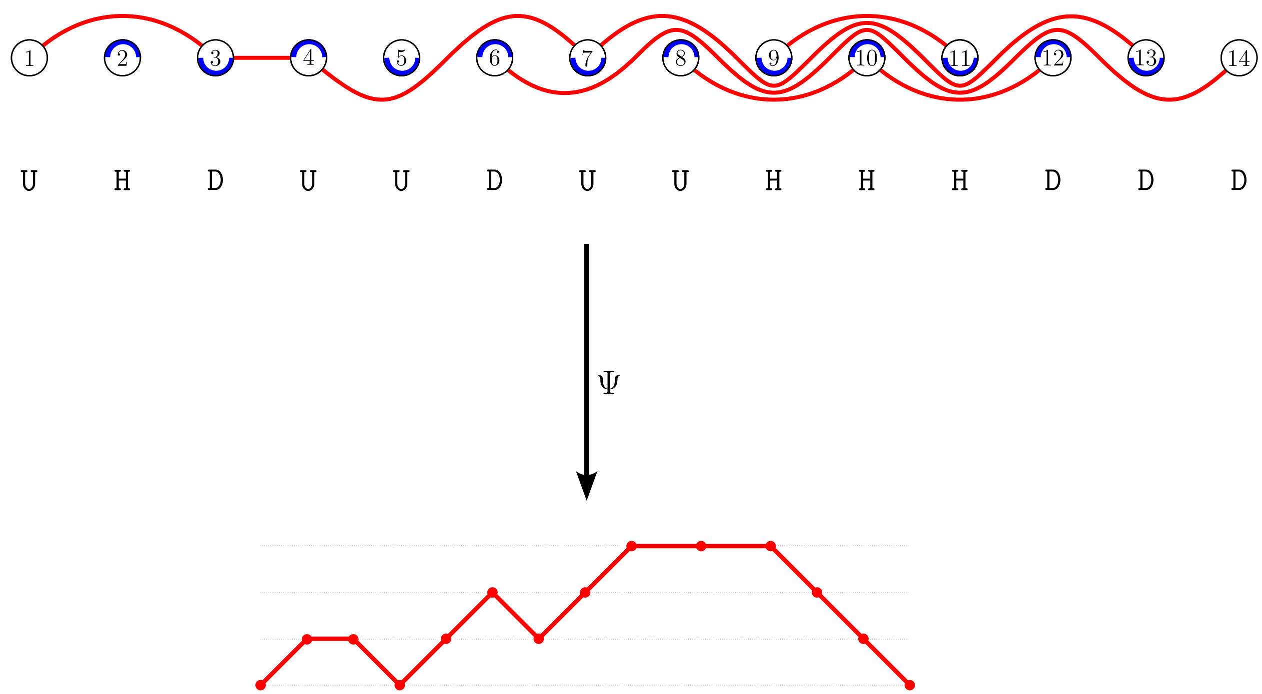
$J(P)$



Main Results

Theorem [B. Defant, Hanson]

The image of $\mathrm{pop}^{\downarrow}$ on the bipartite c-Cambrian lattice is in bijective correspondence with the set of Motzkin paths with no peak at height one.



Main Results

- Each c-Cambrian (of type W) is a *lattice-quotient* of the weak order on W.
- Defant showed the maximum size of any $\mathrm{pop}^{\downarrow}$-orbit is h, the Coxeter number of W.

Main Results

- Each c-Cambrian (of type W) is a *lattice-quotient* of the weak order on W.
- Defant showed the maximum size of any $\mathrm{pop}^{\downarrow}$-orbit is h, the Coxeter number of W.

Theorem [B., Defant, Hanson]

For each lattice-quotient of the weak order on W the maximum size of any $\mathrm{pop}^{\downarrow}$-orbit is h, the Coxeter number of W.

Main Results

- Each c-Cambrian (of type W) is a *lattice-quotient* of the weak order on W.
- Defant showed the maximum size of any $\mathrm{pop}^{\downarrow}$-orbit is h, the Coxeter number of W.

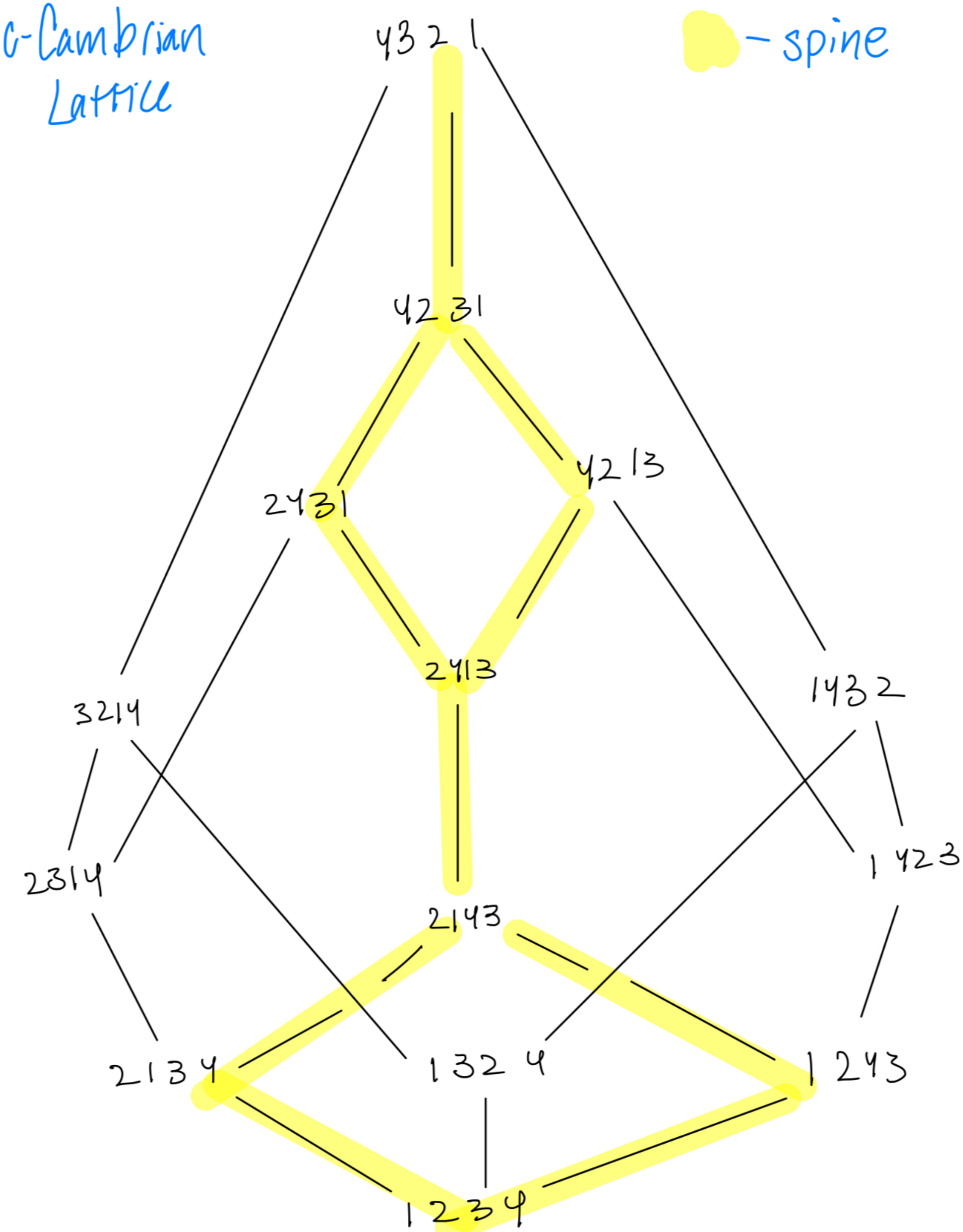
Theorem [B., Defant, Hanson]

For each lattice-quotient of the weak order on W the maximum size of any $\mathrm{pop}^{\downarrow}$-orbit is h, the Coxeter number of W.

Theorem [B., Defant, Hanson]

For each c-Cambrian lattice Camb_c, there exists an element $w \in \mathrm{Camb}_c$ whose $\mathrm{pop}^{\downarrow}$-orbit has size equal to h.

c-Cambrian Lattice



Many many open questions!!

- Compute the image of $\mathrm{pop}^{\downarrow}$ for your favorite lattice!
- Find an upper bound for k, such that $\left(\mathrm{pop}^{\downarrow}\right)^{k}(x) = \hat{0}$ for all elements of your lattice.
- Which elements take the longest to 'sort'? Count them!
- Connections to representation theory!

Representation Theory

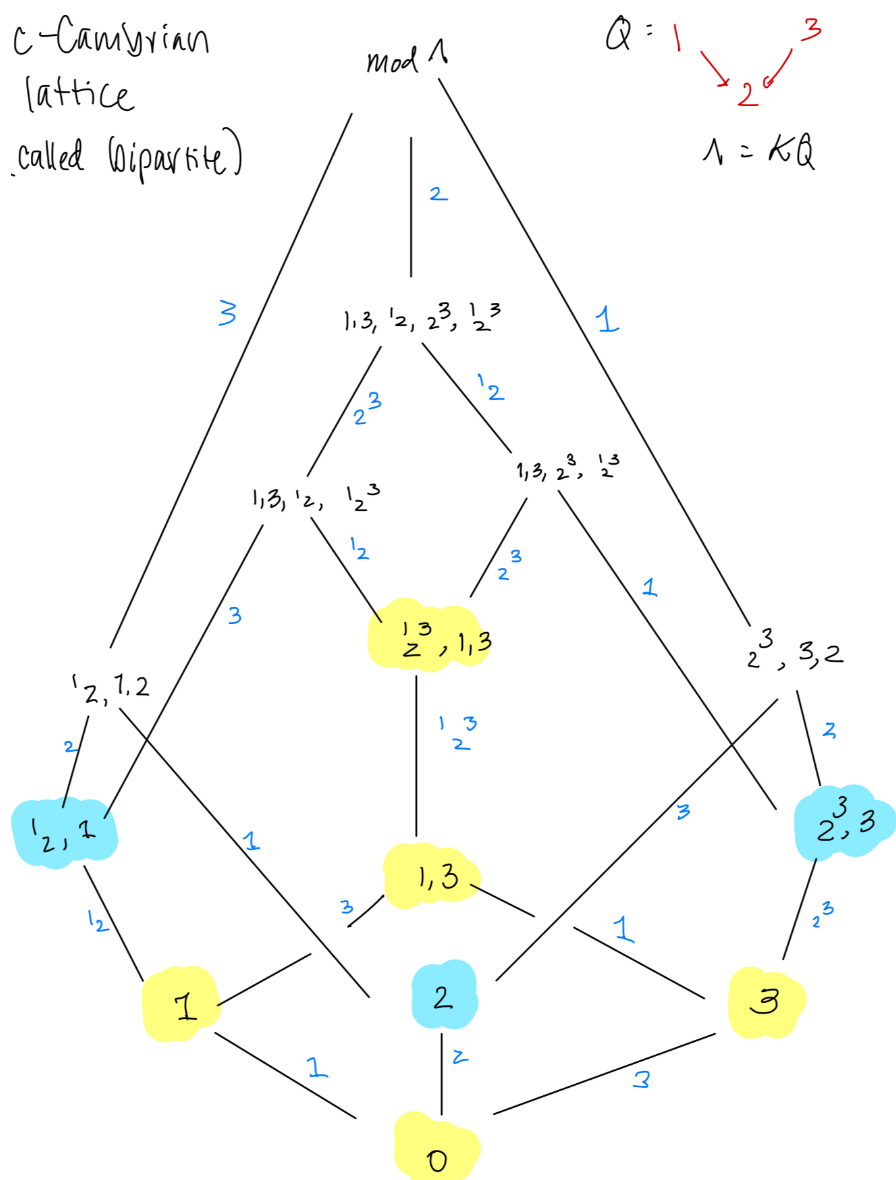
Definition

Let Λ be a finite-dimensional algebra over a field K. A torsion class is a subcategory of modules closed under isomorphism, quotients, and extensions.

The lattice of torsion classes

- We partially order torsion classes by containment, denoted torsΛ.
- The poset torsΛ is a complete lattice sharing many properties with the c-Cambrian lattices.
- Edges of the Hasse diagram of torsΛ can be labeled by certain modules called **bricks**.

Example



Pop-stack sorting

Notation

Let $\mathcal{D}$ is the set of bricks labeling its lower cover relations.

Definition

Given a torsion class $\mathcal{T}$ with lower covers labeled by bricks $\mathcal{D}$, define:

$$\mathrm{pop}^{\downarrow}(\mathcal{T}) = \mathcal{T} \cap {}^{\perp}\mathcal{D}$$

where ${}^{\perp}\mathcal{D}$ is the set of modules which do **not** map to anything in $\mathcal{D}$.

Image of pop-stack (Rep Theory version)

Theorem [B., Defant, Hanson]

Suppose Λ is hereditary, and let $\mathcal{T} \in \mathsf{tors}(\Lambda)$. Then $\mathcal{T}$ is in the image of $\mathrm{pop}^{\downarrow}_{\mathsf{tors}\Lambda}$ if and only if both of the following hold:

1. $\mathrm{Ext}^1_{\Lambda}(X, X') = 0$ for all $X, X' \in \mathcal{D}(\mathcal{T})$.
2. There does not exist a nonzero projective module $P \in \mathcal{T}$.

Thank you!!

Noncrossing Arc Diagrams and Dynamics

Emily Barnard

DePaul University

February 20, 2025