Dirac Index and Associated Cycles of Harish-Chandra Modules

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Minimal Representations and Theta Correspondence
ESI, Vienna, April 11-15, 2022

Supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

EUROPSKI STRUKTURNI I INVESTICIJSKI FONDOVI

Operativni program KONKURENTNOST I KOHEZIJA

Europska unija
Zajedno do fondova EU

Joint work with:

Salah Mehdi, University of Lorraine-Metz

David Vogan, MIT

Roger Zireau, Oklahoma State University

Published in Adv. Math. 361 (2020).

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We study admissible representations of $G_{\mathbb{R}}$ through their Harish-Chandra modules. These are finite length $\mathfrak{g}$-modules with a compatible locally finite action of K.

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Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B:
the associative algebra with 1, generated by $\mathfrak{p}$, with relations

$$xy + yx + 2B(x, y) = 0.$$

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D^2 is the "spin Laplacean".

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Let M be a Harish-Chandra module with infinitesimal character.
Let S be a spin module for $C(\mathfrak{p})$; it is constructed as $S = \bigwedge \mathfrak{p}^+$ for $\mathfrak{p}^+ \subset \mathfrak{p}$ max isotropic.

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If $H_D(M) \neq 0$, then the infinitesimal character of M can be read off from $H_D(M)$ (conjectured by Vogan, proved by Huang-P.)

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To define the Dirac index, assume $\operatorname{rank}\mathfrak{g} = \operatorname{rank}\mathfrak{k}$. Then $\dim\mathfrak{p}$ is even, so the $C(\mathfrak{p})$-module S is graded:

$$S = S^+ \oplus S^- \quad (= \textstyle\bigwedge^{\mathrm{even}}\mathfrak{p}^+ \oplus \bigwedge^{\mathrm{odd}}\mathfrak{p}^+).$$

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Since M has finite length, $I(M)$ is a finite-dimensional virtual $\widetilde{K}$-module and we consider its (virtual) dimension. It is an integer, given by the Weyl dimension formula for $\mathfrak{k}$.

Associated variety

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The associated variety $AV(M)$ of M is the support of $\mathrm{gr}(M)$ – the variety in $\mathfrak{g}^*$ defined by the ideal annihilating $\mathrm{gr}(M)$.

Since a good filtration of M is K-stable, $\mathfrak{k} \subset \mathrm{ann}(\mathrm{gr}(M))$, so $AV(M)$ is a K-stable subset of $(\mathfrak{g}/\mathfrak{k})^* \cong \mathfrak{p}^*$.

Associated variety

In fact, $AV(M)$ is contained in the nilpotent cone $\mathcal{N}_\mathfrak{p}$, so is a union of closures of some K-orbits in $\mathcal{N}_\mathfrak{p}$:

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If M is irreducible, then each $\mathcal{O}_k$ is a 'real form' of a single G-orbit in the nilpotent cone $\mathcal{N}_{\mathfrak{g}}$. In other words, there is a G-orbit $\mathcal{O}^{\mathbb{C}} \subset \mathcal{N}_{\mathfrak{g}}$ such that $G \cdot \mathcal{O}_k = \mathcal{O}^{\mathbb{C}}$, for all $k = 1, \ldots, \ell$.

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In this situation, $\overline{\mathcal{O}^{\mathbb{C}}}$ is the associated variety of the annihilator of M (i.e., the associated variety of the $U(\mathfrak{g})$-module $U(\mathfrak{g})/\operatorname{ann}(M)$).

Associated cycle

The associated cycle of M is a formal integer combination

$$AC(M) = \sum_{k=1}^{\ell} m_k \overline{\mathcal{O}}_k,$$

where the multiplicity m_k is the rank of $\mathrm{gr}(M)$ at a generic point in $\overline{\mathcal{O}}_k$.

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Definitions and basic facts can be found in
D. A. Vogan, *Associated varieties and unipotent representations*, Harmonic Analysis on reductive groups (W. Barker and P. Sally, eds.), Progress in Mathematics, vol. 101, Birkhäuser, 1991, pp. 315–388.

Coherent families

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\lambda_0 \in \mathfrak{h}^*$ be regular. Let Λ be the lattice of G-integral weights.

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1. For each λ, X_λ has infinitesimal character λ;
2. For any finite-dimensional $(\mathfrak{g}, K)$-module F, and for any λ,

$$X_\lambda \otimes F = \sum_{\mu \in \Delta(F)} X_{\lambda+\mu},$$

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See Vogan's Green Book.

Coherent families

A basic fact is that any (virtual) Harish-Chandra module X with regular infinitesimal character λ_0 defines a unique coherent family X_λ with $X_{\lambda_0} = X$.

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In particular, if an invariant of Harish-Chandra modules can be extended to virtual Harish-Chandra modules, then studying how it varies over the coherent family attached to X can lead to new invariants of X.

Dirac index polynomial

Dirac index can be defined for virtual Harish-Chandra modules, and it varies coherently over any coherent family on $\mathfrak{h}^*$, where $\mathfrak{h}$ is the compact Cartan subalgebra of $\mathfrak{g}$.

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This follows from the fact that

$$I(M) = M \otimes S^+ - M \otimes S^-$$

for M with infinitesimal character. The right side of this formula then gives the general definition and one shows that it has the desired properties. (One can also extend the definition of Dirac cohomology and define the index as its Euler characteristc.)

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The function $\lambda \mapsto \dim I(X_\lambda)$ extends to a homogeneous polynomial $DI_p(X)$ on $\mathfrak{h}^*$. It is given by the Weyl dimension formula for $\mathfrak{k}$.

Dirac index polynomial

$DI_p(X)$ is W-harmonic, i.e., it generates a W-representation which does not appear in lower degrees. (W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.)

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Reference:

S. Mehdi, P. Pandžić, D. Vogan, *Translation principle for Dirac index*, Amer. J. Math. **139** (2017), no. 6, 1465–1491.

Multiplicity polynomials

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For a coherent family X_λ, all X_λ have the same associated variety, while the multiplicities m_k vary with λ in a polynomial fashion.

Multiplicity polynomials

In this way, starting from a Harish-Chandra module X and considering its coherent family X_λ, one obtains the multiplicity polynomials $m_k(X)$. They are again W-harmonic.

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In general, the polynomials $m_k(X)$ are quite different from $DI_p(X)$; for example their degrees are usually different.

For a certain interesting class of Harish-Chandra modules we have however found a connection and we showed how $DI_p(X)$ can be obtained as an integer linear combination of the $m_k(X)$.

The main result

Assume that $G_{\mathbb{R}}$ is simple, connected and linear and that $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank.
Assume that the W-representation generated by the Weyl dimension formula for $\mathfrak{k}$ is Springer and corresponds to a G-orbit $\mathcal{O}^{\mathbb{C}} \subset \mathcal{N}_{\mathfrak{g}}$.

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These assumptions are satisfied if

- $G_{\mathbb{R}} = SU(p,q)$;
- $G_{\mathbb{R}} = SO_e(2p, 2q+1)$ with $q \geq p-1$;
- $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$
- $G_{\mathbb{R}} = SO^*(2n)$;
- $G_{\mathbb{R}} = SO_e(2p, 2q)$;
- $G_{\mathbb{R}}$ is any exceptional equal rank group except F_4 with $K_{\mathbb{R}} = \mathrm{Spin}(9)$.

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Then

$$DI_p(X) = \sum_k c_k m_k(X),$$

where c_k are integers independent of X. Moreover, these constants can be explicitly computed.

Remarks about the proof

The first issue is the definition of the associated cycle for virtual Harish-Chandra modules. This was done in

D. A. Vogan, *The method of coadjoint orbits for real reductive groups*, Representation theory of Lie groups (Park City, UT, 1998), IAS/Park City Math. Ser., vol. 8, Amer. Math. Soc., Providence, RI, 2000, pp. 179–238.

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We use a version of this construction in terms of the K-theory of K-equivariant coherent sheaves on the nilpotent cone $\mathcal{N}_{\mathfrak{p}}$.

If M is a Harish-Chandra module, then $\operatorname{gr} M$ is a finitely generated $(S(\mathfrak{p}), K)$-module supported in $\mathcal{N}_{\mathfrak{p}}$. Since $\mathcal{N}_{\mathfrak{p}}$ is affine, such modules correspond to K-equivariant coherent sheaves on $\mathcal{N}_{\mathfrak{p}}$.

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In this way we get a map

$$\operatorname{gr}\colon \{\text{virtual Harish-Chandra modules}\} \to \mathrm{K}^K(\mathcal{N}_\mathfrak{p}),$$

where $\mathrm{K}^K(\mathcal{N}_\mathfrak{p})$ is the Grothendieck group of the abelian category of K-equivariant coherent sheaves on $\mathcal{N}_\mathfrak{p}$.

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Roughly speaking, we now find a nice $\mathbb{Z}$-basis for $\mathrm{K}^K(\mathcal{N}_{\mathfrak{p}})$ – corresponding to certain integer combinations of the discrete series representations – for which everything can be computed.

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Write $\mathcal{O} = K \cdot e = K/K^e$ and let $\tau \in (K^e)^\frown$, where $(K^e)^\frown$ is the set of irreducible algebraic representations of K^e.

Then the sheaf of sections of the homogeneous vector bundle

$$K \underset{K^e}{\times} \tau \to \mathcal{O}$$

extends (although not uniquely) to a K-equivariant coherent sheaf on $\overline{\mathcal{O}}$, then extends by zero to a K-equivariant coherent sheaf $\tilde{\mathcal{E}}_{\mathcal{O}}(\tau)$ on $\mathcal{N}_{\mathfrak{p}}$.

Theorem

$\{[\tilde{\mathcal{E}}_{\mathcal{O}}(\tau)]\}$, *with* $\mathcal{O}$ *running over all* K*-orbits in* $\mathcal{N}_{\mathfrak{p}}$ *and* $\tau \in (K^e)^\frown$, *is a* $\mathbb{Z}$*-basis of* $\mathrm{K}^K(\mathcal{N}_{\mathfrak{p}})$.

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Moreover, for any virtual Harish-Chandra module X, *writing* $gr(X) = \sum_{\mathcal{O},\tau} n_{\mathcal{O},\tau}[\tilde{\mathcal{E}}_{\mathcal{O}}(\tau)]$, *we have*

$$AC(X) = \sum_{\mathcal{O}\ \text{max'l}} \left(\sum_{\tau} n_{\mathcal{O},\tau}\, dim(\tau) \right) \overline{\mathcal{O}}.$$

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An important point is that these "leading coefficients" $n_{\mathcal{O},\tau}$ are independent of the choices of extensions $\tilde{\mathcal{E}}_{\mathcal{O}'}(\tau')$.

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The conjecture is then proved using a simple formula for the Dirac index of a discrete series representation (up to sign, it is the lowest K-type shifted by $\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$).

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The eigenvalues of $\mathrm{ad}(h)$ are integers, therefore give a grading of $\mathfrak{g}$:

$$\mathfrak{g}_m = \{X \in \mathfrak{g} : [h, X] = mX\}, \ m \in \mathbb{Z}.$$

An extension $\tilde{\mathcal{E}}_{\mathcal{O}}(\tau)$ can be constructed as the pushforward of a vector bundle under a resolution of singularities of $\overline{\mathcal{O}}$.

Let $\mathcal{O} = K \cdot x$ be a K-orbit in $\mathcal{N}_{\mathfrak{p}}$. Let $\{x, h, y\}$ be a standard triple with $h \in \mathfrak{h}$.

The eigenvalues of $\mathrm{ad}(h)$ are integers, therefore give a grading of $\mathfrak{g}$:

$$\mathfrak{g}_m = \{X \in \mathfrak{g} : [h, X] = mX\},\ m \in \mathbb{Z}.$$

We set $\mathfrak{p}_m = \mathfrak{g}_m \cap \mathfrak{p}$, the m-eigenspace in $\mathfrak{p}$.

This grading gives a θ-stable parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}, \qquad \text{with } \mathfrak{l} = \mathfrak{g}_0 \text{ and } \mathfrak{u} = \sum_{m \leq -1} \mathfrak{g}_m.$$

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The corresponding parabolic subgroup in G and its Levi decomposition are written as $Q = LU$.

We use two well-known facts due to Kostant and Rallis:

(i) The stabilizer K^x has a Levi decomposition with reductive part
$$(K^x)_{\mathrm{red}} = K \cap L^x = (K \cap L)^x$$
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and $K^x \subset K \cap \overline{Q}$.

(ii) If we set $\mathfrak{p}[2] := \sum_{m \geq 2} \mathfrak{p}_m$, then $\mathfrak{p}[2]$ is stable under $\mathrm{Ad}(K \cap \overline{Q})$ and the morphism

$$\mu \colon K \underset{K \cap \overline{Q}}{\times} \mathfrak{p}[2] \to \overline{\mathcal{O}}, \qquad \mu(k, \xi) := k \cdot \xi, \tag{1}$$

is a resolution of singularities.

The higher direct images $R^i\mu_*$ of the morphism μ give a homomorphism of K^K groups:

$$\begin{aligned} \mu_*\colon \mathsf{K}^K(K \underset{K\cap\overline{Q}}{\times} \mathfrak{p}[2]) &\to \mathsf{K}^K(\mathcal{N}_\mathfrak{p}) \\ \mu_*([\mathcal{S}]) &:= \sum_i (-1)^i [R^i\mu_*(\mathcal{S})]. \end{aligned} \tag{2}$$

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For any representation σ of $K \cap \overline{Q}$ there is a K-equivariant vector bundle

$$K \underset{K\cap\overline{Q}}{\times} (\mathfrak{p}[2] \otimes \sigma) \to K \underset{K\cap\overline{Q}}{\times} \mathfrak{p}[2].$$

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We write $\mathcal{S}_\sigma$ for the sheaf of algebraic sections of this bundle. Using results of Achar and some additional computations we show

(1) $[\mu_*(\mathcal{S}_\sigma)]$ is an extension of $\mathcal{E}_{\mathcal{O}}(\sigma|_{K^\times})$.

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(2) Assume σ is irreducible with highest weight λ_σ, and denote by $X(\mu)$ the discrete series representation with Harish-Chandra parameter μ. Then $\Gamma(\mu_*(\mathcal{S}_\sigma))|_K$ equals

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$$\sum_{\substack{A\subset\Delta_n^+(\mathfrak{l})\\ C\subset\Delta(\mathfrak{p}_1)}} (-1)^{\#A+\#C} X(\lambda_\sigma+\rho_c(\mathfrak{l})-\rho_n(\mathfrak{l})+2\rho(A)-2\rho(C)$$
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This gives the announced formula for the extensions in terms of discrete series representations. (One shows that in this way we indeed obtain extensions of all $\mathcal{E}_\mathcal{O}(\tau)$, $\tau \in (K^\times)\widehat{\ }$.)

More about K-groups

Suppose an algebraic group H acts on a variety X with finitely many orbits. Let $\mathrm{K}^H(X)$ denote the Grothendieck group of the category of H-equivariant coherent sheaves on X.

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Let $Z \subset X$ be closed and H-stable, and let $U = X \setminus Z$. Let $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$ be the embeddings. Then there is an exact sequence

$$\cdots \longrightarrow \mathsf{K}_1^H(U) \longrightarrow \mathsf{K}^H(Z) \xrightarrow{i_*} \mathsf{K}^H(X) \xrightarrow{j^*} \mathsf{K}^H(U) \longrightarrow 0,$$

where i_* is extension by zero and j^* is the restriction from X to U.

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Quillen's definition starts from an abelian (or exact) category $\mathcal{A}$ and attaches to it another category $Q(\mathcal{A})$, with the same objects as $\mathcal{A}$, and with morphisms being isomorphisms onto a subquotient.

Then one considers the nerve of $Q(\mathcal{A})$: it is the simplicial set with p-simplices defined as diagrams $X_0 \to \cdots \to X_p$ in $Q(\mathcal{A})$, and edges defined by deleting one of the X_i.

Then one passes to the classifying space $BQ(\mathcal{A})$ of $Q(\mathcal{A})$; it is a CW complex which is a geometric realization of the nerve of $Q(\mathcal{A})$.

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$$K_i(\mathcal{A}) = \pi_{i+1}(BQ(\mathcal{A})).$$

One shows that the fundamental group $\pi_1(BQ(\mathcal{A}))$ is the Grothendieck group $K(\mathcal{A})$.

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By surjectivity of j^*, $\mathcal{E}_{\mathcal{O}}(\tau)$ extends to $\overline{\mathcal{O}}$. We extend this extension further by zero and obtain an H-equivariant coherent sheaf $\tilde{\mathcal{E}}_{\mathcal{O}}(\tau)$ on X.

We list the H-orbits on X as $\mathcal{O}_1, \ldots, \mathcal{O}_M$ in a way compatible with closure ($\mathcal{O}_k \subset \overline{\mathcal{O}}_j$ implies $k \leq j$).

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Choose base points x_k in each orbit; so $\mathcal{O}_k = H \cdot x_k$.

Applying the construction above to each orbit and each isotropy representation we obtain coherent sheaves $\tilde{\mathcal{E}}_{\mathcal{O}_k}(\tau_{kl})$.

Lemma

$\mathrm{K}^H(X)$ is spanned (over $\mathbb{Z}$) by
$\{\tilde{\mathcal{E}}_{\mathcal{O}_k}(\tau_{kl}) : k = 1, \ldots, M, \tau_{kl} \in (H^{x_k})\widehat{\ }\}$.

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This is proved by induction on the number M of orbits in X.

The inductive step uses Basic Fact 2 applied to $U = \mathcal{O}_M$ (which has to be open), and $Z = X \setminus U$.

Theorem

In the above setting, let
$\mathcal{B} := \{[\tilde{\mathcal{E}}_{\mathcal{O}_k}(\tau_{kl})] : k = 1, \dots, M, \tau_{kl} \in (H^{x_k})\widehat{\ }\} \subset \mathrm{K}^H(X)$.

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Then

1. for any closed embedding $i \colon Z \hookrightarrow X$ of an H-stable subset Z of X, $i_* \colon \mathrm{K}^H(Z) \to \mathrm{K}^H(X)$ is injective
2. $\mathcal{B}$ is a $\mathbb{Z}$-basis of $\mathrm{K}^H(X)$.

Sketch of proof

Induction on the number of orbits.

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If there is just one orbit, then $X = \mathcal{O} = H \cdot x$ and the coherent sheaves $\mathcal{E}_{\mathcal{O}}(\tau)$ are a basis; this is Basic Fact 1. (1) holds trivially.

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This follows from the exact sequence of Basic Fact 2, and the fact that there are no nonzero homomorphisms from $\mathrm{K}_1^H(\mathcal{O}_M)$ to $\mathrm{K}^H(Z)$.

To see this last fact, we compute $\mathrm{K}_1^H(\mathcal{O}_M)$ and see that it is equal to a direct sum of copies of $\mathbb{C}^\times$ (one copy for each representation of H^{x_M}).

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This follows from Quillen's Devissage Theorem which says that any abelian category has the same K-groups as its semisimplification, and from the well known fact that the K_1 group of the category of finite-dimensional vector spaces is $\mathbb{C}^\times$.

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On the other hand, our induction hypothesis says that $\mathrm{K}^H(Z)$ is a free abelian group and there are no nonzero homomorphisms from a divisible group into a free abelian group.

THANK YOU AND HAPPY BIRTHDAY TO GORDAN!