Cartan geometries
lecture 2

Andreas Čap

University of Vienna
Faculty of Mathematics

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- This second lecture first explains a general perspective on the conformal Cartan connection as discussed in lecture 1.
- This leads to parabolic geometries and to more general Cartan geometries associated to certain filtered G_0-structures and I want to give you an impression of what this looks like.
- Then I will discuss some of the general tools for Cartan geometries, with a focus on cases related either to conformal geometry or to the geometric theory of differential equations.
- On the one hand, we will discuss constructions relating geometries of different types, on the other hand constructions of invariant differential operators.

Contents

The construction of the conformal Cartan connection seemed to depend on specific aspects of the underlying structure. But there are indications that things may be simplified. E.g. the kernel P_+ of $P \to CO(n)$ is contractible, so $\mathcal{G} \cong \mathcal{G}_0 \times P_+$. Starting from the homogeneous model rather than the underlying structure, the important features of the conformal situation generalize:

- conformal: $\mathfrak{g} = \mathfrak{so}(n+1,1)$ is simple and admits a grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_0 = \mathfrak{co}(n)$, $\mathfrak{g}_1 = \mathfrak{p}_+$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.
- This generalizes to parabolic subalgebras in semisimple Lie algebras which come from gradings $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ for some $k \geq 1$. They are classified by subsets of simple roots.

One crucial ingredient for the construction is related to the algebraic prolongations of $\mathfrak{co}(n)$. Now the negative part $\mathfrak{g}_-$ of the grading is a Lie subalgebra represented on $\mathfrak{g}$. The facts on prolongations can be rephrased in terms of $H^1(\mathfrak{g}_-, \mathfrak{g})$. This can be computed in all cases using Kostant's theorem.

The second crucial ingredient is the normalization condition (torsion-freeness plus vanishing Ricci type contraction). This is also closely connected to $H^*(\mathfrak{g}_-, \mathfrak{g})$ and via a duality between $\mathfrak{g}_-$ and $\mathfrak{p}_+ = \oplus_{i>0}\mathfrak{g}_i$ this can be obtained (for any $(\mathfrak{g}, \mathfrak{p})$) from a Lie algebra homology differential ("Kostant codifferential").

For any parabolic pair (G, P), a Cartan geometry $(\mathcal{G} \to M, \omega)$ now determines an underlying structure given by a principal bundle $\mathcal{G}_0 \to M$ with structure group $G_0 := P/P_+$ and a family of partially defined differential forms induced by ω. This admits an interpretation as a filtered analog of a G_0-structure.

Example: Put $G = SU(n+1, 1)$ and $P \subset G$ the stabilizer of a complex isotropic line. Then $G/P \cong S^{2n+1}$ and this inherits a contact structure $H \subset TS^{2n+1}$. In addition H naturally is a complex vector bundle of rank n. This is the standard CR structure on S^{2n+1} and indeed the filtered G_0-structures obtained here are strictly pseudoconvex almost CR structures.

Now there is a general proof for existence and uniqueness of Cartan geometries inducing a given underlying structure as follows:

- Starting from $\mathcal{G}_0$, define $\mathcal{G} := \mathcal{G}_0 \times_{G_0} P$.
- Choose a principal principal connection on $\mathcal{G}$; use this and the data given on $\mathcal{G}_0$ to define a Cartan connection $\hat{\omega}$ on $\mathcal{G}$. Then $(\mathcal{G}, \hat{\omega})$ induces the given underlying structure.
- Cartan connections on $\mathcal{G}$ with fixed underlying structure form an affine space. There is a notion of homogeneity for Cartan connections and curvatures. In lowest homogeneity, the dependence of curvature is tensorial (explicit map).
- Using the normalization condition discussed above, one can normalize $\hat{\omega}$ homogeneity by homogeneity to obtain a normal Cartan connection ω on $\mathcal{G}$ inducing the given structure.
- The information on $H^1(\mathfrak{g}_-, \mathfrak{g})$ then implies that two normal Cartan connections inducing the same underling structure related by an automorphism covering $\mathrm{id}_{\mathcal{G}_0}$.

For parabolic geometries (with two exceptions), this establishes a categorical equivalence between Cartan geometries and underlying structures. No reference to a specific construction but only normality of the curvature is needed for uniqueness.

More general procedure [Č., '17]:

- Start with G/P with a filtration on $\mathfrak{g}$ nicely related to P, which also gives $P_+ \subset P$.
- Any Cartan geometry of type (G, P) then determines an underlying filtered G_0-structure, where $G_0 = P/P_+$.
- Specify the algebraic input (prolongation + normalization condition) needed to establish categorical equivalence between Cartan geometries and these underlying structures.
- In [Č., Doubrov, The, '17] this input was provided for the geometries related to (systems of) ODE as listed below.

The underlying structures for which such an equivalence has been established include (most of them parabolic):

- almost Grassmannian and almost quaternionic structures
- classical projective structures and a contact analog of those
- path geometries (equivalent to systems of 2nd order ODE)
- single ODE of order $k \geq 3$ (non-parabolic for $k \geq 4$)
- systems of $m > 1$ ODE of order $k \geq 3$ (non-parabolic)
- Legendrean contact structures (related to geometry of Monge-Ampère type equations)
- (split-) quaternionic contact structures, in particular generic rank 4 distributions in dimension 7.
- generic rank k Distributions in dimension n for $(k, n) = (2, 5)$, $(3, 6)$, $(k, k(k+1)/2)$.

Fefferman constructions

Start with with the CR-sphere S^{2n+1} (complex null-lines in $\mathbb{C}^{(n+1,1)} \cong \mathbb{R}^{(2n+2,2)}$). Real null-lines then form the total space of an S^1-bundle over S^{2n+1}. This is a homogeneous bundle, so there is a corresponding associated bundle $\tilde{M} \to M$ for any Cartan geometry of type (G, P) over M, where $G = SU(n+1, 1)$.

Take $(\tilde{G}, \tilde{P})$ with $\tilde{G} = SO_0(2n+2, 2)$. Extension of structure group leads to a Cartan geometry of type $(\tilde{G}, \tilde{P})$ on $\tilde{M}$ and hence to a conformal structure on $\tilde{M}$.
The Cartan geometry is the canonical one associated to the conformal structure iff one starts from an integrable CR structure.
The conformal structures obtained by this construction can be characterized by "conformal holonomy" contained in G.

This generalizes a classical construction of Fefferman and Burns-Diederich-Shnider.

There are several analogs related to other inclusions $G \hookrightarrow \tilde{G}$. For example Nurowski's conformal structure associated to a $(2,3,5)$-distribution and D.J.F. Fox's extension of a contact projective structure to a projective structure can be obtained in this way. Holonomy characterizations are always available and the geometry on $\tilde{M}$ is flat iff the one on M is flat.

There is a generalization that starts from a non-flat, homogeneous Cartan geometry of some type (H, K) on G/P. This gives rise to a homomorphism $i : P \to K$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{h}$ and the pair (i, α) then determines an extension functor mapping Cartan geometries of type (G, P) to Cartan geometries of type (H, K). This can be used to explicitly describe the path geometry determined by the chains of a CR structure. One gets control on the curvature of that path geometry which leads to a conceptual proof of the fact that a diffeomorphism between CR manifolds that maps chains to chains must be (anti-)CR. [Č., Žadník, '09]

Holonomy of Cartan geometries

A general theory of holonomy reductions of Cartan geometries was developed in [Č., Gover, Hammerl, '14]:

- For type (G, P) reductions are defined for $H \subset G$, the simplest instances are given by parallel sections of tractor bundles.
- $H \subset G$ acts on G/P with orbits of different dimensions. E.g. for $SO(p, q) \subset SL(n, \mathbb{R})$ and G/P a Grassmannian, orbits are determined by the restriction of the inner product.
- For a holonomy reduction of $(\mathcal{G} \to M, \omega)$, M accordingly decomposes into "curved orbits", which inherit Cartan geometries of type $(H, H \cap \hat{P})$. Here $\hat{P}$ is an appropriate conjugate of P.
- **Example**: For $\overline{M} = M \cup \partial M$, a Poincaré–Einstein metric g on M defines a conformal structure with reduced holonomy on $\overline{M}$. Its curved orbits are M endowed with g and ∂M endowed with its conformal infinity.

Natural differential operators

The motivating example here are conformally invariant operators. Direct methods (make choices and analyze dependence) get out of hand quickly, so look for invariant constructions.

- On G/P: G-invariant differential operators; connection to representation theory.
- Parabolic case, irreducible bundles: Correspondence to homomorphism of generalized Verma modules leads to precise descriptions for G/P and strong restrictions in general.
- For natural constructions, one has to go beyond usual geometric objects and involve e.g. tractor bundles.

So one often knows that on G/P there is an invariant DO between two bundles that is unique up to multiples. The question is how to construct this, if possible in a way that extends to curved geometries. This is true for all constructions described below.

The fundamental derivative

Recall that for a geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P), the adjoint tractor bundle is $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$.

- Sections of $\mathcal{A}M$ correspond to P-invariant functions $\mathcal{G} \to \mathfrak{g}$, via ω, these correspond to P-invariant vector fields on $\mathcal{G}$.
- These can be used to differentiate the equivariant functions corresponding to sections of a general associated vector bundle E. This defines $D : \Gamma(\mathcal{A}M) \times \Gamma(E) \to \Gamma(E)$, $(s, \sigma) \mapsto D_s\sigma$, with naturality properties like a Levi-Civita connection.
- Tractor connections can be easily written via D
- For parabolic cases, a scheme for explicit descriptions is available; completely worked out for important examples.

This operator can be used as a fundamental ingredient in several construction schemes for invariant differential operators for parabolic geometries.

curved Casimir's and splitting operators

Viewing D as an operator $\Gamma(E) \to \Gamma(\mathcal{A}^*M \otimes E)$, it can be iterated. The Killing form of $\mathfrak{g}$ induces a natural (indefinite) bundle metric on $\mathcal{A}^*M$. Combining this with D^2, we obtain $\mathcal{C} : \Gamma(E) \to \Gamma(E)$ that specializes to the action of the Casimir if $M = G/P$.

- $\mathcal{C}$ is a differential operator of order ≤ 1. If E is induced by an irreducible representation of P, $\mathcal{C}$ acts by a scalar.
- This scalar is computable via representation theory. Forming polynomials in $\mathcal{C}$ one obtains generalizations of tractor-D operators and of the BGG splitting operators.
- One may also use $\mathcal{C}$ to directly construct invariant operators between bundles induces by irreducible representations.

Carrying these constructions out in specific cases only needs input from representation theory (decomposing restrictions of representations of P or G to G_0, Casimir eigenvalues, etc.).

A toy example (compare with tractor D)

Let $\mathbb{V}$ be a representation of P that contains an irreducible subrepresentation $\mathbb{W}_1 \subset \mathbb{V}$ such that $\mathbb{W}_0 := \mathbb{V}/\mathbb{W}_1$ is irreducible, too. Let us denote by $\mathcal{V}$, E_0 and E_1 the corresponding natural bundles and by $\Pi : \Gamma(\mathcal{V}) \to \Gamma(E_0)$ the projection. For $i = 0, 1$ let λ_i be the Casimir eigenvalue on $\Gamma(E_i)$.

- $\mathcal{C} : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ satisfies $\mathcal{C}|_{\Gamma(E_1)} = \lambda_1 \operatorname{id}$ by naturality.
- Hence $\mathcal{C} - \lambda_1 \operatorname{id}$ factorizes to an operator $S : \Gamma(E_0) \to \Gamma(\mathcal{V})$.
- By naturality $\Pi \circ S = (\lambda_0 - \lambda_1) \operatorname{id}$. If $\lambda_0 \neq \lambda_1$, then $\frac{1}{\lambda_0 - \lambda_1} S$ is a *splitting operator*.
- If $\lambda_0 = \lambda_1$, then S defines an invariant differential operator $\Gamma(E_0) \to \Gamma(E_1)$.
- Twisting by a density bundle $\mathcal{E}[w]$, the λ_i become affine functions $\lambda_i(w)$; often $\exists! w_0$ such that $\lambda_0(w_0) = \lambda_1(w_0)$.

BGG sequences

A tractor bundle $\mathcal{V} \to M$ gives rise to a twisted de-Rham sequence $(\Omega^*(M, \mathcal{V}), d^{\mathcal{V}})$. The Kostant codifferential defines a tensorial differential on $\Lambda^* T^*M \otimes \mathcal{V}$ lowering degrees by one. Its (pointwise) homology is a sequence $\mathcal{H}_*$ of bundles associated to (completely reducible) representations of P on homology spaces.

$d^{\mathcal{V}}$ mixes differential and tensorial parts that relate in a specific way to the natural filtration on the bundles $\Lambda^* T^*M \otimes \mathcal{V}$. This allows one to "compress" to a sequence of higher order natural differential operators $D_i : \Gamma(\mathcal{H}_i) \to \Gamma(\mathcal{H}_{i+1})$.

On locally flat geometries, the twisted de-Rham sequence is a resolution and then also $(\mathcal{H}_*, D_i)$ is a resolution computing the same cohomology. D_0 always defines an overdetermined system whose solutions are related to parallel sections of $\mathcal{V}$.

References

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