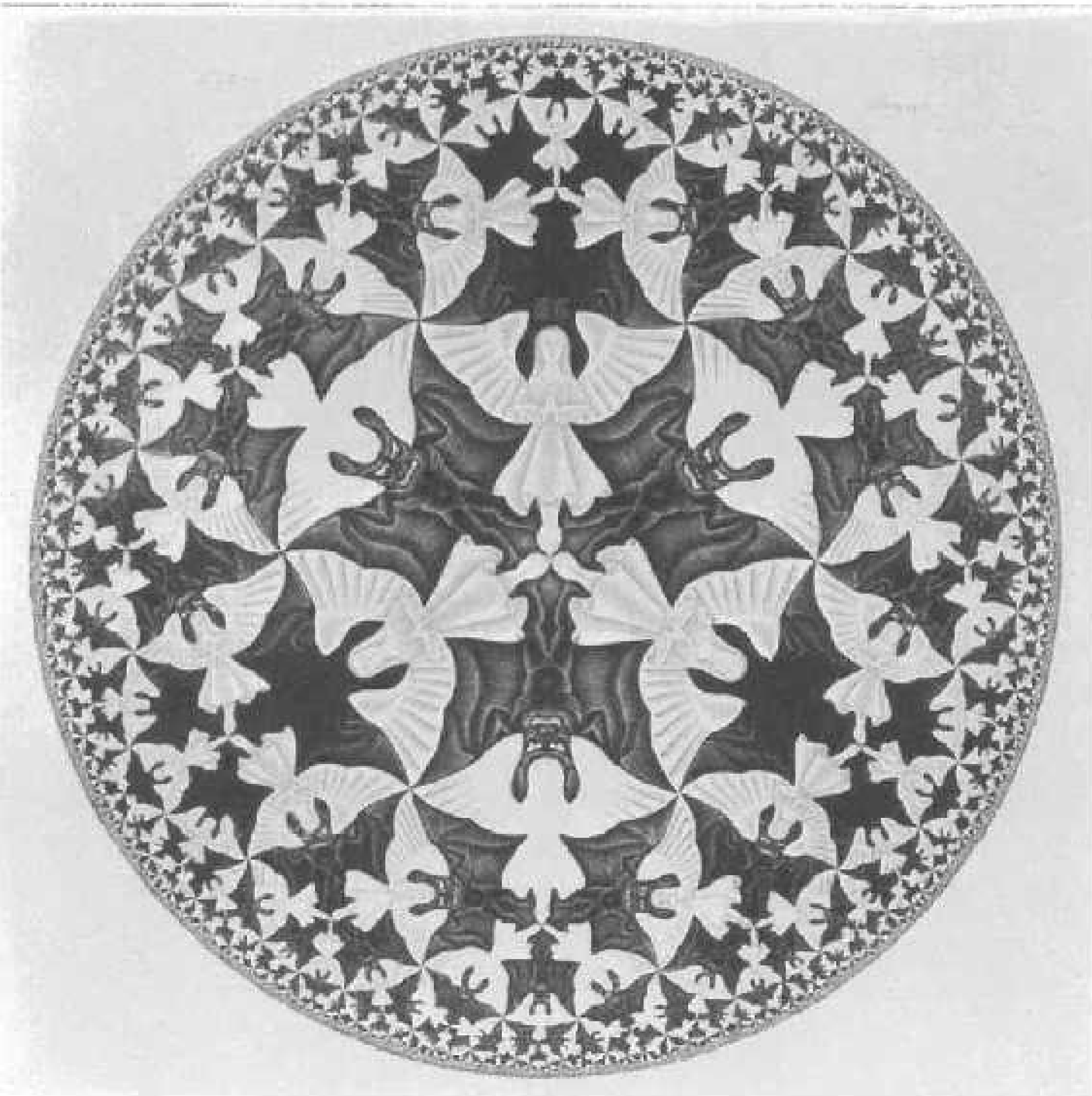
An introduction to dynamics on hyperbolic manifolds

(Vienna, 28.09.2015)

Kurt Falk, Universität Bremen



(M.C. Escher "Heaven & Hell")

§1 Hyperbolic Geometry
§2 Discrete subgroups of $\mathrm{Isom}^+(\mathbb{H})$
§3 Patterson-Sullivan measures
§4 The ergodicity of the geodesic flow

§1 Hyperbolic Geometry

n-dimensional hyperbolic space $\mathbb{H}^n = \mathbb{H}^n$, $n \geq 2$

Poincaré models (introduced by Beltrami ...)

unit ball	upper half space
$\mathbb{B}^n = \{x \in \mathbb{R}^n : \lvert x \rvert < 1\}$	$\mathbb{U}^n = \{x \in \mathbb{R}^n : x_n > 0\}$

endowed with Riemannian metrics given by

$$ds = \frac{2\lvert dx\rvert}{1-\lvert x\rvert^2} = \frac{2\sqrt{dx_1^2+\dots+dx_n^2}}{1-x_1^2-\dots-x_n^2} \qquad \Bigg| \qquad ds = \frac{\lvert dx\rvert}{x_n} = \frac{\sqrt{dx_1^2+\dots+dx_n^2}}{x_n}$$

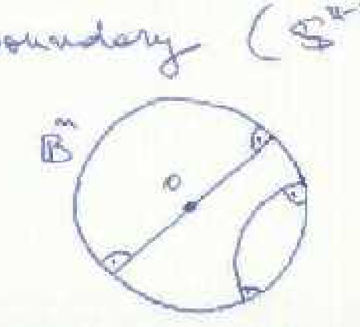
This means that for any smooth curve $\gamma : [a,b] \to \mathbb{B}^n$ (or $\mathbb{U}^n$) we have:

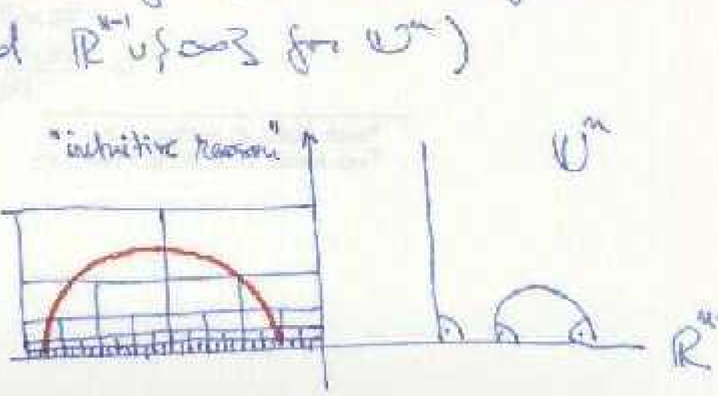
$$\ell(\gamma) = 2\int_a^b \frac{\left((\dot{x}_1(t))^2+\dots+(\dot{x}_n(t))^2\right)^{\frac{1}{2}}}{1-x_1^2(t)-\dots-x_n^2(t)}\,dt \qquad \Bigg| \qquad \ell(\gamma) = \int_a^b \frac{\left((\dot{x}_1(t))^2+\dots+(\dot{x}_n(t))^2\right)^{1/2}}{x_n(t)}\,dt$$

The metric (hyperbolic metric) is defined as usual:

$$d(u,v) = \inf\{\ell(\gamma) : \gamma \text{ smooth curve connecting } u,v\}$$

It can be shown that the geodesics in both models are circle and line segments orthogonal to the respective boundary ($\mathbb{S}^{n-1}$ for $\mathbb{B}^n$ and $\mathbb{R}^{n-1} \cup \{\infty\}$ for $\mathbb{U}^n$)





There is also the hyperbolic volume form

$$dV = 2^n \frac{dx_1 \dots dx_n}{(1-|x|^2)^n} \qquad \Bigg| \qquad dV = \frac{dx_1 \dots dx_n}{(x_n)^n}$$

One can check that $\eta = \sigma \circ \varsigma : \mathbb{U}^n \to \mathbb{B}^n$ is an isometry,

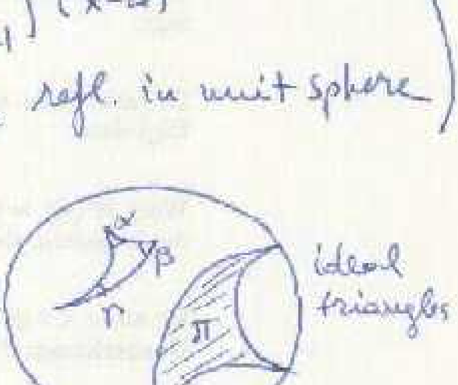
where $\begin{cases} \sigma = \text{reflection in the sphere } S(e_n, \sqrt{2}) \\ \varsigma = \text{reflection in } \mathbb{R}^{n-1} \text{ (swapping } \mathbb{U}^n \text{ and } -\mathbb{U}^n) \end{cases}$

(reflection in $S(a,r)$: $x \mapsto a + \left(\frac{r}{|x-a|}\right)^2 (x-a)$

special case $a=0, r=1$: $x \mapsto \frac{x}{|x|^2}$ refl. in unit sphere)

<u>n=2</u>: Gauss-Bonnet formula

$$\boxed{\text{Area}(\Delta) = \pi - \alpha - \beta - \gamma}$$



<u>Two important properties of $\mathbb{H}^n$</u>

1) The <u>reverse triangle inequality</u>:

If $\alpha \geq \text{Const} > 0$, then

$$\boxed{d(v,w) \geq d(u,v) + d(u,w) - \text{Const}'}$$

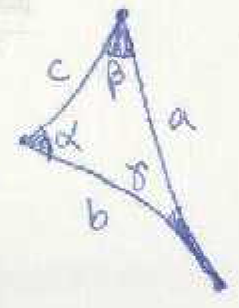
where Const' depends on Const

This follows from hyperbolic sine/cosine rules:

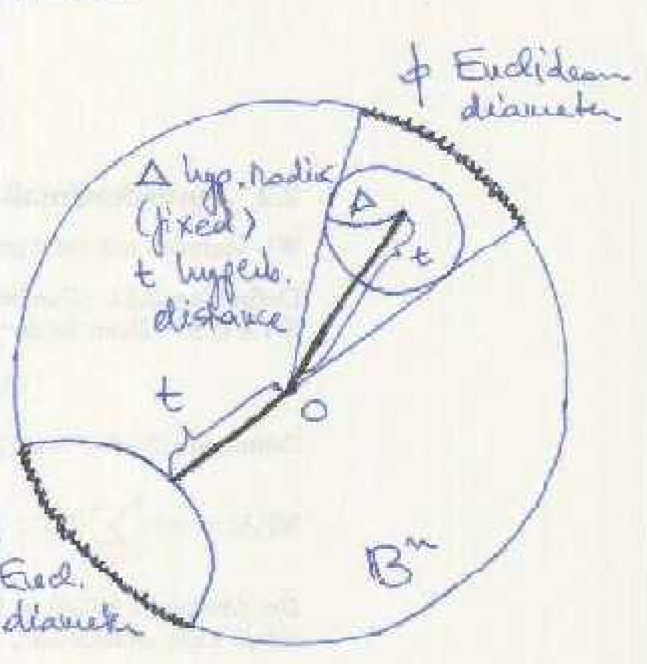
$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$$

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \ \sin \beta}$$



2) Also from hyperbolic & Euclidean trigonometry one obtains:



$$\boxed{e^{-t} \asymp \phi}$$

where "$\asymp$" means that the ratio of the two quantities is uniformly bd. away from 0 and ∞.

$\mathrm{Isom}^+(\mathbb{H})$ group of orientation-preserving isometries of $\mathbb{H}$ ($\mathbb{B}^n$ or $\mathbb{U}^n$)

$\mathcal{M}(\mathbb{R}^n)$ group of Möbius transformations of $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, generated by:

- similarities $\mathbb{R}^n \to \mathbb{R}^n$ (note, $\infty \mapsto \infty$) given by

$$x \mapsto mx + b, \quad m \text{ conformal matrix (i.e. const} \cdot \text{orthogonal)}, \quad b \in \mathbb{R}^n$$

- the reflection in the unit sphere

$$x \mapsto \frac{x}{|x|^2} \quad (\text{note, } 0 \mapsto \infty, \ \infty \mapsto 0)$$

By definition, Möbius maps preserve the class of circles and lines in $\mathbb{R}^n$.

Two important properties of Möbius maps

1) $\forall \gamma \in \mathcal{M}(\mathbb{R}^n)$, $\forall x, y \in \mathbb{R}^n$:

$$|\gamma(x)-\gamma(y)|^2 = |\gamma'(x)|\,|\gamma'(y)|\,|x-y|^2$$

where $|\gamma'(x)|$ is the conformal derivative of γ at x, i.e. the uniquely determined positive number $|\gamma'(x)|$ s.t. $\frac{1}{|\gamma'(x)|}\gamma'(x)$ is orthogonal (here, $\gamma'(x)$ denotes the differential of γ at x).

2) The 1st property implies the invariance of absolute cross-ratios : with

$$|a,b,c,d| = \frac{|a-c|}{|a-d|}\frac{|b-d|}{|b-c|}$$

we have $\forall \gamma \in \mathcal{M}(\mathbb{R}^n)$, $\forall a,b,c,d \in \mathbb{R}^n$, that

$$|\gamma(a),\gamma(b),\gamma(c),\gamma(d)| = |a,b,c,d|$$

One can show that

$$\mathrm{Isom}^+(\mathbb{B}^n) = \left\{ \gamma \in \mathcal{M}(\mathbb{R}^n) : \begin{array}{l}\gamma \text{ orientation-preserving,} \\ \gamma(\mathbb{B}^n) = \mathbb{B}^n\end{array} \right\}$$

$$\mathrm{Isom}^+(\mathbb{U}^n) = \left\{ \gamma \in \mathcal{M}(\mathbb{R}^n) : \begin{array}{l}\text{— '' — '' —} \\ \gamma(\mathbb{U}^n) = \mathbb{U}^n\end{array} \right\}$$

$$\mathrm{Isom}^+(\mathbb{U}^2) \cong PSL_2(\mathbb{R}) = \left\{ z \mapsto \frac{az+b}{cz+d} : \begin{array}{l}a,b,c,d \in \mathbb{R} \\ ad-bc=1\end{array} \right\}$$

$$\mathrm{Isom}^+(\mathbb{U}^3) \cong PSL_2(\mathbb{C})$$

$$\mathrm{Isom}^+(\mathbb{H}^n) \cong PSO(n-1,1) \quad \text{(hyperboloid model)}$$

For instance:

Assuming we have already shown that all transformations $z \mapsto \frac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$ are isometries of $\mathbb{U}^2$, we now show that any (or.-pres.) isometry of $\mathbb{U}^2$ is of this form:

- isometries map geodesics to geodesics;
- given $\phi \in \mathrm{Isom}^+(\mathbb{U}^2)$, there exists an isometry
  $$g: z \mapsto \frac{az+b}{cz+d}, \ ad-bc=1,$$
  so that $g\circ\phi$ leaves $i\mathbb{R}_+$ invariant (choose g to map $\phi(i\mathbb{R}_+)$ to $i\mathbb{R}_+$, where $\mathbb{R}_+ = \{x\in\mathbb{R} : x>0\}$);
- by applying $z \mapsto kz$ with suitably chosen $k>0$, we can assume that $g\circ\phi$ $\begin{cases} \text{fixes } i \\ \text{leaves } (0,i) \text{ and } (i,\infty) \text{ invariant;} \end{cases}$
- by using
  $$d(ip, iq) = \left|\log \frac{p}{q}\right|$$
  which follows from
  $$d(z,w) = \log \frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|},$$
  we conclude that $g\circ\phi$ fixes each point of $i\mathbb{R}_+$;
- finally, set $z = x+iy$ and $g\circ\phi(z) = g(\phi(z)) = u+iv$ and observe that for all $t>0$ we have
  $$d(z,it) = d(g\circ\phi(z), g\circ\phi(it)) = d(u+iv, it)$$
  which implies that $y=v$ and $x=u$ via
  $$\sinh\frac{d(z,w)}{2} = \frac{|z-w|}{2\sqrt{\operatorname{Im} z \operatorname{Im} w}}.$$

This of course implies that $\phi = g^{-1} \in PSL_2(\mathbb{R})$

§2 Discrete subgroups of $\mathrm{Isom}^+(\mathbb{H})$

A *Kleinian grp* is a grp $G < \mathrm{Isom}^+(\mathbb{H})$ which is

- *discrete* (we had seen that $\mathrm{Isom}^+(\mathbb{H})$ is $\cong$ to certain matrix grps and thus inherits topology)

or, equivalently,

- *acts discontinuously on* $\mathbb{H}$, i.e.

$\forall$ cpt $K \subset \mathbb{H}$: $g(K) \cap K \neq \emptyset$ only for finitely many $g \in G$.

In the case $n=2$ such grps are called *Fuchsian*.
We shall assume G to be *torsion-free*, i.e. G contains no elliptic el's (doesn't fix any pts in $\mathbb{H}$).

$F \subset \mathbb{H}$ is a *fundamental domain* for G if

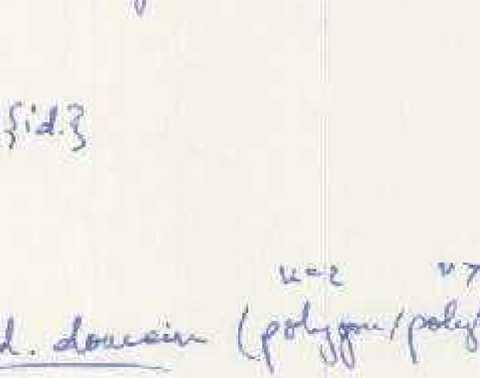
(i) F closed in $\mathbb{H}$, $F \neq \emptyset$

(ii) $g(\mathring{F}) \cap \mathring{F} = \emptyset \quad \forall g \in G \setminus \{id\}$

(iii) $\bigcup_{g \in G} g(F) = \mathbb{H}$

Standard construction: *Dirichlet fund. domain* (polygon/polytope)
($n=2$ / $n>2$)

$$F_x = \bigcap_{g \in G \setminus \{id\}} H_{x,g}, \text{ where}$$



$$H_{x,g} = \{ y \in \mathbb{H} : d(y,x) \leq d(y, g(x)) \}$$

("=" common perpendicular)
Mittelsenkrechte

(This works also with elliptic el's, one assumes x *not* to be an elliptic fixed pt.)

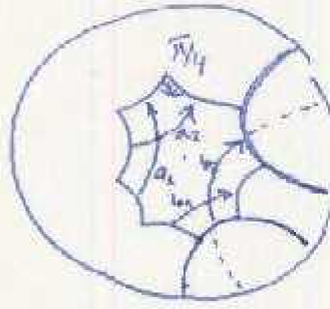
Define $M = \mathbb{H}/G$ simply by the equivalence relation

$x \sim y \iff \exists\, g \in G : g(x) = y$

G torsion-free $\Rightarrow$ M is a <u>hyperbolic manifold</u> with the Riemannian metric of ct neg. curv. -1 inherited from $\mathbb{H}$.

<u>Examples</u>:

1)



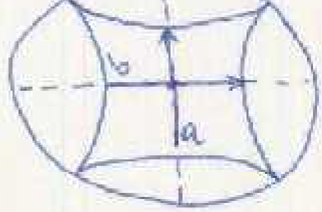
regular octagon of area 4π

4 side-pairings (hyperbolic isometries)

$G = \langle a_1, a_2, b_1, b_2 \rangle$ is a Fuchsian grp

$M = \mathbb{H}^2/G$ $(\mathbb{B}^2/G)$ is a hyperbolic cpct surface of genus 2

2)



2 side-pairings

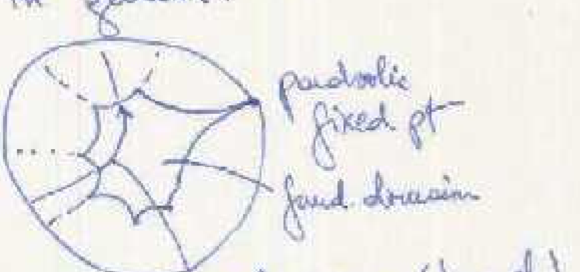
$G = \langle a, b \rangle$

punctured torus group

$M = \mathbb{H}^2/G$

(topologically: torus with one pt missing)

3) In general:



<u>Poincaré's theorem</u> allows one to define Fuchsian grps which give hyp surfaces with a given nr of holes, punctures (and branching pts)
(see e.g. Beardon)

<u>Uniformisation theorem</u>

Any (simply connected) Riemann surface is conformal to $\mathbb{C}$, $\hat{\mathbb{C}}$ or $\mathbb{B}^2$
(flat) (spherical) (hyp)

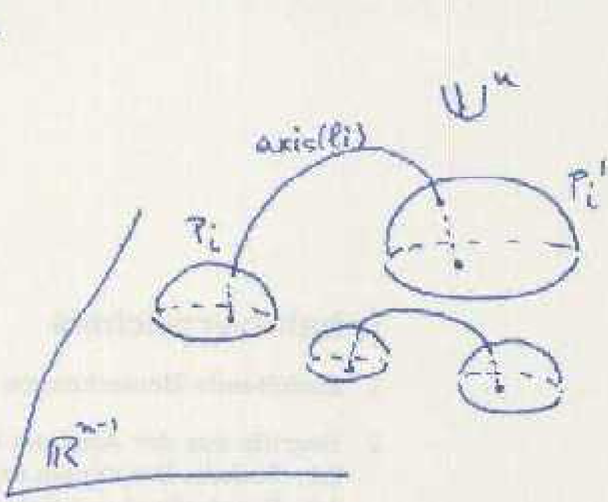
(quotient of $\mathbb{C}$, $\hat{\mathbb{C}}$ or $\mathbb{B}^2$ by a discrete grp of isometries)

4) Schottky groups:

$P_1, P_1', \dots, P_k, P_k'$

hyperbolic planes in $\mathbb{U}^n$

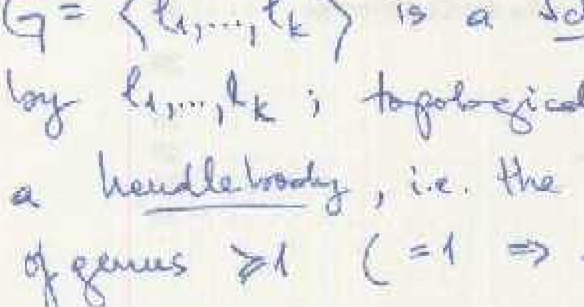
(i.e. spheres or hyperplanes in $\mathbb{R}^n$ which are orthogonal to $\partial\mathbb{U}^n$, intersected with $\mathbb{U}^n$)

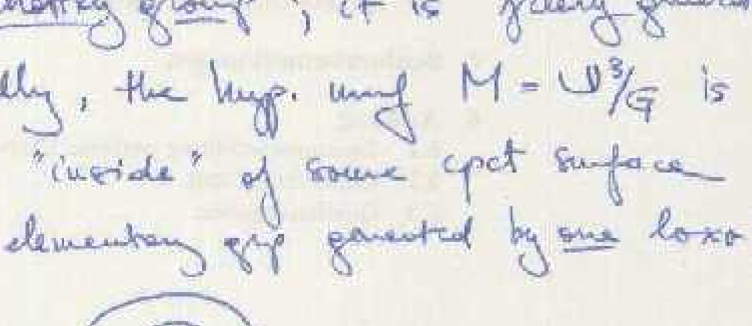


so that, for each of them, all others are in the same connected component of $\mathbb{U} \setminus$ hyperbolic plane.

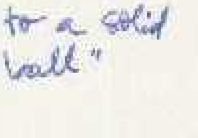
$l_1, \dots, l_k$ loxodromic isometries s.t. $\begin{cases} \text{axis}(l_i) \perp P_i, P_i' \\ l_i(P_i) = P_i' \end{cases}$

$G = \langle l_1, \dots, l_k \rangle$ is a Schottky group, it is freely generated by $l_1, \dots, l_k$; topologically, the hyp. mfd $M = \mathbb{U}^3/G$ is a handlebody, i.e. the "inside" of some cpct surface of genus ≥ 1 ($=1 \Rightarrow$ elementary grp generated by one loxo):





simplest example of a Schottky grp (in $\mathbb{B}^2$):

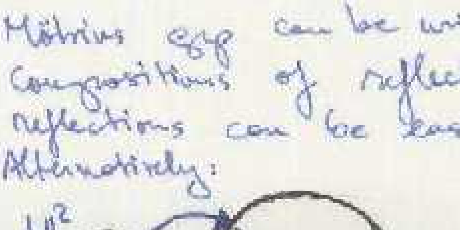


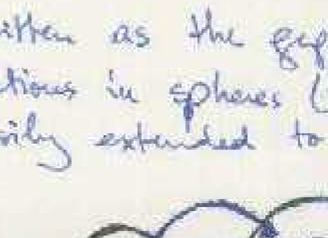
2 side-pairings $G = \langle a, b \rangle$ free

5) Fuchsian grps as Kleinian grps, via the Poincaré-extension:

- Möbius grp can be written as the grp of all finite compositions of reflections in spheres (or planes) and reflections can be easily extended to one more dimension
- Alternatively:

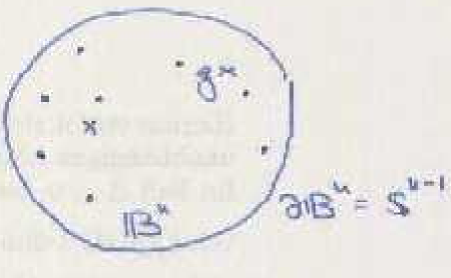
hyperbolic isometry:





(extensions of parabolic isometries analogous)

For a Kleinian grp $G < \mathrm{Isom}^+(\mathbb{H})$ the <u>limit set</u> $L(G)$ is defined as



$$L(G) = \overline{Gx} \setminus Gx$$

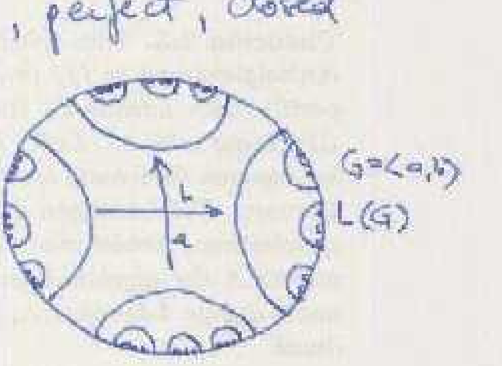
for some $x \in \mathbb{H}$ (definition independent of choice of x), i.e. as the set of accumulation pts of Gx.

Disc. action of G on $\mathbb{H}$ means that $L(G) \subset \partial\mathbb{H}$.

It turns out that there are 3 possibilities for $L(G)$:

- $\# L(G) \leq 2$ (G is <u>elementary</u>)
- $L(G) = \partial\mathbb{H}$ (G is <u>of the 1st kind</u>)
- $L(G) \neq \partial\mathbb{H}$ is uncountable, perfect, closed

$L(G)$ can for instance be a "Cantor-like set" as in the case of a Schottky grp acting on $\mathbb{B}^2$ (see page 9)



$\mathrm{Isom}^+(\mathbb{H})$ acts on $\partial\mathbb{H}$ as well (Möbius transformations), so one defines the <u>set of discontinuity</u>

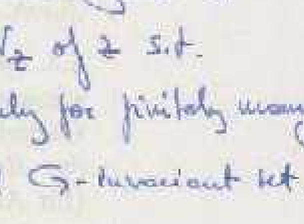
$$\Omega(G) = \partial\mathbb{H} \setminus L(G) = \{ z \in \partial\mathbb{H} : G \text{ acts } \underline{\text{discontinuously}} \text{ at } z \}$$

$\left(\begin{array}{l} \text{i.e. } \exists \text{ nbhd } V_z \text{ of } z \text{ s.t.} \\ gV_z \cap V_z \neq \emptyset \text{ only for finitely many } g \end{array} \right)$

<u>Properties</u>:
$L(G)$ closed, $\Omega(G)$ open, both G-invariant; $L(G)$ is the smallest closed G-invariant set.

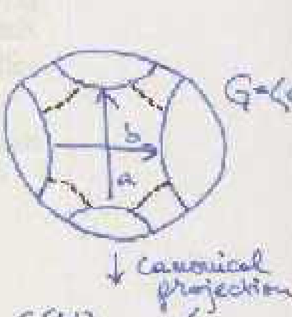
The <u>convex hull of the limit set</u> is defined as

$$CH(G) = \begin{array}{c}\text{hyperbolic} \\ \text{convex hull}\end{array} \left(\bigcup_{\substack{\gamma \text{ geod.} \\ \text{connecting} \\ \text{pts in } L(G)}} \gamma \right)$$



and the <u>convex core</u> as

$$C(M) = CH(G)/G = \begin{array}{l}\text{hyperbolic convex hull} \\ \text{of the union of all} \\ \underline{\text{closed}} \text{ geodesics in } M = \mathbb{H}/G\end{array}$$



($CH(G) \subset \mathbb{H}$ is G-invariant and G acts discontinuously on $CH(G)$.)

G is called $\begin{cases} \underline{\text{cocompact}} \text{ if } M \text{ is compact (Expl. 1. p. 8)} \\ \underline{\text{Cofinite}} \text{ if } \mathrm{vol}(M) < \infty \text{ (Expl. 2. p. 8)} \\ \underline{\text{Convex cocompact}} \text{ if } C(M) \text{ is cpct (Expl. 4. p. 9)} \\ \underline{\text{geometrically finite}} \text{ if } \mathrm{vol}\left(\mathcal{N}_\varepsilon(C(M))\right) < \infty \\ \text{for some, and thus all, } \varepsilon > 0 \end{cases}$

In dimension $n=2$ (and only there) we have:

$$G \text{ geometrically finite} \iff G \text{ finitely generated}$$

(So if G is ∞-ly generated, e.g. $M = \mathbb{H}^2/G$  ..., then G is geom. infinite.)

The radial / conical limit set is defined as

$$L_r(G) = \left\{ \xi \in L(G) : \begin{array}{l} \exists\, r_\xi \text{ geod. ray towards } \xi \\ \exists\, c > 0 \text{ s.t. } d(go, r_\xi) \le c \\ \text{for } \infty\text{-ly many } g \in G \end{array} \right\}$$

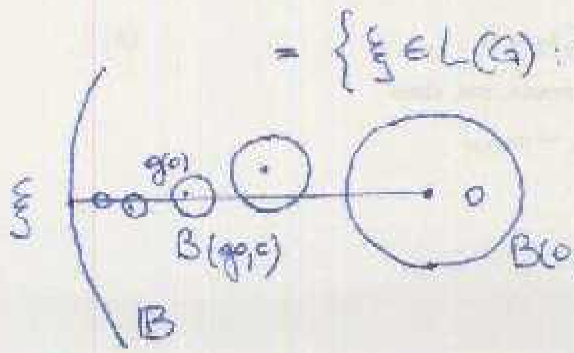
Euclidean cone

$$= \left\{ \xi \in L(G) : \begin{array}{l} \forall\, r_\xi \text{ geod. ray towards } \xi \\ \pi(r_\xi) \text{ returns } \infty\text{-ly often} \\ \text{to } \underline{\text{some}} \text{ cpct set in } M \end{array} \right\}$$

$$= \bigcup_{c>0} L_r^{(c)}(G),$$

where $L_r^{(c)}(G) = \left\{ \xi \in L(G) : \begin{array}{l} r_{o,\xi} \text{ geod. ray from } o \text{ to } \xi \\ \text{intersects } B(go,c) \text{ for } \infty\text{-ly many } g \in G \end{array} \right\}$

$$= \left\{ \xi \in L(G) : \pi(r_{o,\xi}) \text{ returns } \infty\text{-ly often to } \pi(B(go,c)) \right\}$$

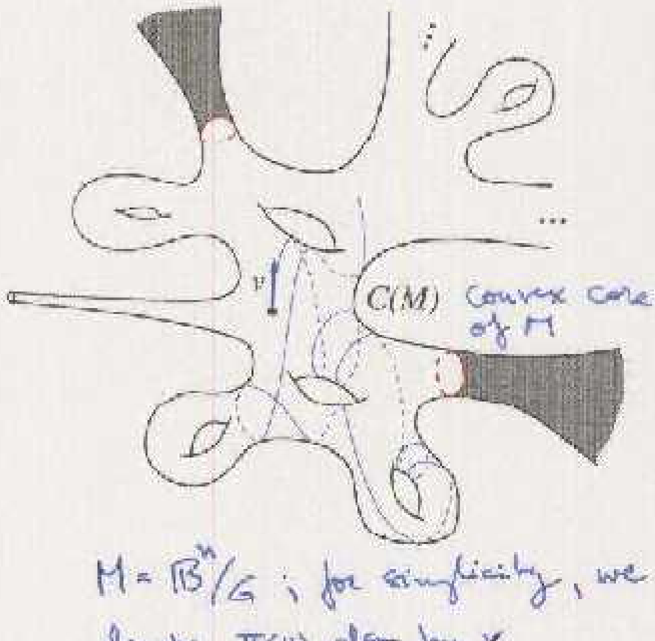


How to understand $L(G)$ and $L_r(G)$ in terms of dynamics in $M = \mathbb{H}/G$. (Recall, $\pi: \mathbb{H} \to M$ is the canonical projection, which also extends to $\pi: T_1\mathbb{H} \to T_1 M$ where T_1 denotes the unit tangent bundle.)

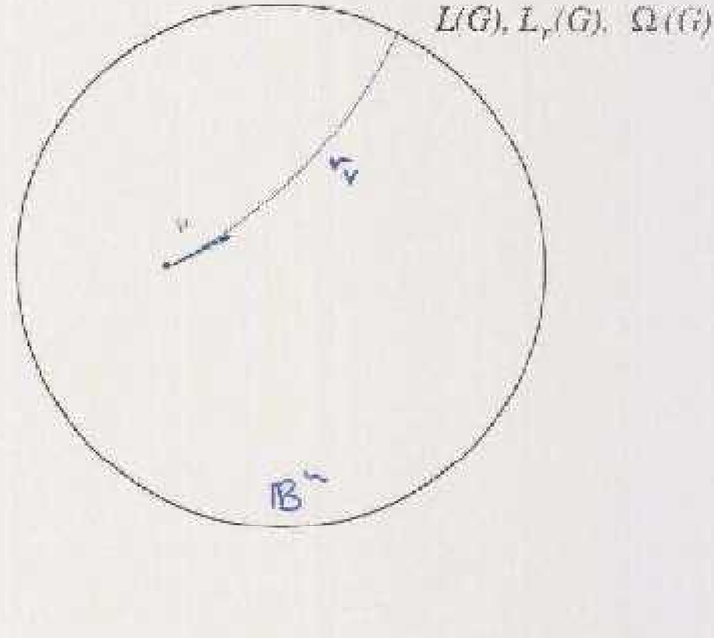
Consider some $v \in T_1\mathbb{H}$; for simplicity, denote $\pi(v) \in T_1 M$ also by v; in both instances, v gives rise to a geodesic ray r_v. We then have:

$r_v \subset C(M) \iff r_v$ ends at some $\xi \in L(G)$

r_v returns ∞-ly often to some cpct in $C(M)$ $\iff r_v$ ends at some $\xi \in L_r(G)$



$M = \mathbb{B}^n/G$; for simplicity, we denote $\pi(v)$ also by v



Thus, $L_r(G)$ corresponds to recurrent dynamics in $C(M)$, and $L(G) \setminus L_r(G)$ corresponds to non-recurrent dynamics, also within the convex core $C(M)$.

For our purposes, $L_p(G)$ denotes the set of all <u>bounded parabolic fixed points</u> of G; a fixed point p of some parabolic isometry in G is called <u>bounded</u>, if $\exists$ horoball H_p at p s.t. $\mathrm{vol}\left(\mathcal{N}_\varepsilon(\pi(H_p) \cap C(M))\right) < \infty$ for some (and thus all) $\varepsilon > 0$.

As a refinement of the radial limit set, one can also introduce the <u>uniformly radial limit set</u> of a Kleinian group G:

$$L_{ur}(G) = \left\{ \xi \in L(G) : \begin{array}{l} \exists\, r_\xi \text{ geod. ray towards } \xi \\ \exists\, c > 0 \text{ s.t. } r_\xi \subset \bigcup_{g \in G} B(go, c) \end{array} \right\}$$

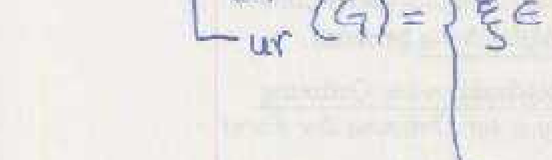
$$= \left\{ \xi \in L(G) : \begin{array}{l} \forall\, r_\xi \text{ geod. ray towards } \xi \\ \exists\, K \subset M \text{ cpct s.t. } \pi(r_\xi) \subset K \end{array} \right\}$$

$$= \bigcup_{c>0} L_{ur}^{(c)}(G),$$

where

$$L_{ur}^{(c)}(G) = \left\{ \xi \in L(G) : \begin{array}{l} r_{o,\xi} \subset \bigcup_{g \in G} B(go, c), \text{ where} \\ r_{o,\xi} \text{ is the geod. ray from } o \text{ to } \xi \end{array} \right\}$$

$$= \{ \xi \in L(G) : \pi(r_{o,\xi}) \subset \pi(B(o,c)) \}.$$



With the <u>transient limit set</u> $L_t(G) = L(G) \setminus L_r(G)$, for which clearly $L_p(G) \subset L_t(G)$, we have the "correspondences":

$L(G) \leftrightarrow$ dynamics in $C(M)$

$L_r(G) \leftrightarrow$ recurrent dynamics in $C(M)$

$L_{ur}(G) \leftrightarrow$ confined dynamics in $C(M)$
(to <u>some</u> cpct in M)

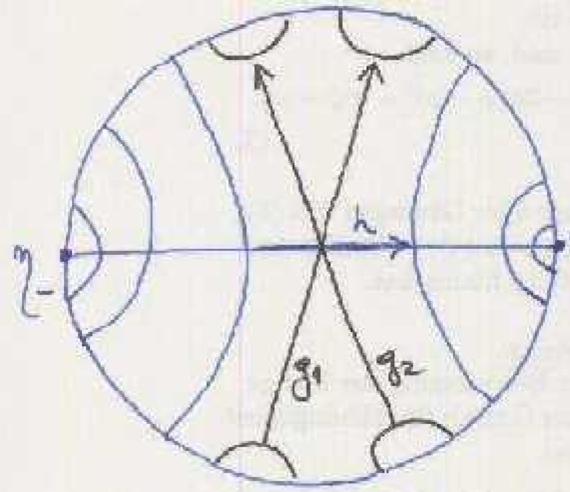
$L_t(G) \leftrightarrow$ non-recurrent dynamics in $C(M)$

<u>Theorem</u> (Beardon & Maskit 1974)

G geometrically finite $\Longleftrightarrow L(G) = L_r(G) \,\dot\cup\, L_p(G)$

Instead of a proof, we give a simple <u>example</u> of a geometrically infinite grp, for which

$$L(N) \setminus (L_r(N) \cup \underbrace{L_p(N)}_{\overset{\shortparallel}{\phi} \text{ here.}}) \neq \phi$$



h, g_1, g_2 hyperbolic isometries

$G = \langle h, g_1, g_2 \rangle$, $H = \langle h \rangle$, $G_1 = \langle g_1, g_2 \rangle$

$G = H * G_1$ free product

$$N = \langle h^k g_i h^{-k} : i = 1,2,\ k \in \mathbb{Z} \rangle = \ker \varphi$$

where $\varphi : G \to H$ is the canonical homomorphism.

We have that $N \triangleleft G$ and $G/_N \cong H \cong \mathbb{Z}$.

If the axis of h goes from η_- to η_+, it is easy to see that $\eta_-, \eta_+ \in L(N)$ but $\eta_-, \eta_+ \notin L_r(N)$.

Since $L_p(N) = \phi$ ($\nexists$ parabolics in H, G_1 and G), it follows that $\eta_-, \eta_+ \in L(N) \setminus L_r(N) \neq \phi$.

N is infinitely generated and geometrically infinite, and $\eta_-, \eta_+ \in L_J(N)$, where the <u>Jørgensen limit set</u> is defined as

$$L_J(N) = \left\{ \xi \in L(N) : \exists \text{ geod. ray } r_\xi \text{ s.t. } r_\xi \subset D \text{ with } D \text{ some Dirichlet fund. dom. for } N \right\}$$

G Kleinian grp acting on $\mathbb{H}^n$, $n \geq 2$, usually $\mathbb{H}^n = \mathbb{B}^n$.
The Poincaré series

$$P(x,s) = \sum_{g \in G} e^{-s\, d(x, go)}, \quad x \in \mathbb{H}, \; s \in \mathbb{R},$$

has critical exponent

$$\delta = \delta(G) = \inf \{ s \in \mathbb{R} : P(x,s) \text{ converges} \}$$
$$= \sup \{ s \in \mathbb{R} : P(x,s) \text{ diverges} \},$$

which does not depend on the choice of x.

Remark:
$$\delta(G) = \limsup_{R \to \infty} \frac{\log \#(Gx \cap B(x,R))}{R} = \lim_{R \to \infty} \cdots$$

Theorem (Roblin 2002)

G is of divergence type if $P(x, \delta)$ diverges, and of convergence type otherwise.
A priori, we do not know whether G is of conv. or divergence type – this turns out to be very relevant information about G.

Some facts:

- $0 < \delta(G) \leq n-1$ (G acting on $\mathbb{H}^n$);
  if G non-elementary
  second inequality for instance given by a volume argument:
  $$\mathrm{Vol}_{hyp}(B(x,R)) = \mathrm{Vol}_{Eucl}(\text{unit ball}) \int_0^R (\sinh t)^{n-1}\, dt$$
- G cocompact $\Rightarrow$ $\delta = n-1$ and G of div. type
  (also a volume argument, see Nicholls)
- G cofinite $\Rightarrow$ –"– –"–
- G convex cocompact, geom. finite : later! (need Patterson measure for that!)

We will now define the Patterson-Sullivan measure using the Poincaré series of a Kleinian group G. For reasons becoming clear later, we need the Poincaré series to diverge at $\delta = \delta(G)$, if the Patterson-Sullivan measure is to be concentrated on the limit set $L(G)$. If the Poincaré series converges, Patterson devised a method to overcome this difficulty. One can define inductively a so-called Patterson function (or slowly varying function) $h: \mathbb{R}_+ \to \mathbb{R}_+$ so that:

1) $$P_h(x,s) = \sum_{g\in G} h(e^{d(x,go)}) e^{-s\,d(x,go)} < \infty, \text{ for } s > \delta$$

$$P_h(x,s) \text{ diverges for } s \leq \delta$$

2) $\forall\, \varepsilon > 0 \; \exists\, r_\varepsilon > 0$ so that
$$h(rt) \leq t^\varepsilon h(r), \quad \forall r > r_\varepsilon, \forall t > 1$$

From the first two condition the following can be deduced:

3) $\forall$ bounded interval $I \subset \mathbb{R}$ we have
$$\frac{h(e^{r+t})}{h(e^r)} \longrightarrow 1$$
uniformly for $t \in I$, as $r \to \infty$.

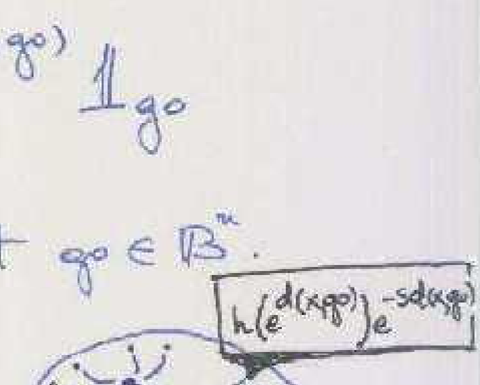
Note that the afore-mentioned inductive procedure produces a bounded function h if the group G is already of divergence type.

We shall call
$$P_h(x,s) = \sum_{g\in G} h(e^{d(x,go)}) e^{-s\,d(x,go)}, \quad x \in H, s \in \mathbb{R},$$
the modified Poincaré series of G.

For $s > \delta$ and $x \in \mathbb{B}^n$ define the following measure on the compact $\mathbb{B}^n \cup \mathbb{S}^{n-1}$

$$\mu_{x,s} = \frac{1}{P_h(0,s)} \sum_{g \in G} h\left(e^{d(x,go)}\right) e^{-s\, d(x,go)} \mathbb{1}_{go}$$

where $\mathbb{1}_{go}$ is the Dirac point mass at $go \in \mathbb{B}^n$.



Consider some sequence $s_n \searrow \delta$; applying the Theorem of Helly-Bray cited/stated below, one concludes that $\exists$ subsequence $(n_k)_{k \in \mathbb{N}}$ s.t.

$$\mu_{x, s_{n_k}} \xrightarrow{\ *\ } \mu_x \quad \text{measure on } \mathbb{B}^n \cup \mathbb{S}^{n-1}$$

Since G acts discontinuously on $\mathbb{B}^n \cup \Omega(G)$ (recall, $\Omega(G) = \mathbb{S}^{n-1} \setminus L(G)$ is the set of discontinuity of G), and since $P_h(0,s)$ diverges at $s = \delta$, it follows that

$$\operatorname{Supp} \mu_x = L(G).$$

We call μ_x, $x \in \mathbb{B}^n$, the Patterson measure, or Patterson-Sullivan measure, based at $x \in \mathbb{B}^n$.

[Thm (Helly-Bray).
Ω separable, locally cpct (cpct); $\mathcal{B} = \sigma(\mathcal{T})$ Borel σ-algebra; then,

$$\mathcal{M}_c = \{ \mu \text{ Borel measure} : \mu(\Omega) \leq c \}, \quad c > 0,$$

is sequentially cpct w.r.t. vague (weak) topology, i.e. $\forall (\mu_n)$ in $\mathcal{M}_c$ $\exists \mu_{n_k}$ s.t. $\mu_{n_k} \xrightarrow{*} \mu \in \mathcal{M}_c$ as $k \to \infty$, i.e.

$$\int f \, d\mu_{n_k} \xrightarrow{k \to \infty} \int f \, d\mu, \quad \forall f \in \underset{\text{cont. fcts with cpct. support}}{\mathcal{C}_K(\Omega)} \quad \Big(\text{or } \underset{\text{just cont. (bd.) fcts}}{\underline{\mathcal{C}(\Omega)}}\Big)$$
]

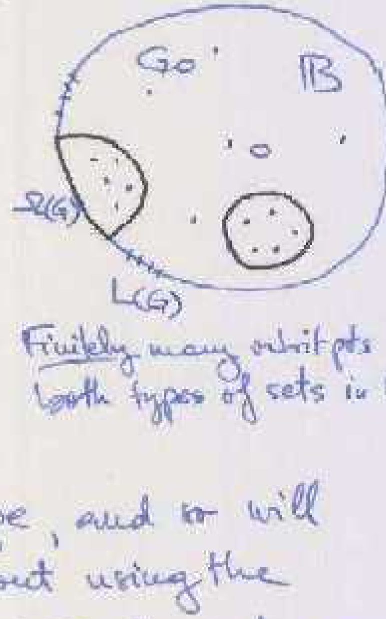
Remarks:

1) A priori, μ_x is <u>not</u> uniquely determined, but depends on the choice of sequence $s_n \searrow \delta$ and on the subsequence s_{n_k} given by the Helly-Bray Theorem.

2) However, the μ_x, $x \in \mathbb{B}^n$, can be chosen in such a way that they are all absolutely continuous to μ_0 (which is, by definition, a probability measure). The family of measures $(\mu_x)_{x \in \mathbb{B}^n}$ is called a <u>conformal density</u>.
(See Theorem 3.4.1 and discussion thereafter in Nicholls, and also Chapter 4 in Nicholls.)

3) The fact that the divergence of $P_h(0,\delta)$ implies $\operatorname{supp} \mu_x = L(G)$ follows from the following properties of weak-$*$-limits of measures: if $\mu_n \xrightarrow{*} \mu$ then

$$\mu(K) \geq \limsup \mu_n(K), \ \forall \text{ cpt } K$$
$$\mu(O) \leq \liminf \mu_n(O), \ \forall \text{ open } O$$

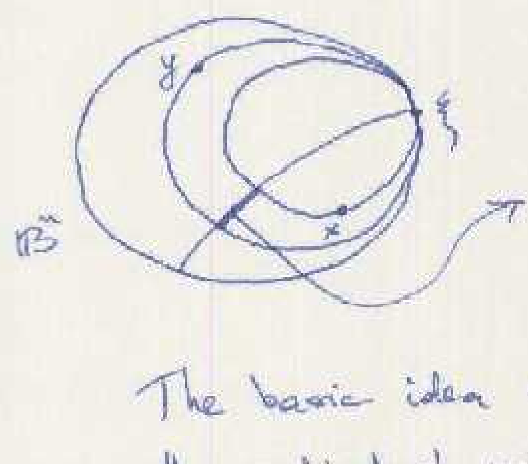


Finitely many orbit pts in both types of sets in $\mathbb{B}^n \cup \Omega(G)$

4) In several (sketches of) proofs hereafter we shall pretend that the group is already of divergence type, and so will perform the calculations/estimates without using the <u>modified</u> Poincaré series. The general case is not very much more difficult, and can be tackled using properties 2) and 3) of Patterson functions (page 16).

Next, we give the main two properties of Patterson-Sullivan measures, namely <u>δ-harmonicity</u> and <u>δ-conformality</u>.

δ-harmonicity $\qquad \frac{d\mu_x}{d\mu_y}(\xi) = e^{\delta d_\xi(y,x)}$



horospherical distance
(Busemann fct)

$d_\xi(y,x) > 0$

$(d_\xi(y,x) = -d_\xi(x,y))$

The basic idea of proof:
the quotient of weights of $g(o)$
w.r.t. μ_{x,s_n} and μ_{y,s_n} is

$$\frac{e^{-s_n d(x,g(o))}}{e^{-s_n d(y,g(o))}} = e^{s_n(d(y,g(o)) - d(x,g(o)))}$$

But a calculation in hyp. geom.
shows that $d(y,g(o)) - d(x,g(o)) \to d_\xi(y,x)$ as $g(o) \to \xi$

Applied to dy's of μ_{x,s_n} and
μ_{y,s_n}, this gives the
required R-N-derivative ⊠

Remark: $\frac{d\mu_x}{d\mu_y}(\xi) = e^{\delta d_\xi(y,x)} = \left(\frac{P(x,\xi)}{P(y,\xi)}\right)^\delta$

where $P(x,\xi) = \frac{1-|x|^2}{|x-\xi|^2}$ is the Poisson kernel; it can be checked
that $P(x,\xi) = \text{const}$ describes horospheres at ξ.

In particular: $\frac{d\mu_x}{d\mu_o}(\xi) = e^{\delta d_\xi(o,x)} = (P(x,\xi))^\delta$ because $P(0,\xi) = 1$

δ-conformality

$\forall \gamma \in G \ \forall A \subset \partial\mathbb{H}$ measurable:

$$\mu_0(\gamma A) = \int_A |\gamma'(\xi)|^\delta \, d\mu_0(\xi)$$

(in general: $\mu_x(\gamma A) = \int_A \left(\frac{P(\gamma^{-1}(x),\xi)}{P(x,\xi)} \right)^\delta d\mu_x(\xi)$)

where $|\gamma'(\xi)|$ is the conformal derivative of γ at ξ (i.e. the uniquely determined (positive) number $|\gamma'(\xi)|$ so that $\frac{1}{|\gamma'(\xi)|}\gamma'(\xi)$ is orthogonal).

Note, $|\gamma'(\xi)| = P(\gamma^{-1}(0),\xi) = \dfrac{1-|\gamma^{-1}(0)|^2}{|\gamma^{-1}(0)-\xi|^2}$

↳ follows from $|\gamma(x)-\gamma(y)|^2 = |\gamma'(x)|\,|\gamma'(y)|\,|x-y|^2$

Idea of proof:

Consider set A: 

and $s_n \searrow \delta$ as before (def. of P-S-meas.)

$$\mu_{\gamma(0),s_n}(A) = \frac{1}{P_h(0,s_n)} \sum_{g\in G} e^{-s_n d(\gamma o, g o)} \mathbb{1}_{go}(A) =$$

$$= \frac{1}{P_h(0,s_n)} \sum_{g\in G} e^{-s_n d(o, \gamma^{-1} g o)} \mathbb{1}_{go}(A) =$$

$$= \frac{1}{P_h(0,s_n)} \sum_{h\in G} e^{-s_n d(o, h o)} \mathbb{1}_{ho}(A) = \mu_{0,s_n}(\gamma^{-1}A)$$

$\left(\begin{array}{l} \mathbb{1}_{go}(A) = 1 \\ \Leftrightarrow g(o) \in A \\ \Leftrightarrow \gamma^{-1}g(o) \in \gamma^{-1}(A) \\ \Leftrightarrow \mathbb{1}_{\gamma^{-1}go}(\gamma^{-1}A) = 1 \end{array} \right)$

By taking the weak limit, it follows that

$$\mu_{\gamma o}(A) = \mu_o(\gamma^{-1}A), \quad \forall \gamma\in G \ \forall A \text{ measurable}$$

Now combine this with δ-harmonicity to obtain:

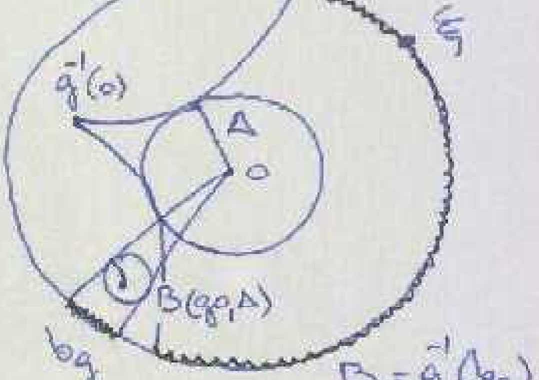
$$\frac{d(\mu_0 \circ \gamma)}{d\mu_0}(\xi) = \frac{d\mu_{\gamma^{-1}(0)}}{d\mu_0}(\xi) = \left(P(\gamma^{-1}(0),\xi)\right)^\delta = |\gamma'(\xi)|^\delta \quad \blacksquare$$

Sullivan's Shadow lemma

For sufficiently large $\Delta > 0$ (depending on G) we have

$$\mu_o(b_g) \asymp e^{-\delta\, d(o, go)} \asymp |b_g|^\delta$$

where b_g is the shadow from o of $B(go, \Delta)$ and $|b_g|$ the Euclidean diameter of b_g.

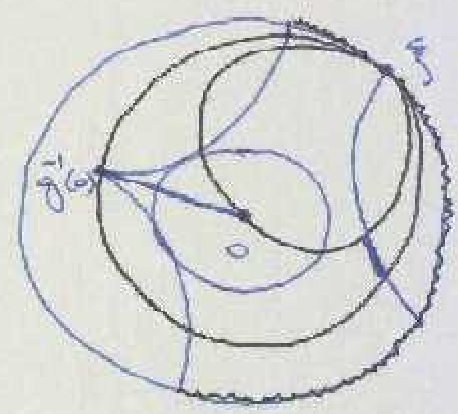


Ideas used in proof:

- δ-conformality
- $d_\xi(o, g^{-1}(o)) \asymp_+ -d(o, go)$

  (where $\asymp_+$ denotes additive comparability, i.e. quantities coincide up to an additive const)

  for $\xi \in B_g = g^{-1}(b_g)$.

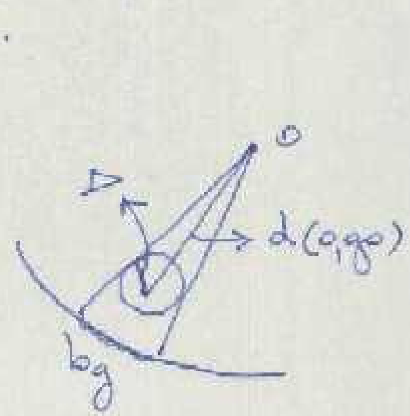


- for sufficiently large Δ we have that

  $$\mu_o(B_g) \asymp \text{const}.$$

  (independently of $g \in G$)

- $|b_g| \asymp e^{-d(o, go)}$



$\blacksquare$

Remark: One of the first uses for the Sullivan Shadow lemma is determining Hausdorff dimensions of limit sets. Hence, we need to look at Hausdorff measures & dimension.

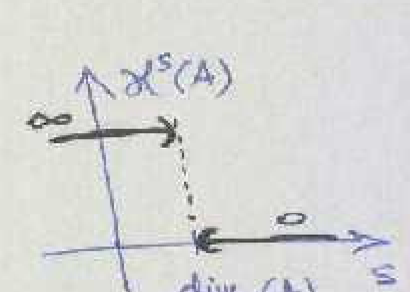
A crash course in Hausdorff measure & dimension

For $A \in \mathcal{B}$ ($\mathcal{B}$ Borel σ-algebra in $\mathbb{R}^n$) and $s \in \mathbb{R}$ define

$$\mathcal{H}^s(A) = \lim_{\delta \searrow 0} \inf \Big\{ \sum_{i \in \mathbb{N}} |O_i|^s : A \subset \bigcup_{i \in \mathbb{N}} O_i \,,\ |O_i| < \delta \Big\}.$$

It can be shown that the limit exists and that $\mathcal{H}^s$ is a Borel measure on $(\mathbb{R}^n, \mathcal{B})$; it is called the s-dim'al Hausdorff measure on $\mathbb{R}^n$.

Also, $\exists \dim_H(A)$ defined by



$$\dim_H(A) = \inf\{ s \in \mathbb{R} : \mathcal{H}^s(A) = 0 \}$$
$$= \sup\{ s \in \mathbb{R} : \mathcal{H}^s(A) = \infty \}$$

This is called the Hausdorff dimension of A.

For instance, the Hausdorff dim of the middle third Cantor set is $\log 2/\log 3$, and in general, the Hausdorff dim of the limit set of an iterated function system (IFS) with contraction rates r_i is given by the Hutchinson formula:

$$\sum r_i^s = 1 \quad , \ s = \dim_H$$

One of the standard methods of computing/estimating Hausdorff dim's is to put a suitable measure on the respective set and the apply the

Frostman lemma: let μ be a Borel measure on $\mathbb{R}^n$ s.t. $0 < \mu(\mathbb{R}^n) < \infty$. Let $F \subset \mathbb{R}^n$ be a Borel set and $0 < c < \infty$ some constant. Then,

(a) $\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} < c, \ \forall x \in F \Rightarrow \mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$.

(b) $\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} > c, \ \forall x \in F \Rightarrow \mathcal{H}^s(F) \leq 2^s \frac{\mu(\mathbb{R}^n)}{c}$.

Remarks.

1) If we assume that $\mu(F)>0$, then part (a) extends to:

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} < C, \ \forall x\in F \Rightarrow \mathcal{H}^s(F) \geq \frac{\mu(F)}{C} > 0 \Rightarrow s \leq \dim_H(F)$$

2) (b) extends to:

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} < c, \ \forall x\in F \Rightarrow \mathcal{H}^s(F) \leq 2^s \frac{\mu(\mathbb{R}^n)}{c} < \infty \Rightarrow s \geq \dim_H(F)$$

3) In particular, if $\mu(F)>0$, then we have the implication

$$\mu(B(x,r)) \asymp r^s, \ \forall x\in F \ \Rightarrow \ s = \dim_H(F)$$

These observations allow us to prove the following results:

Theorem (Sullivan, Tukia)

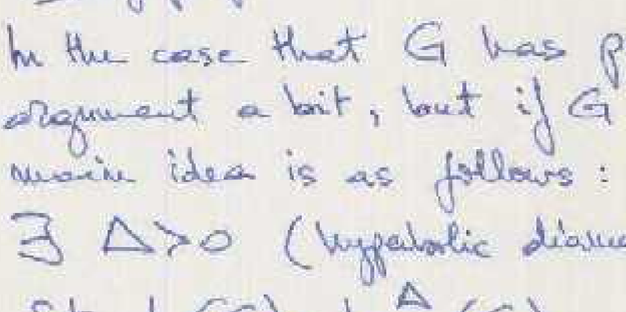
G geometrically finite $\Rightarrow \dim_H(L(G)) = \delta(G)$

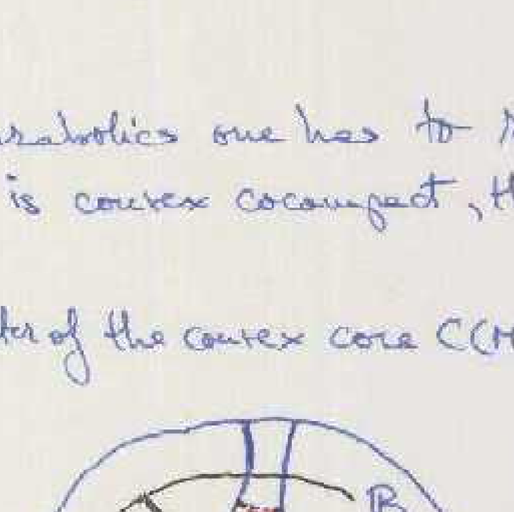
Idea of proof:

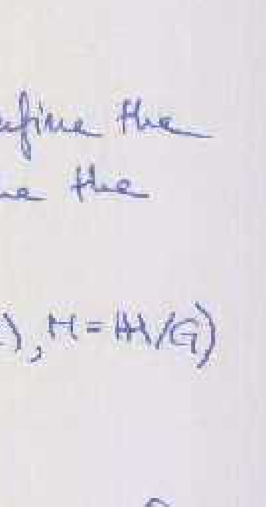
In the case that G has parabolics one has to refine the argument a bit, but if G is convex cocompact, the the main idea is as follows:

$\exists\ \Delta>0$ (hyperbolic diameter of the convex core $C(M)$, $M=\mathbb{H}/G$)

s.t. $L(G) = L^{\Delta}_{ur}(G)$







$\xi\in L(G)$

For each $\xi\in L(G)$ we thus obtain a sequence of eccentric balls "zooming down" on ξ. The Sullivan Shadow lemma gives the "correct" scaling behaviour of μ(ball), and thus, modulo eccentricity (which can be dealt with), Frostman's lemma proves the claim

Theorem: μ does not have atoms in $L_r(G)$.

Idea of proof: application of the shadow lemma ▫

Theorem: μ does not have atoms in $L_p(G)$.

Remark: Remember that we denoted by $L_p(G)$ the set of bounded parabolics! Thus, μ might have atoms in unbounded parabolic fixed points. (I do not know any examples, though...)

Corollary: G geom. finite $\Rightarrow$ μ has no atoms

Idea of proof: theorems above & Beardon-Maskit $L = L_r \cup L_p$ ▫

Theorem G geometrically finite $\Rightarrow$ G of divergence type.

Proof (by contradiction):
Assume G is of convergence type, i.e. $\sum_{g\in G} e^{-\delta d(o,go)} < \infty$.
As before in proof of Thm (Sullivan, Tukia), we argue that if G is geometrically finite, then $\exists\, \Delta > 0$ sufficiently large s.t.

$$L_r(G) = L_r^{(\Delta)}(G) = \limsup \{ b_g : g \in G \},$$

where b_g is the shadow from o of $B(go, \Delta)$, and the limsup set of a sequence $(A_n)_{n\in\mathbb{N}}$ of sets is defined as the set of all points which are in ∞-ly many of the A_n.

On the other hand, the Sullivan Shadow lemma tells us that

$$\mu(b_g) \asymp e^{-\delta d(o,go)}$$

and so

$$\sum_{g\in G} e^{-\delta d(o,go)} \asymp \sum_{g\in G} \mu(b_g) < \infty$$

By the Borel-Cantelli lemma we now conclude that

$$\mu(L_r(G)) = \mu(\limsup\{b_g : g\in G\}) = 0$$

Since $L_p(G)$ is countable and μ has no atoms in $L_p(G)$, we conclude that $\mu(L(G)) = \mu(L_r(G)) + \mu(L_p(G)) = 0 + 0 = 0$ ↯ ▫