Descendants of the mother of all continued fractions

Slade Sanderson
with Karma Dajani and Cor Kraaikamp

Uniform Distribution of Sequences
ESI University of Vienna
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Regular CFs

Each $x \in \mathbb{R}$ has an (essentially) unique **regular continued fraction** (RCF) expansion

$$x = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

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where $a_n \in \mathbb{Z}$ with $a_n > 0$ for $n > 0$. Denote RCF-**convergents** by

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n] \in \mathbb{Q}.$$

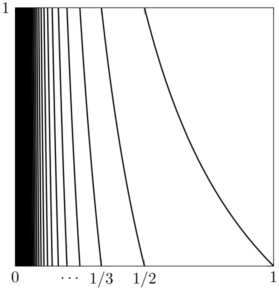
The Gauss map

The **Gauss map** $G : [0,1] \to [0,1]$ defined by $G(0) = 0$ and for $x \neq 0$,

$$G(x) = \frac{1}{x} - a(x),$$

with $a(x) = \lfloor 1/x \rfloor$ generates RCF-digits:

$$a_n = a(G^{n-1}(x)), \quad n > 0.$$



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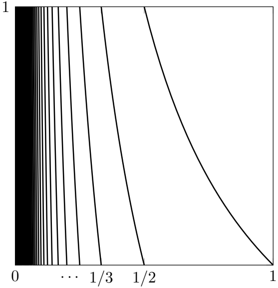

The dynamical system $([0,1], \mathcal{B}, \nu_G, G)$ is ergodic, where the **Gauss measure** ν_G is the a.c., G-invariant probability measure with density $1/(\log 2(1+x))$.

The natural extension of the Gauss map

In the 1970s and 80s, Nakada, Ito and Tanaka introduced an explicit **natural extension** $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$ of $([0,1], \mathcal{B}, \nu_G, G)$, i.e., the 'smallest' invertible dynamical system of which $([0,1], \mathcal{B}, \nu_G, G)$ is a factor, or 'subsystem.'

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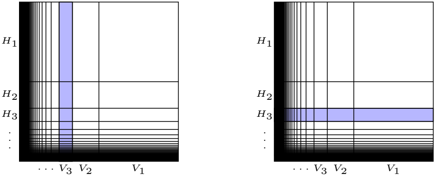

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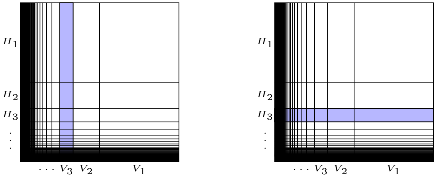
With $\pi : \Omega \to [0,1]$ the projection to the first coordinate, $\nu_G(A) = \bar{\nu}_G(\pi^{-1}A)$, where $\bar{\nu}_G$ has density $1/(\log 2(1+xy)^2)$.



Other CF-algorithms

There are several other algorithms producing **generalised** CF (GCF) expansions

$$x = [\beta_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots] = \beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \ddots}}$$

with $\beta_0 \in \mathbb{Q}$ and for $n > 0$, $\alpha_n, \beta_n \in \mathbb{Z}$, $\alpha_n \neq 0$.

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- Backward
- Nearest Integer
- Singular
- Regular
- Optimal
- Diagonal

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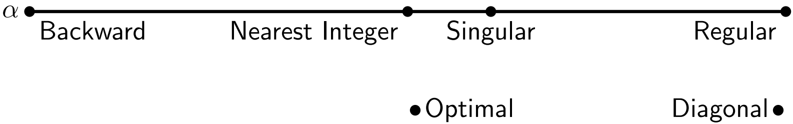

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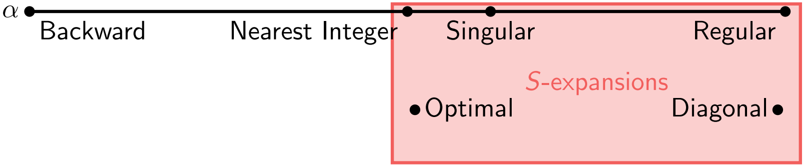
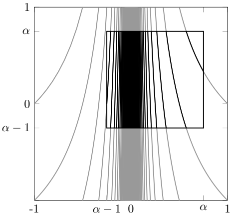


Nakada's α-CFs

For each $\alpha \in [0,1]$, define $G_\alpha : [\alpha-1, \alpha) \to [\alpha-1,\alpha)$ by $G_\alpha(0) = 0$ and

$$G_\alpha(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor, \qquad x \neq 0.$$

Each G_α has a unique, a.c. invariant measure ρ_α, and $([\alpha-1,\alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$ is ergodic.



Kraaikamp's S-expansions

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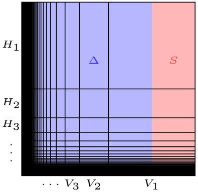

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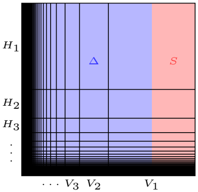
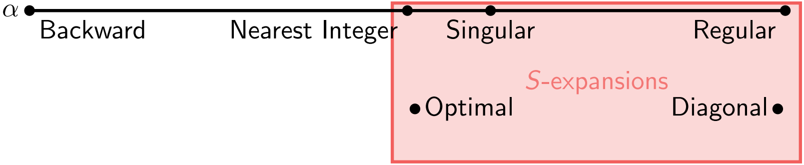
S-expansions use induced transformations to govern singularisations: remove p_n/q_n iff $\mathcal{G}^n(x, 0) \in S$.



Unifying family?



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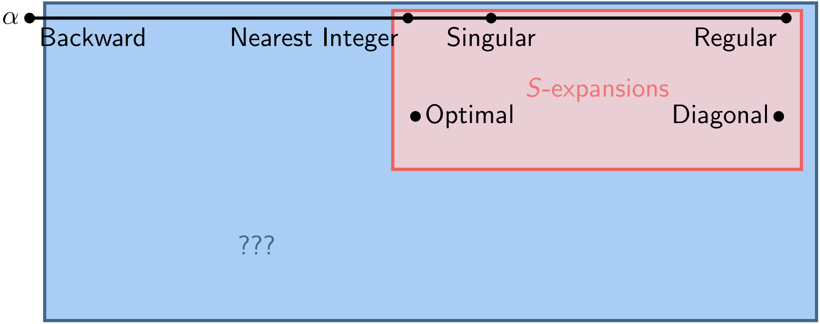

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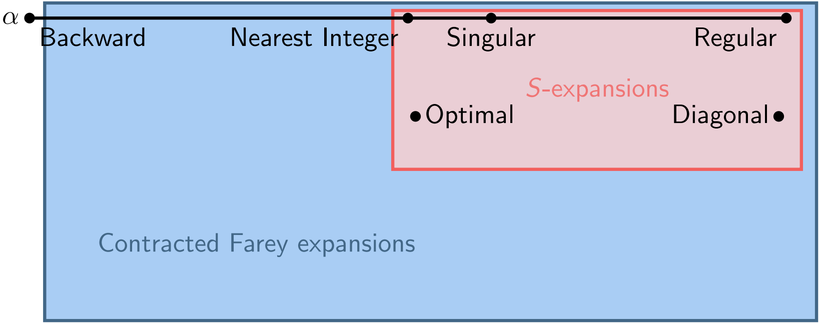

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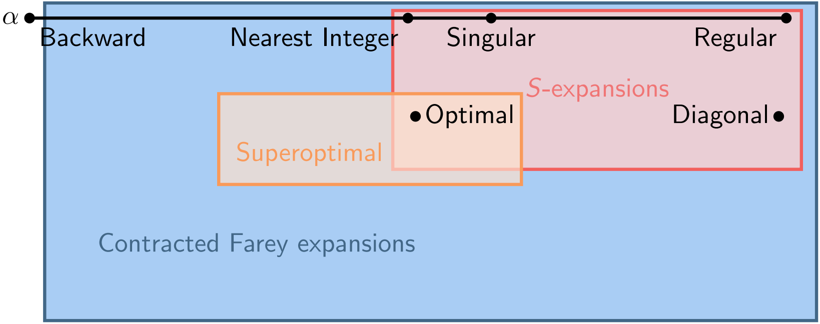

Singularisation vs. contraction

Singularisation is well-known and dates back to Lagrange (1798), but it is limited:

(i) can only remove p_n/q_n if $a_{n+1} = 1$, and

(ii) cannot remove p_n/q_n and p_{n+1}/q_{n+1}.

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(ii) cannot remove p_n/q_n and p_{n+1}/q_{n+1}.

But there's a more general acceleration technique called **contraction**:

Theorem (Seidel 1855)

Let $[\beta_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$ *be a* GCF *with convergents* $P_n/Q_n = [\beta_0; \alpha_1/\beta_1, \dots, \alpha_n/\beta_n]$*, and let* $(n_k)_{k\geq 0}$ *be any strictly increasing sequence of non-negative integers. Under mild assumptions, there is an explicit* GCF $[\beta_0'; \alpha_1'/\beta_1', \alpha_2'/\beta_2', \dots]$ *whose convergents are precisely* P_{n_k}/Q_{n_k}.

All α-CFs realised as 'S-expansions with contraction'?

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The non-monotonicity of the entropy of α-continued fraction transformations

Hitoshi Nakada[1] and Rie Natsui[2]

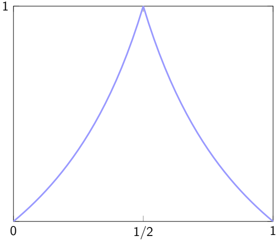
between the extensions of any α and $\alpha_{\frac{1}{2}}$ for $\alpha \in [\sqrt{2}-1, \frac{1}{2}]$ as a generalization of [1]. Here we note that the natural extension of T_α cannot be obtained by a simple induced transformation, in the sense of [1], of the natural extension of T_1. This is related to the fact that a convergent of the continued fraction expansion of x by T_α may not be a convergent of the simple continued fraction expansion of x.

Farey tent map

The **Farey tent map** $F : [0,1] \to [0,1]$ is

$$F(x) = \begin{cases} \frac{x}{1-x}, & x \leq 1/2, \\ \frac{1-x}{x}, & x > 1/2. \end{cases}$$

The dynamical system $([0,1], \mathcal{B}, \mu, F)$ is ergodic, where μ is the infinite, σ-finite, a.c. invariant measure with density $1/x$.



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'THE MOTHER OF ALL CONTINUED FRACTIONS'

BY

KARMA DAJANI (UTRECHT) AND COR KRAAIKAMP (DELFT)

Farey expansions and convergents

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$$n = a_1 + \cdots + a_{j_n} + \lambda_n, \qquad 0 \leq \lambda_n < a_{j_n+1}.$$

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The map F generates GCF-expansions called **Farey expansions** whose **Farey convergents** are

$$\frac{P_{n-1}}{Q_{n-1}} = \frac{u_n}{s_n} = \frac{\lambda_n p_{j_n} + p_{j_n-1}}{\lambda_n q_{j_n} + q_{j_n-1}},$$

i.e., all RCF-convergents and **mediant convergents** of x.

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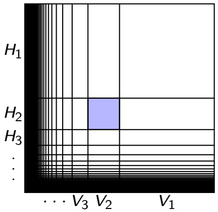

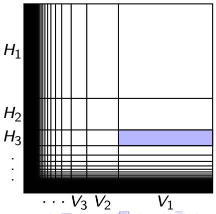

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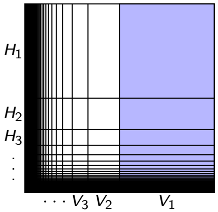
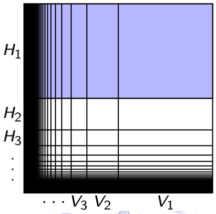
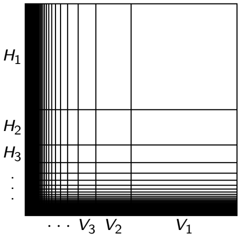




$\mathcal{F}$-orbits and Farey convergents

Letting $z = (x, 1)$ and $z_n := \mathcal{F}^n(z)$, we have a 1-1 correspondence

$$z_n \in V_{a_{j_n+1}-\lambda_n} \cap H_{\lambda_n+1} \quad \longleftrightarrow \quad \frac{u_n}{s_n} = \frac{\lambda_n p_{j_n} + p_{j_n-1}}{\lambda_n q_{j_n} + q_{j_n-1}}.$$



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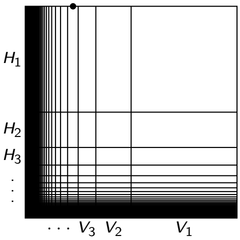
Example

Let $x = [0; \overline{4, 2, 3}]$. We have

$$z_0 = ([0; 4, \overline{2, 3, 4}], [0; 1]),$$

$$\left(\frac{u_n}{s_n}\right)_{n\geq 0} = \Bigg(\frac{p_{-1}}{q_{-1}}, \frac{p_0 + p_{-1}}{p_0 + q_{-1}}, \frac{2q_0 + p_{-1}}{2q_0 + q_{-1}}, \frac{3p_0 + p_{-1}}{3q_0 + q_{-1}}, \frac{p_0}{q_0}, \frac{p_1 + p_0}{q_1 + q_0}, \frac{p_1}{q_1}, \frac{p_2 + p_1}{q_2 + q_1}, \frac{2p_2 + p_1}{2q_2 + q_1}, \dots\Bigg)$$



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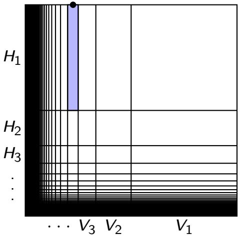

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Example

Let $x = [0; \overline{4, 2, 3}]$. We have

$$z_1 = ([0; 3, \overline{2, 3, 4}], [0; 2]),$$

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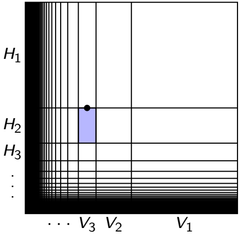

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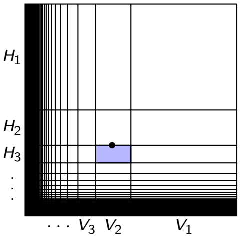

$\mathcal{F}$-orbits and Farey convergents

Letting $z = (x, 1)$ and $z_n := \mathcal{F}^n(z)$, we have a 1-1 correspondence

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Example

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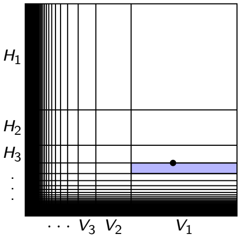

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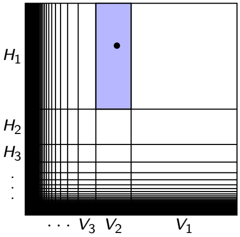

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Example

Let $x = [0; \overline{4, 2, 3}]$. We have

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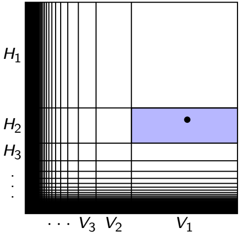

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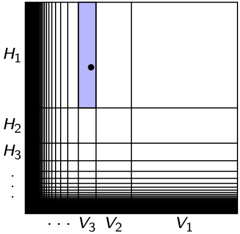

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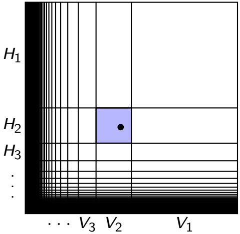

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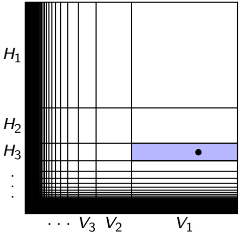

Inducing Ito's natural extension

For $R \subset \Omega$ with $0 < \bar{\mu}(R) < \infty$, define $\mathcal{F}_R := \mathcal{F}^{N_R} : \Omega \to R$, where

$$N_R(z) := \inf\{n \geq 1 \mid \mathcal{F}^n(z) \in R\}$$

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Let $z = (x, 1)$. When $\bar{\mu}(\mathrm{int}R) > 0$, $\mathcal{F}^n(z) \in R$ i.o. for a.e. x. The map $\mathcal{F}_R$ determines a subsequence $(z_k^R)_{k\geq 0} = (z_{N_k^R})_{k\geq 0}$ of $(z_n)_{n\geq 0}$ and, via $z_n \longleftrightarrow u_n/s_n$, a subsequence $(u_k^R/s_k^R)_{k\geq 0} = (u_{N_k^R}/s_{N_k^R})_{k\geq 0}$ of $(u_n/s_n)_{n\geq 0}$.

Inducing contractions of the mother

Definition

The **contracted Farey expansion** (CFE) of x w/r/t $R \subset \Omega$, denoted $[\beta_0^R; \alpha_1^R/\beta_1^R, \alpha_2^R/\beta_2^R, \dots]$, is the contraction of the Farey expansion of x w/r/t $(N_{k+1}^R - 1)_{k \geq 0}$.

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Proposition

The contracted Farey expansion of x w/r/t R has convergents $(u_n^R/s_n^R)_{n\geq 0}$. Moreover, the digits α_n^R, β_n^R may be described explicitly in terms of the dynamics of $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$.

Two-sided shift space

Let $A_{[0,N_R(z)]} =: \begin{pmatrix} u_R(z) & t_R(z) \\ s_R(z) & r_R(z) \end{pmatrix}$, and suppose R is bounded away from $y = 0$ and that $s_R = 1$ ($\implies u_R = 0, 1$).

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$$\varphi_R(z) := \begin{cases} \left(x, \frac{1-y}{y}\right), & u_R(z) = 0, \\ (x - 1, 1 - y), & u_R(z) = 1. \end{cases}$$

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Theorem (Dajani, Kraaikamp, S. 2025)

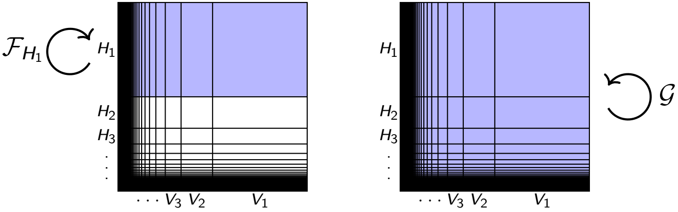
If $z = (x, 1)$ with $x = [\beta_0^R; \alpha_1^R/\beta_1^R, \alpha_2^R/\beta_2^R, \dots]$ and $(X, Y) = \varphi_R(z)$, then

$$\tau_R^n(X, Y) = \left([0; \alpha_{n+1}^R/\beta_{n+1}^R, \alpha_{n+2}^R/\beta_{n+2}^R, \dots], [0; 1/\beta_n^R, \alpha_n^R/\beta_{n-1}^R, \dots, \alpha_2^R/\beta_1^R]\right).$$

Moreover, $\bar{\nu}_R = \bar{\mu}_R \circ \varphi_R^{-1}$ has density $1/(\bar{\mu}(R)(1 + XY)^2)$.

Regular CFs

Let $R = H_1$. Brown–Yin ('96) showed $(H_1, \mathcal{B}, \bar{\mu}_{H_1}, \mathcal{F}_{H_1}) \cong (\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$.



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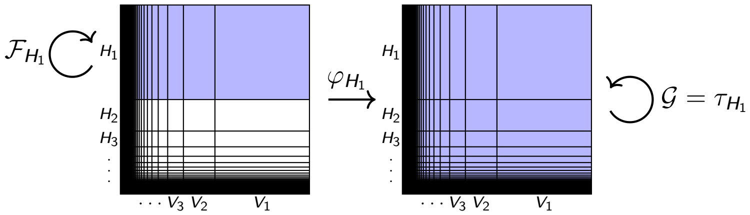

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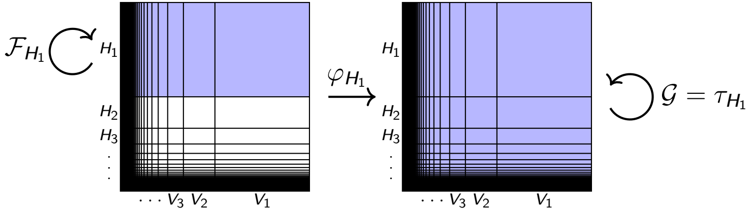
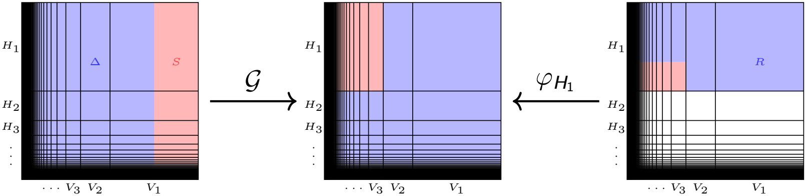
$$x = [\beta_0^{H_1}; \alpha_1^{H_1}/\beta_1^{H_1}, \alpha_2^{H_1}/\beta_2^{H_1}, \ldots] = [0; \beta_1^{H_1}, \beta_2^{H_1}, \ldots].$$



Kraaikamp's S-expansions

Let S be a **singularisation area**, $\Delta = \Omega \backslash S$, and $R := \varphi_{H_1}^{-1} \circ \mathcal{G}(\Delta)$.



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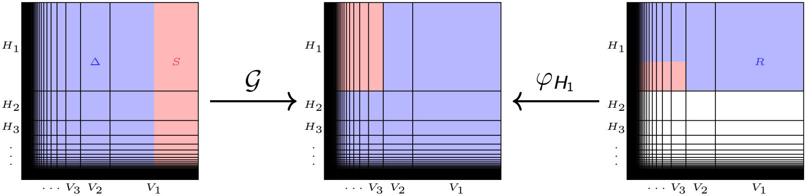
Let S be a **singularisation area**, $\Delta = \Omega \backslash S$, and $R := \varphi_{H_1}^{-1} \circ \mathcal{G}(\Delta)$. Then the CFE of x w/r/t R is the S-expansion of x, and $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ coincides with the two-sided shift space for S-expansions introduced by Kraaikamp.



Nakada's α-CFs

Fix $0 < \alpha \leq 1$. Let $k(z) := \inf\{j > 0 \mid \mathcal{F}_{H_1}^{-j}(z) \in [0, \alpha) \times [1/2, 1]\}$

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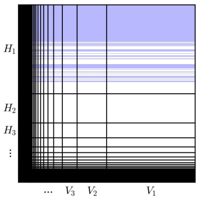

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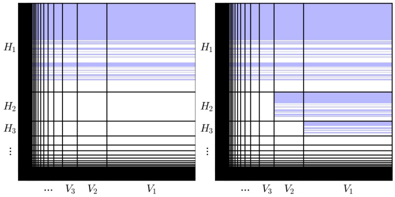

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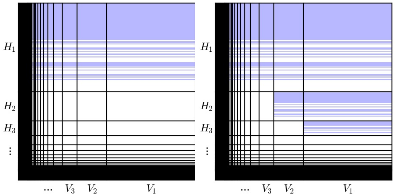

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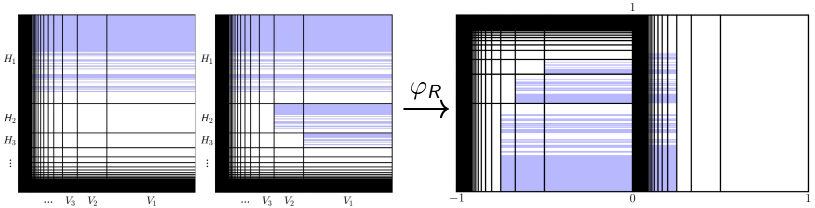
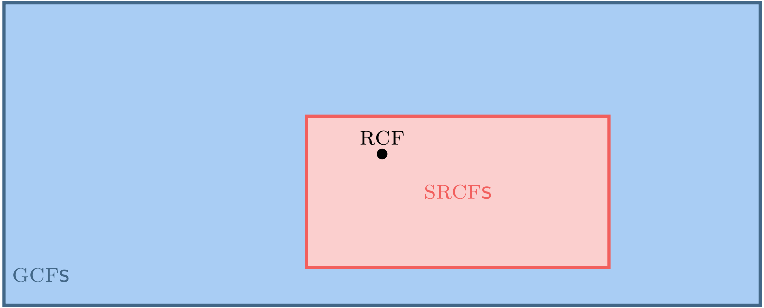


Semi-regular CFs



Bosma's optimal CFs

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where $q_{n(k)} \leq Q_k < q_{n(k)+1}$ and $G := (\sqrt{5}+1)/2$.

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For x irrational and p, q relatively prime, set $\Theta(x, p/q) := q^2|x - p/q|$.
For a.e. x and any SRCF-expansion with (reduced) convergents P_k/Q_k,

$$\text{(i)} \quad \sup_{k\geq 1} \Theta\left(x, \frac{P_k}{Q_k}\right) \geq \frac{1}{2} \quad \& \quad \text{(ii)} \quad \limsup_{k\to\infty} \frac{n(k)}{k} \leq \frac{\log 2}{\log G} \approx 1.4404\ldots,$$

where $q_{n(k)} \leq Q_k < q_{n(k)+1}$ and $G := (\sqrt{5}+1)/2$.
In 1987, Bosma introduced an algorithm producing **optimal** CFs (introduced by Selenius, 1960) which satisfy

$$\text{(i)} \quad \Theta\left(x, P_k/Q_k\right) < \frac{1}{2}\ \forall k \quad \& \quad \text{(ii)} \quad \lim_{k\to\infty} \frac{n(k)}{k} = \frac{\log 2}{\log G}.$$

Superoptimal CFs

Let $\varepsilon, C > 0$. A GCF-exp'n with (reduced) convergents P_k/Q_k is (ε, C)-**superoptimal** if both

$$\text{(i)} \quad \Theta\left(x, P_k/Q_k\right) \leq \varepsilon \ \forall k \quad \& \quad \text{(ii)} \quad \limsup_{k\to\infty} \frac{n(k)}{k} \geq C.$$

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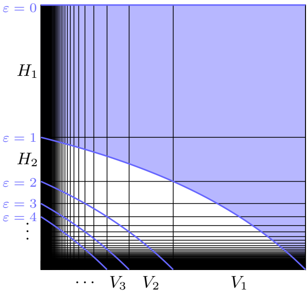
Proposition

$\Theta(x, u_n/s_n) < \varepsilon$ iff $z_n \in S_\varepsilon$, where

$$S_\varepsilon := \left\{ z = (x, y) \;\middle|\; \frac{1-y}{x+y-xy} < \varepsilon \right\}.$$



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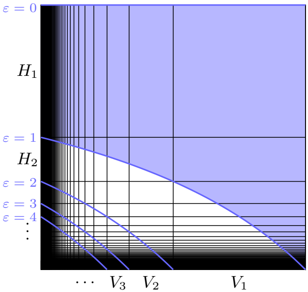

Theorem (S. 2025+)

If $R \subset S_\varepsilon$ with $\bar{\mu}(R) \leq \frac{\log 2}{C}$, then the CFE of x w/r/t R is (ε, C)-superoptimal.

Legendre–Hurwitz CFs

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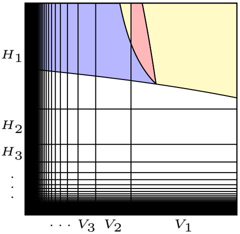
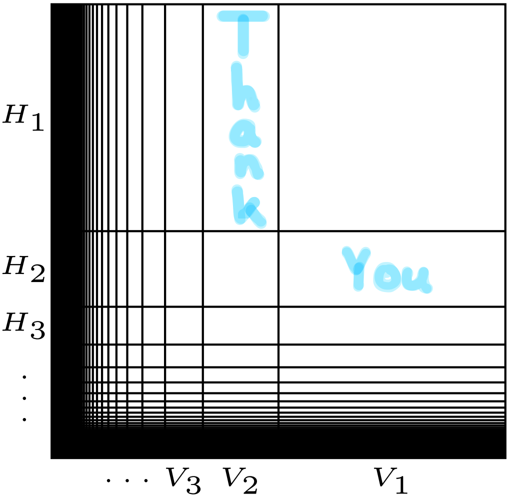
Corollary

The CFE of <u>any</u> irrational x w/r/t $R = S_{1/\sqrt{5}}$ exists, is (ε, C)-superoptimal for $\varepsilon = 1/\sqrt{5} \approx 0.4472\ldots$, $C = \sqrt{5}\log 2 \approx 1.5499\ldots$, and the Farey convergents u_n^R/s_n^R of x are <u>precisely</u> the rationals p/q from Hurwitz's theorem.





Thank
You
H_1
H_2
H_3
$\vdots$
$\cdots V_3$ V_2 V_1