Recent progress on certain problems related to local Arthur packets of classical groups

Baiying Liu

Purdue University

April 11, 2022

HAPPY BIRTHDAY, GORDAN!

Notation

- F a nonarchimedean local field of char. zero with Weil group W_F
- G_n quasi-split classical group $\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}_{2n+1}(F)$, $\mathrm{SO}_{2n}^{\alpha}(F)$
- $\widehat{\mathrm{G}}_n(\mathbb{C})$ the Langlands dual group $\mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n}(\mathbb{C})$
- ${}^L G_n$ the Langlands L-group

$$ {}^L\mathrm{G}_n = \begin{cases} \widehat{\mathrm{G}}_n(\mathbb{C}) & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \\ \mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_F & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n}^{\alpha}. \end{cases} $$

Notation

- F a nonarchimedean local field of char. zero with Weil group W_F
- G_n quasi-split classical group $\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}_{2n+1}(F)$, $\mathrm{SO}_{2n}^{\alpha}(F)$
- $\widehat{\mathrm{G}}_n(\mathbb{C})$ the Langlands dual group $\mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n}(\mathbb{C})$
- ${}^L G_n$ the Langlands L-group

$$
{}^L\mathrm{G}_n = \begin{cases} \widehat{\mathrm{G}}_n(\mathbb{C}) & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \\ \mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_F & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n}^{\alpha}. \end{cases}
$$

- To parametrize local components of square-integrable automorphic representations, Arthur introduced local Arthur packets which are finite (multi)-sets of irreducible unitary representations

Local Arthur parameters

- $$\psi : W_F \times \mathrm{SL}_2^D(\mathbb{C}) \times \mathrm{SL}_2^A(\mathbb{C}) \to {}^L\mathrm{G}_n$$
  $$\psi = \bigoplus_{i=1}^{r} \phi_i \otimes S_{m_i} \otimes S_{n_i},$$
  satisfying the following conditions:
  (1) $\phi_i(W_F)$ is *bounded* and consists of semi-simple elements, and $\dim(\phi_i) = k_i$;
  (2) the restrictions of ψ to the two copies of $\mathrm{SL}_2(\mathbb{C})$ are analytic, S_k is the k-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$, and
  $$\sum_{i=1}^{r} k_i m_i n_i = N := \begin{cases} 2n+1 & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \\ 2n & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}. \end{cases}$$
- ψ is tempered if all $n_i = 1$.

Local L parameters attached to local Arthur parameters

- $$\phi_\psi(w,x) = \psi\left(w, x, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right),$$

  where,

  $$\phi_i(w) \otimes S_{m_i}(x) \otimes S_{n_i}\left(\begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right) = \bigoplus_{j=-\frac{n_i-1}{2}}^{\frac{n_i-1}{2}} |w|^j \phi_i(w) \otimes S_{m_i}(x).$$

  Arthur showed that $\psi \mapsto \phi_\psi$ is injective.

Local L parameters attached to local Arthur parameters

- $$\phi_\psi(w,x) = \psi\left(w, x, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right),$$

  where,

  $$\phi_i(w) \otimes S_{m_i}(x) \otimes S_{n_i}\left(\begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right) = \bigoplus_{j=-\frac{n_i-1}{2}}^{\frac{n_i-1}{2}} |w|^j \phi_i(w) \otimes S_{m_i}(x).$$

  Arthur showed that $\psi \mapsto \phi_\psi$ is injective.

- $$\psi^\Delta : W_F \times \mathrm{SL}_2(\mathbb{C}) \to W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$$
  $$(w,x) \mapsto (w,x,x).$$

Construction of local Arthur packets

- Arthur defined Π_ψ using character identity relations.
  Π_ψ is a finite (multi)-set of irreducible unitary representations of G_n.

$$\Pi_{\phi_\psi} \subset \Pi_\psi.$$

Construction of local Arthur packets

- Arthur defined Π_ψ using character identity relations.
  Π_ψ is a finite (multi)-set of irreducible unitary representations of G_n.

$$\Pi_{\phi_\psi} \subset \Pi_\psi.$$

- Moeglin constructed Π_ψ explicitly and showed that it is multiplicity free.
  Moeglin and Xu's work showed that the constructed packets satisfies Arthur's character identity relations.
  Xu gave the nonvanishing criterions, L-data in certain cases, cardinality of Arthur packets.

Construction of local Arthur packets

- Arthur defined Π_ψ using character identity relations.
  Π_ψ is a finite (multi)-set of irreducible unitary representations of G_n.

$$\Pi_{\phi_\psi} \subset \Pi_\psi.$$

- Moeglin constructed Π_ψ explicitly and showed that it is multiplicity free.
  Moeglin and Xu's work showed that the constructed packets satisfies Arthur's character identity relations.
  Xu gave the nonvanishing criterions, L-data in certain cases, cardinality of Arthur packets.
- Atobe gave a refinement on the Mœglin's construction, using the new derivatives introduced by himself and Minguez, which makes it relatively easier to compute the L-data.

Wave front sets of representations in Π_ψ?

Conjecture (Shahidi's conjecture)

For any quasi-split reductive group G, tempered local Arthur-packets have generic members.

Conjecture (Enhanced Shahidi's conjecture)

For any quasi-split reductive group G, local Arthur packets are tempered if and only if they have generic members.

Conjecture (Jiang's conjecture)

Generalization of Shahidi's conjecture to nontempered local Arthur packets.

Intersection of local Arthur packets

- Local Arthur packets could have nontrivial intersections.
  When $\Pi_{\psi_1} \cap \Pi_{\psi_2} \neq \emptyset$?

Intersection of local Arthur packets

- Local Arthur packets could have nontrivial intersections. When $\Pi_{\psi_1} \cap \Pi_{\psi_2} \neq \emptyset$?
- Moeglin: $\Pi_{\psi_1} \cap \Pi_{\psi_2} \neq \emptyset \Rightarrow \psi_1^{\Delta} = \psi_2^{\Delta}$.

Intersection of local Arthur packets

- Local Arthur packets could have nontrivial intersections. When $\Pi_{\psi_1} \cap \Pi_{\psi_2} \neq \emptyset$?
- Moeglin: $\Pi_{\psi_1} \cap \Pi_{\psi_2} \neq \emptyset \Rightarrow \psi_1^{\Delta} = \psi_2^{\Delta}$.
- Fix any two local Arthur packets, one can certainly check the intersection by Moeglin's construction or by Atobe's refinement. Difficulty: find a systematic way to determine the intersection.

Question

Fix a local Arthur parameter ψ, how to systematically determine the set

$$\{\psi' \mid \Pi_{\psi} \cap \Pi_{\psi'} \neq \emptyset\}?$$

Wave front sets of representations

- Harish-Chandra/Howe character expansion: π irr. adm. rep. of G_n, $\exists$ a n.b.h.d. U of 1 such that for any $f \in C_c^\infty(U)$,

$$\mathrm{tr}\,\pi(f) = \sum_{\mathcal{O}\in\mathcal{N}_{\mathfrak{g}n(F)}} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(f),$$

where

$$\hat{\mu}_{\mathcal{O}}(f) = \int_{CO} \hat{\tilde{f}}(u) d\mu_{\mathcal{O}}(u),$$

$\tilde{f} = f \circ exp$, $\hat{\tilde{f}}$ Fourier transform of $\tilde{f}$.

Wave front sets of representations

- Harish-Chandra/Howe character expansion: π irr. adm. rep. of G_n, $\exists$ a n.b.h.d. U of 1 such that for any $f \in C_c^\infty(U)$,

$$\mathrm{tr}\,\pi(f) = \sum_{\mathcal{O}\in\mathcal{N}_{\mathfrak{g}_n(F)}} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(f),$$

  where

$$\hat{\mu}_{\mathcal{O}}(f) = \int_{CO} \hat{\tilde{f}}(u) d\mu_{\mathcal{O}}(u),$$

  $\tilde{f} = f \circ exp$, $\hat{\tilde{f}}$ Fourier transform of $\tilde{f}$.

- $\mathfrak{n}(\pi) = \{\mathcal{O} \in \mathcal{N}_{\mathfrak{g}_n(F)} | c_{\mathcal{O}}(\pi) \neq 0\}$.
  $\mathfrak{n}^m(\pi)$ maximal orbits in $\mathfrak{n}(\pi)$, wave front set of π.
  $\mathfrak{p}(\pi)$ partitions corresponding to orbits in $\mathfrak{n}(\pi)$.
  $\mathfrak{p}^m(\pi)$ maximal partitions in $\mathfrak{p}(\pi)$ under the dominance order.

Conjecture/Properties of wave front sets

Conjecture (Moeglin-Waldspurger, Kawanaka)

Let G be a connected reductive group defined over F. Assume that π is an irreducible admissible representation of $\mathrm{G}(F)$. Then nilpotent orbits in $\mathfrak{n}^m(\pi)$ belong to a unique geometric orbit over $\overline{F}$.

- Moeglin: orbits in $\mathfrak{n}^m(\pi)$ are admissible. For classical groups, orbits in $\mathfrak{n}^m(\pi)$ are special.
  Moeglin-Waldspurger, Varma: for $\mathcal{O} \in \mathfrak{n}^m(\pi)$, $c_{\mathcal{O}} = \dim Wh_{\mathcal{O}}(\pi)$, the generalized Whittaker model for π associated to $\mathcal{O}$.
  Gomez-Gourevitch-Sahi: for $\mathcal{O} \in \mathfrak{n}(\pi)$, $Wh_{\mathcal{O}}(\pi) \twoheadrightarrow DWh_{\mathcal{O}}(\pi)$, degenerate Whittaker models for π. $\mathcal{O} \in \mathfrak{n}^m(\pi)$ are quasi-admissible,

Raising of nilpotent orbits in wave front sets

- Jiang-L.-Savin: raising phenomenon in $\mathfrak{n}(\pi)$,

  $$\mathcal{O} \in \mathfrak{n}(\pi), \text{ not special } \Rightarrow \mathcal{O}^G \in \mathfrak{n}(\pi),$$

  where $\mathcal{O}^G$ is the smallest $\mathrm{G}(F)$ special orbit which is bigger than $\mathcal{O}$.
  $\Rightarrow$
  $\mathcal{O} \in \mathfrak{n}^m(\pi)$ are special (except 10 nonspeical orbits in exceptional groups which can not be ruled out).

Raising of nilpotent orbits in wave front sets

- Jiang-L.-Savin: raising phenomenon in $\mathfrak{n}(\pi)$,

$$\mathcal{O} \in \mathfrak{n}(\pi), \text{ not special } \Rightarrow \mathcal{O}^G \in \mathfrak{n}(\pi),$$

  where $\mathcal{O}^G$ is the smallest $\mathrm{G}(F)$ special orbit which is bigger than $\mathcal{O}$.
  $\Rightarrow$
  $\mathcal{O} \in \mathfrak{n}^m(\pi)$ are special (except 10 nonspeical orbits in exceptional groups which can not be ruled out).
- Loke-Savin: the minimal orbit of G_2 can not be in $\mathfrak{n}^m(\pi)$.

Recall local Arthur parameters

- $$\psi : W_F \times \mathrm{SL}_2^D(\mathbb{C}) \times \mathrm{SL}_2^A(\mathbb{C}) \to {}^L\mathrm{G}_n$$
$$\psi = \bigoplus_{i=1}^{r} \phi_i \otimes S_{m_i} \otimes S_{n_i},$$

 $\dim(\phi_i) = k_i$;

$$\sum_{i=1}^{r} k_i m_i n_i = N := \begin{cases} 2n+1 & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \\ 2n & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}. \end{cases}$$

 Let $a_i = k_i m_i$, $b_i = n_i$, $\underline{p}(\psi) = [b_1^{a_1} \cdots b_r^{a_r}]$, $b_1 \geq \cdots \geq b_r$. $\underline{p}(\psi)$ is partition of $\widehat{\mathrm{G}}_n(\mathbb{C})$.

Conjecture (Jiang's conjecture)

Let ψ be a local Arthur parameter of G_n, and let Π_ψ be the local Arthur packet attached to ψ. Then the followings hold.

(1) *For any $\pi \in \Pi_\psi$, any partition $\underline{p} \in \mathfrak{p}^m(\pi)$ has the property that $\underline{p} \leq \eta_{\hat{\mathfrak{g}}_n,\mathfrak{g}_n}(\underline{p}(\psi))$.*

(2) *There exists at least one member $\pi \in \Pi_\psi$ having the property that $\eta_{\hat{\mathfrak{g}}_n,\mathfrak{g}_n}(\underline{p}(\psi)) \in \mathfrak{p}^m(\pi)$.*

Here $\eta_{\hat{\mathfrak{g}}_n,\mathfrak{g}_n}$ denotes the Barbasch-Vogan duality map from the partitions for the dual group $\widehat{\mathrm{G}}_n(\mathbb{C})$ to the partitions for G_n.

e.g. for Sp_{2n}, $\eta_{\hat{\mathfrak{g}}_n,\mathfrak{g}_n}(\underline{p}(\psi)) = [(\underline{p}(\psi)^t)^-]_{Sp_{2n}}$.

- ψ naturally gives a representation π_ψ of $\mathrm{GL}_N(F)$.

Theorem (L.-Shahidi)

$$\mathfrak{p}^m(\pi_\psi) = \{\underline{p}(\psi)^t\}.$$

- ψ naturally gives a representation π_ψ of $\mathrm{GL}_N(F)$.

Theorem (L.-Shahidi)

$$\mathfrak{p}^m(\pi_\psi) = \{\underline{p}(\psi)^t\}.$$

- Let $\theta(g) = {}^t g^{-1}$, $\tilde{J} = \begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & \cdots & & \\ (-1)^{N+1} & & & 0 \end{pmatrix}$, $\tilde{\theta}(N) = Int(\tilde{J}) \circ \theta$.

  Let $\widetilde{\mathrm{G}}^+(N) = \mathrm{G}(N) \rtimes \langle \tilde{\theta}(N) \rangle$, $\widetilde{\mathrm{G}}(N) = \mathrm{G}(N) \rtimes \tilde{\theta}(N)$. Arthur defined a canonical extension $\tilde{\pi}_\psi$ to $\widetilde{\mathrm{G}}(N)(F)$. By Clozel's character expansion of disconnected groups, can define similarly $\mathfrak{n}(\tilde{\pi}_\psi)$, $\mathfrak{n}^m(\tilde{\pi}_\psi)$, $\mathfrak{p}(\tilde{\pi}_\psi)$, $\mathfrak{p}^m(\tilde{\pi}_\psi)$.

Conjecture (L.-Shahidi)

$$\mathfrak{p}^m(\tilde{\pi}_\psi) = \{(\underline{p}(\psi)^t)_{\widehat{\mathrm{G}}_n}\}.$$

Theorem (L.-Shahidi)

Assume the conjecture on $\mathfrak{p}^m(\tilde{\pi}_\psi)$. Then the followings hold.

1. *For any $\underline{p} > \eta_{\hat{\mathfrak{g}}_n,\mathfrak{g}_n}(\underline{p}(\psi))$, $\underline{p} \notin \cup_{\pi\in\Pi_\psi}\mathfrak{p}^m(\pi)$.*
2. *Enhanced Shahidi's conjecture is true.*
3. *Assume further the uniqueness conjecture of $\mathfrak{p}^m(\pi)$. Let $\underline{p}_1 = \left[\lfloor \frac{b_1}{2} \rfloor^{a_1} \lfloor \frac{b_2}{2} \rfloor^{a_2} \cdots \lfloor \frac{b_r}{2} \rfloor^{a_r}\right]^t$, and $n^* = \lfloor \frac{\sum_{b_i \text{ odd}} a_i}{2} \rfloor$. Then Jiang's conjecture holds for the following cases.*
    1. *When $\mathrm{G}_n = \mathrm{Sp}_{2n}$, and $([\underline{p}_1\underline{p}_1(2n^*)]^t)_{\mathrm{Sp}_{2n}} = ([b_1^{a_1}\cdots b_r^{a_r}]^-)_{\mathrm{Sp}_{2n}}$.*
    2. *When $\mathrm{G}_n = \mathrm{SO}_{2n+1}$, and $([\underline{p}_1\underline{p}_1(2n^*+1)]^t)_{\mathrm{SO}_{2n+1}} = ([b_1^{a_1}\cdots b_r^{a_r}]^+)_{\mathrm{SO}_{2n+1}}$.*
    3. *When $\mathrm{G}_n = \mathrm{SO}_{2n}^\alpha$, and $[\underline{p}_1\underline{p}_1(2n^*-1)1]^{\mathrm{SO}_{2n}} = ([b_1^{a_1}\cdots b_r^{a_r}]^t)_{\mathrm{SO}_{2n}}$.*

These identities hold when b_i are of the same parity.

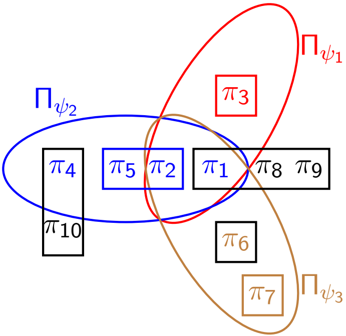
Idea of the proof

- Constructed a representation $\sigma \in \Pi_\psi$ following the work of Jiang-Soudry, L., Jantzen-L.
  $\mathfrak{p}^m(\sigma)$ achieves the upper bound when those identities hold (used the work of Gomez-Gourevitch-Sahi and Jiang-L.-Savin).
- Used the character identities of Arthur, the matching method of Shahidi 1990, and certain dimension identities for nilpotent orbits of different groups.
- Okada, Ciubotaru-Mason-Brown-Okada recently computed the wave front set of irreducible Iwahori-spherical representations of split connected reductive p-adic groups with "real infinitesimal characters", proved many special cases of Jiang's conjecture.

An example

- Let ρ be the trivial representation. Consider three local Arthur parameters of $\mathrm{Sp}_{10}(F)$,

$$\psi_1 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_2 \otimes S_2,$$
$$\psi_2 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_1 \otimes S_1 + \rho \otimes S_3 \otimes S_1,$$
$$\psi_3 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_1 \otimes S_3 + \rho \otimes S_1 \otimes S_1.$$



Nontempered GGP

Let $G_1 = \mathrm{SO}_{2n+1}$ and $G_2 = \mathrm{SO}_{2n}$ and ψ_1 and ψ_2 be a relevant pair of local Arthur parameters for G_1 and G_2 respectively.

Conjecture (Gan-Gross-Prasad)

There exists a unique pair of representations $\pi_1 \times \pi_2 \in \Pi_{\phi_{\psi_1}} \times \Pi_{\phi_{\psi_2}}$ *such that* $\mathrm{dim}\mathrm{Hom}_{\mathrm{SO}_{2n}}(\pi_1 \otimes \pi_2, \mathbb{C}) \neq 0$.

However, if considering Vogan A-packet $\Pi_{\psi_1} \times \Pi_{\psi_2}$, then the uniqueness may not hold (Gan-Gross-Prasad, 2020, "Branching laws for Classical Groups: the non-tempered case", Remark 7.8), due to the nontrivial intersection of certain tempered Arthur packets and nontempered Arthur packets.

Mœglin's Construction

- Mœglin reduced the general case to the good parity case.

$$\psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}$$

  where

  - ρ is an irreducible unitary supercuspidal representation of some GL_d which is identified with an irreducible bounded representation of W_F via the local Langlands correspondence for GL_d;
  - S_a is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a;
  - I_ρ is an appropriate indexing set.

- ψ is of good parity if every summand $\rho \otimes S_a \otimes S_b$ is self-dual and of the same type as ψ.

Mœglin's Construction

Theorem (Mœglin)

Let ψ be a local Arthur parameter. We have the decomposition

$$\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee$$

where ψ_1 is a local Arthur parameter which is not of good parity, ψ_0 is a local Arthur parameter of good parity, and $\psi_1^\vee$ denotes the dual of ψ_1. Furthermore, for $\pi \in \Pi_{\psi_0}$ the induced representation $\pi_{\psi_1} \rtimes \pi$ is irreducible, independent of choice of ψ_1, and we have

$$\Pi_\psi = \{\pi_{\psi_1} \rtimes \pi \,|\, \pi \in \Pi_{\psi_0}\}.$$

- Hence, if we know the construction of local Arthur packets of good parity, then we know the general case.

Mœglin's Construction

- The rest of Mœglin's construction is as follows:

$$\left\{ \begin{array}{c} \text{discrete} \\ \text{tempered} \end{array} \right\} \to \{\text{elementary}\} \to \left\{ \begin{array}{c} \text{discrete} \\ \text{diagonal} \\ \text{restriction} \end{array} \right\} \to \{\text{good parity}\}$$

- Elementary parameters: $a_i = 1$ or $b_i = 1$ for every summand. To obtain elementary local Arthur packets from tempered local Arthur packets, Mœglin uses generalized Aubert involutions.
- Discrete diagonal restriction parameters: $\left[\frac{a_i+b_i}{2} - 1, \left|\frac{a_i-b_i}{2}\right|\right]$ are disjoint for any $i \in I_\rho$. To obtain these packets, Mœglin takes certain socles (i.e. maximal semisimple subrepresentations).
- Finally, local Arthur packets of good parity can be recovered from those of discrete diagonal restriction by taking certain derivatives.

Atobe's reformulation

- The computation of L-data for representations of the local Arthur packets for elementary and discrete diagonal restriction cases are difficult in general.
- From now on, let $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$.

Atobe's reformulation

- The computation of L-data for representations of the local Arthur packets for elementary and discrete diagonal restriction cases are difficult in general.
- From now on, let $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$.
- As a remedy, Atobe gave a refinement of Mœglin's construction:

$$\left\{ \begin{array}{c} \text{discrete} \\ \text{tempered} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{non}-\text{negative} \\ \text{discrete} \\ \text{diagonal} \\ \text{restriction} \end{array} \right\} \rightarrow \{\text{good parity}\}$$

- We say that a local Arthur parameter ψ is non-negative if $a_i \geq b_i$ for any $i \in I_\rho$ and every ρ.

Atobe's Reformulation

- *Extended multi-segment* for G_n:

$$\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$$

  - $A_i + B_i \geq 0$ for all ρ and $i \in I_\rho$, $0 \leq l_i \leq \frac{b_i}{2}$, $\eta_i = \pm 1$. I_ρ has a admissible total order: $A_i > A_j, B_i > B_j \Rightarrow i > j$.
  - As a representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$,

$$\psi_{\mathcal{E}} = \bigoplus_\rho \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}$$

    where $(a_i, b_i) = (A_i + B_i + 1, A_i - B_i + 1)$, is a local Arthur parameter for G_n of good parity.
  - The sign condition

$$\prod_\rho \prod_{i \in I_\rho} (-1)^{[\frac{b_i}{2}] + l_i} \eta_i^{b_i} = 1.$$

Atobe's Reformulation

- Let ρ be the trivial representation. The pictograph

$$\mathcal{E} = \begin{pmatrix} & -1 & 0 & 1 & 2 & 3 \\ & \lhd & \ominus & \oplus & \ominus & \rhd \\ & & & & \lhd & \rhd \end{pmatrix}_\rho$$

corresponds to the extended multi-segment $\mathcal{E} = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i=1<2}$ of Sp_{26} where $A_1 = A_2 = 3$, $B_1 = -1$, $B_2 = 2$, $l_1 = l_2 = 1$, $\eta_1 = -1$, and $\eta_2 = 1$. The A_i's and B_i's denote the endpoints of the pictograph, l_i's denote the number of triangles, and η_i's denote the first sign.
The associated local Arthur parameter is

$$\psi_{\mathcal{E}} = \rho \otimes S_3 \otimes S_5 + \rho \otimes S_6 \otimes S_2.$$

Atobe's Reformulation

- Atobe: $\mathcal{E} \to \pi(\mathcal{E})$, irreducible or zero, reformulated Xu's nonvanishing criterion.

Theorem (Atobe)

Let ψ be a local Arthur parameter of good parity and $\Psi(\psi)$ be the set of extended multi-segments $\mathcal{E} = \cup_{\rho}\{([A_i, B_i]_{\rho}, l_i, \eta_i)\}_{i \in (I_{\rho}, >)}$ such that $\psi_{\mathcal{E}} = \psi$ and if $B_i < 0$ for some $i \in I_{\rho}$, then I_{ρ} satisfies $B_i > B_j \Rightarrow i > j$. Then

$$\Pi_{\psi} = \{\pi(\mathcal{E}) | \mathcal{E} \in \Psi(\psi)\} \setminus \{0\}.$$

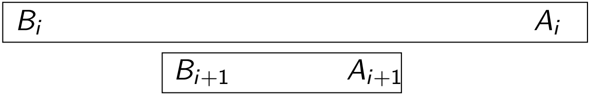
Theorem (Atobe; Hazeltine-L.-Lo)

1. *There exists algorithms to determine whether a given representation is in any local Arthur packet or not.*
2. *Assume $\pi \in \Pi_\psi$, there exists algorithms to determine all the local Arthur packets containing π.*

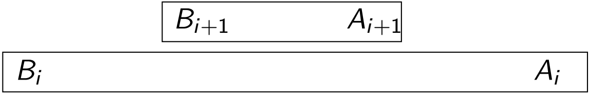
- There exists a complete set of operators on $\mathcal{E}$ which preserve representations and can be used to exhaust the set $\{\mathcal{E}' | \pi(\mathcal{E}') = \pi(\mathcal{E})\}$.

Row exchange

- Swap rows in a pictograph:

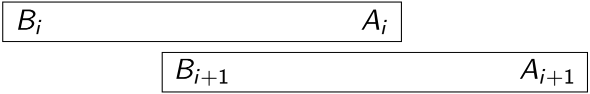


$\Rightarrow$

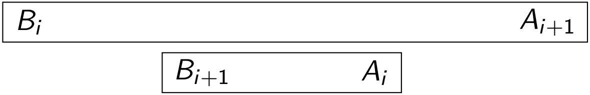


Xu: row exchange preserves representations

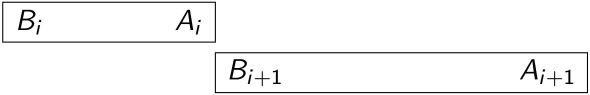
Union-Intersection



$\Rightarrow$



or



$\Rightarrow$

Atobe: union-intersection preserves representations.

$dual \circ ui \circ dual$

Theorem (Atobe)

Suppose $\pi(\mathcal{E}) \neq 0$. Then there exists an extended multi-segment, denoted $dual(\mathcal{E})$, such that $\widehat{\pi(\mathcal{E})} = \pi(dual(\mathcal{E}))$, where $\widehat{\pi(\mathcal{E})}$ denoted the Aubert-Zelevinsky dual of $\pi(\mathcal{E})$.

- The effect of *dual* on $\mathcal{E}$ is to change a segment $[A_i, B_i]$ to $[A_i, -B_i]$.
- $\pi((ui \circ dual)(\mathcal{E})) = \pi(dual(\mathcal{E}))$, and $\widehat{\widehat{\pi(\mathcal{E})}} = \pi(\mathcal{E}) \Rightarrow$ $\pi((dual \circ ui \circ dual)(\mathcal{E})) = \pi(\mathcal{E})$, i.e., $dual \circ ui \circ dual$ preserves representations.

Partial Dual

- In the case that every $A_i, B_i \in \mathbb{Z}$, row exchange, union and intersection, *dual* $\circ$ *ui* $\circ$ *dual*, their compositions and inverses are enough to exhaust the set $\{\mathcal{E}'|\pi(\mathcal{E}) = \pi(\mathcal{E}')\}$.
- If $A_i, B_i \in \frac{1}{2}\mathbb{Z}$. Also need partial dual: $[A_i, \frac{1}{2}] \Rightarrow [A_i, \frac{-1}{2}]$.

Main Theorem

Theorem (Hazeltine-L.-Lo)

1. *Like the operators row exchange, union-intersection, and dual $\circ$ ui $\circ$ dual, the partial dual also preserves representations.*
2. *Suppose that $\pi(\mathcal{E}) = \pi(\mathcal{E}') \neq 0$. Then $\mathcal{E}$ and $\mathcal{E}'$ are related by a composition of these four operators and their inverses.*
3. *There is a precise formula to compute the set*

$$\{\mathcal{E}' | \pi(\mathcal{E}) = \pi(\mathcal{E}')\}.$$

Applications

Theorem (Hazeltine-L.-Lo)

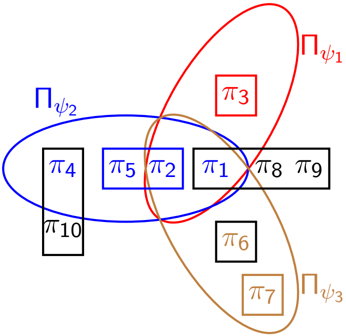
1. *Given any local Arthur parameter ψ, give a formula to count the number of tempered representations inside Π_ψ and describe their L-data.*
2. *The enhanced Shahidi conjecture is true for $\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$. That is, a local Arthur packet Π_ψ contains a generic member if and only if ψ is tempered.*
3. *Determine all $\mathcal{E}$ such that $\pi(\mathcal{E})$ is in the L-packet associated with $\psi_{\mathcal{E}}$.*
4. *For a representation π of Arthur type, give a definition of "the" local Arthur parameter $\psi(\pi)$ of π, such that*
    1. $\pi \in \Pi_{\psi(\pi)}$.
    2. *If $\pi \in \Pi_{\phi_\psi}$, then $\psi(\pi) = \psi$.*

- "The" local Arthur parameter roughly corresponds to taking all possible ui^{-1}, $dual \circ ui \circ dual$, and possibly a partial dual.

Back to the example

- Let ρ be the trivial representation. Consider three local Arthur parameters of $\mathrm{Sp}_{10}(F)$,

$$\psi_1 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_2 \otimes S_2,$$
$$\psi_2 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_1 \otimes S_1 + \rho \otimes S_3 \otimes S_1,$$
$$\psi_3 = \rho \otimes S_1 \otimes S_7 + \rho \otimes S_1 \otimes S_3 + \rho \otimes S_1 \otimes S_1.$$



Happy Birthday, Gordan!