Infinite-dimensional Geometry: Theory and Applications
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# Banach-Lie groupoids part III

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Some problems in infinite dimensional geometry

1. No dimension counting arguments: existence of injective automorphisms, which are not surjective.
2. Image of a linear map may not be closed, closed subspaces may not be complemented (split).
3. Many classical theorems fail or require non-trivial modifications (special assumptions).
4. Double dual of the a Banach space may not be canonically isomorphic to the original space.
5. ....

In effect many structures has various non-equivalent definitions: e.g. strong/weak riemannian structures, strong/weak symplectic structures, sub/weak Poisson groupoid, sub/almost sub Poisson morphism, partial Poisson manifold....

Example: Poisson bracket $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ Leibniz identity doesn't imply

$$\{f, g\}(m) = \pi_m(Df(m), Dg(m))$$

Even if $\pi$ exists it is in $\Gamma^\infty(\bigwedge^2 T^*M)$, which implies $X_f(m) = \pi_m(Df(m), \cdot) \in T_m^{**}M \neq T_mM$

We fix the terminology related to Banach geometry:
a smooth map $f : N \to M$ between two Banach manifolds will be called

• a *submersion* if for each $x \in N$ the tangent map $T_x f : T_x N \to T_{f(x)} M$ is a surjection and $\ker T_x f$ is a split subspace of $T_x N$;

• a *weak immersion* if $T_x f$ is an injection;

• an *immersion* if its range is additionally closed in $T_{f(x)} M$;

• an immersion is *split* if its range admits a closed complement.

Recall $\mathcal{L}(\mathcal{H})$ is the lattice of orthogonal projectors in Hilbert space $\mathcal{H}$

$$\mathcal{L}(\mathcal{H}) = \{p \in \mathcal{L}^\infty(\mathcal{H}) \mid p^2 = p^* = p\}.$$

It is useful to identify the projector with its image.

So, $\mathcal{L}(\mathcal{H})$ is the Grassmannian of all closed subspaces of $\mathcal{H}$.

The construction of this differential structure goes through a family of charts $\phi_W : U_W \to \mathcal{L}^\infty(W, W^\perp)$ indexed by $W \in \mathcal{L}(\mathfrak{M})$ defined by

$$\phi_W(V) = P_{W^\perp}(P_W|_V)^{-1}, \tag{1}$$

where

$$U_W = \{V \in \mathcal{L}(\mathfrak{M}) \mid V \oplus_B W^\perp = \mathcal{H}\}, \tag{2}$$

$P_W$ is the orthogonal projection on $W$ and $\oplus_B$ denotes a direct sum in the sense of Banach spaces (i.e. not necessarily orthogonal). Note that the condition for $V$ to belong to the chart domain $U_W$ is equivalent to the projection $(P_W|_V)^{-1}$.
The inverse map $\phi_W^{-1} : \mathcal{L}^\infty(W, W^\perp) \to \mathcal{L}(\mathfrak{M})$ to a chart assigns to a bounded operator its graph in $W \oplus W^\perp = \mathcal{H}$, i.e.

$$\phi_W^{-1}(A) = \{(w, Aw) \in W \times W^\perp \mid w \in W\} \tag{3}$$

for $A \in L^\infty(W, W^\perp)$.

• fix an orthogonal decomposition (called polarization) of the Hilbert space $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

onto infinite dimensional Hilbert subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$
• block decomposition of an operator $A$ acting on $\mathcal{H}$:

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

## Definition

The *restricted Grassmannian* $\mathrm{Gr}_{\mathrm{res}}$ is defined as a set of Hilbert subspaces $W \subset \mathcal{H}$ such that:

1. the orthogonal projection $p_+ : W \to \mathcal{H}_+$ is a Fredholm operator;
2. the orthogonal projection $p_- : W \to \mathcal{H}_-$ is a Hilbert–Schmidt operator.

• identify the Hilbert subspace $W$ with a projector $P_W$ onto this subspace

## Fact

$$W \in \mathrm{Gr}_{\mathrm{res}} \Longleftrightarrow P_W - P_+ \in L^2$$

• $P_+$, $P_-$: the orthogonal projectors onto $\mathcal{H}_+$ and $\mathcal{H}_-$
Thus, we identify $\mathrm{Gr}_{\mathrm{res}}$ with the set of projectors $\{P_W \mid W \in \mathrm{Gr}_{\mathrm{res}}\} \subset \mathcal{L}(\mathcal{H})$.

The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ (also known as the Sato Grassmannian) is a strongly symplectic (or even Kähler) manifold modeled on a Hilbert space.

It appeared in the study of the KdV and KP equations, it has many other applications, including quantum field theory, loop groups, Banach Lie–Poisson spaces, integrable Hamiltonian systems, Banach Poisson–Lie groups.

• Unitary restricted group $\mathrm{U}_{\mathsf{res}}(\mathcal{H})$

$$\mathrm{U}_{\mathsf{res}}(\mathcal{H}) := \{u \in \mathrm{U}(\mathcal{H}) \mid [u, P_+] \in L^2\}$$

is Banach Lie group [A.B.Tumpach 2020]
• $\mathrm{U}_{\mathsf{res}}(\mathcal{H})$ acts transitively on $\mathrm{Gr}_{\mathrm{res}}$ and the stabilizer of $\mathcal{H}_+$ for this action is $U_+ \times U_-$, which is a Banach Lie subgroup of $\mathrm{U}_{\mathsf{res}}(\mathcal{H})$ [D.Beltita 2006]. In this way, $\mathrm{Gr}_{\mathrm{res}}$ can also be seen as a smooth homogeneous space $\mathrm{U}_{\mathsf{res}}(\mathcal{H})/(U_+ \times U_-)$.

The differential structure on $\mathrm{Gr}_{\mathrm{res}}$ is obtained using the same charts as for $\mathcal{L}(\mathcal{H})$ taking value in $L^2(W, W^\perp)$ and transition functions are smooth in $L^2$ topology.

- the chart $\phi_W : U_W \to L^2(W, W^\perp)$ on $\mathrm{Gr}_{\mathrm{res}}$, for $W \in \mathrm{Gr}_{\mathrm{res}}$

$$\phi_W(p) = (P_W)^\perp p P_W (P_W p P_W)^{-1},$$

where

$$U_W = \{p \in \mathrm{Gr}_{\mathrm{res}} \mid P_W p P_W \text{ is invertible in } W\}.$$

- In particular for $W = P_+$ this formula can be written as

$$\phi_{\mathcal{H}_+}(p) = p_{-+}(p_{++})^{-1} = p(p_{++})^{-1} - P_+.$$

This map takes values in $L^2_{+-}$ since $P_- p$ is Hilbert–Schmidt by the definition of the restricted Grassmannian.

- for particular case of $W = \mathcal{H}_+$ the inverse to a chart is

$$\phi^{-1}_{\mathcal{H}_+}(A) = \begin{pmatrix} (1 + A^*A)^{-1} & (1 + A^*A)^{-1}A^* \\ A(1 + A^*A)^{-1} & A(1 + A^*A)^{-1}A^* \end{pmatrix}$$

- equivalently:

$$\phi^{-1}_{\mathcal{H}_+}(A) = (P_+ + A)(1 + A^*A)(P_+ + A^*P_-)$$

$$\phi^{-1}_{W}(A) = (P_W + A)(1 + A^*A)(P_W + A^*P_{W^\perp})$$

### Proposition

The transition maps for the atlas $\{\phi_W, \tilde{U}_W\}_{W\in \mathrm{Gr}_{\mathrm{res}}}$ on the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ are

$$\psi_{V,W}(A) = \phi_V \circ \phi_W^{-1}(A) = P_V^{\perp}(1_W + A)\big(P_V(P_W + A)\big)^{-1}$$

and are smooth with respect to $L^2$ topology.

Let us stress at this point that $^{-1}$ in above formula denotes the inverse in the space $L^\infty(W, V)$ and in consequence it is smooth. It can be also seen as inverse in the groupoid of partially invertible elements.

One defines a groupoid $\mathcal{U}_{\mathsf{res}}(\mathcal{H}) \rightrightarrows \mathrm{Gr}_{\mathrm{res}}$ as

$$\mathcal{U}_{\mathsf{res}}(\mathcal{H}) = s^{-1}(\mathrm{Gr}_{\mathrm{res}}) \cap t^{-1}(\mathrm{Gr}_{\mathrm{res}})$$

$$= \{u \in \mathcal{U}(\mathcal{H}) \mid u^*u, uu^* \in \mathrm{Gr}_{\mathrm{res}}\}$$

**Fact**

For $u \in \mathcal{U}_{\mathsf{res}}(\mathcal{H})$ we have $u_{+-}, u_{-+} \in L^2$.

- It is a subgroupoid (in algebraic sense).

## Proposition

For every point $W \in \mathrm{Gr}_{\mathrm{res}}$ there exists a neighbourhood $\Omega_W \subset \mathrm{Gr}_{\mathrm{res}}$ and a smooth map $\sigma_W : \Omega_W \to \mathrm{U}_{\mathsf{res}}(\mathcal{H})$ such that

$$\forall\, W' \in \Omega_W \quad W' = \sigma_W(W')\mathcal{H}_+$$

- Using these local sections we construct injective maps:

$$\mathcal{U}_{\mathsf{res}}(\mathcal{H}) \supset s^{-1}(W') \cap t^{-1}(W) \rightrightarrows \mathrm{Gr}_{\mathrm{res}} \times U_+ \times \mathrm{Gr}_{\mathrm{res}}$$

$$u \mapsto (uu^*, \sigma_W(uu^*)^{-1} u \sigma_{W'}(u^*u)_{|\mathcal{H}_+}, u^*u)$$

**Proposition**

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$$u \mapsto (uu^*, \sigma_W(uu^*)^{-1} u \sigma_{W'}(u^*u)_{|\mathcal{H}_+}, u^*u)$$

- couple this map with charts on the respective manifolds

$$\Phi_{\alpha\beta\gamma}(u) = (\tilde{\psi}_\gamma(uu^*), \psi_\alpha(\sigma_\gamma(uu^*)^{-1} u \sigma_\beta(u^*u))_{|\mathcal{H}_+}, \tilde{\psi}_\beta(u^*u))$$

$$\in L^2(W_\gamma, W_\gamma^\perp) \times \mathfrak{u}(\mathcal{H})_+ \times L^2(W_\beta, W_\beta^\perp)$$

**Theorem**

The family $(\Omega_{\alpha\beta\gamma}, \Phi_{\alpha\beta\gamma})$ defines a smooth atlas on $\mathcal{U}_{\text{res}}(\mathcal{H})$.

The manifold $\mathcal{U}_{\text{res}}(\mathcal{H})$ is a Banach–Lie groupoid with respect to the defined maps.

• Obviously, $\mathcal{U}_{\mathsf{res}}(\mathcal{H})$ is not a Banach–Lie subgroupoid of the Banach–Lie groupoid of all partial isometries $\mathcal{U}(\mathcal{H})$: the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ is not a submanifold of Grassmannian . It is only a weakly immersed submanifold as the image of the tangent of the inclusion map is not closed.
• Unlike $\mathcal{U}(\mathcal{H})$, the groupoid $\mathcal{U}_{\mathsf{res}}(\mathcal{H})$ is transitive and pure, i.e. the map $(s,t) : \mathcal{U}_{\mathsf{res}}(\mathcal{H}) \to \mathrm{Gr}_{\mathrm{res}} \times \mathrm{Gr}_{\mathrm{res}}$ is surjective and both base and total space are modeled on a single (up to isomorphism) Banach space.

Few words about tangent group $TU(\mathcal{H})$.

THANK YOU