Topological order, tensor networks and subfactors

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Topological order and operator algebras

Topological order in terms of gapped Hamiltonians and their ground states arising as MPS and PEPS has caught much attention recently. We study tensor networks arising from subfactor thory of Jones in operator algebras and clarify the roles of higher relative commutants.

Outline of the talk:

1. Gapped Hamiltonian, MPS and PEPS
2. Work of Bultinck et al.
3. Subfactors and bi-unitary connections
4. MPO, a tube algebra and a modular tensor category
5. PMPO and PEPS
6. PMPO and higher relative commutants of a subfactor

Gapped Hamiltonian, MPS and PEPS

From a viewpoint of operator algebras, gapped Hamiltonians are studied in the context of an infinite tensor product of matrix algebras $M_d(\mathbb{C})$. A gapped Hamiltonian is a sequence of self-adjoint matrices in finite tensor products and they have a fixed gap between the lowest eigenvalues and the next eigenvalues. Various functional analytic studies have been made.

A matrix product state (MPS) is a certain expression of a state using a trace of a matrix product. It is useful to express a ground state of a gapped Hamiltonian explicitly for a one-dimensional system. A projected entangled pair state (PEPS) is its higher dimensional analogue. We are now interested in the 2-dimensional case here.

Work of Bultinck et al.

Bultinck-Mariëna-Williamson-Şahinoğlu-Haegemana-Verstraete (2017) considered an algebra of matrix product operators (MPOs) arising from a system of physically nice tensor networks. Such an algebra is called a matrix product operator algebra (MPOA). They considered a fusion category of matrix product operators and presented a graphical method to construct an interesting system of anyons. They get a modular tensor category and discuss its physical significance.

They are aware of its similarity to an old work of Ocneanu in subfactor theory. We explain that their construction is really the same as Ocneanu's mathematically.

Subfactor theory

A certain nice infinite dimensional operator algebra is called a type II_1 factor. We consider an inclusion of type II_1 factors $N \subset M$ and simply call it a subfactor. We have a natural notion of the Jones index $[M : N]$ of a subfactor, which is analogous to the index of a subgroup or the degree of an extension of a field. The set of possible Jones index values is

$$\left\{4\cos^2\frac{\pi}{n} \mid n = 3, 4, 5, \dots\right\} \cup [4, \infty].$$

The square root of the Jones index value is similar to the quantum dimension for a quantum group or the statistical dimension in quantum field theory.

Fusion categories of bimodules

Let $N \subset M$ be a subfactor with finite Jones index. We set $N = M_0$, $M = M_1$ and apply the Jones basic construction to get

$$M_0 \subset M_1 \subset M_2 \subset M_3 \cdots .$$

We consider four kinds of bimodules ${}_N M_{kN}$, ${}_N M_{kM}$, ${}_M M_{kN}$, ${}_M M_{kM}$ for all k and make their irreducible decompositions. Generically, we would get countably many, mutually inequivalent bimodules, but in some nice situations, we have only finitely many ones, then we say that the subfactor is of finite depth. This condition is analogous to rationality of a conformal field theory. We then have a fusion category of N-N bimodules.

Commuting squares and subfactors

Fix a subfactor $N \subset M$ with finite Jones index and finite depth. The diagram

$$\begin{array}{ccc} M' \cap M_k & \subset & M' \cap M_{k+1} \\ \cap & & \cap \\ N' \cap M_k & \subset & N' \cap M_{k+1} \end{array}$$

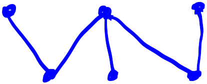
gives a square of finite dimensional C^*-algebras. Such a square satisfies a special property called a commuting square. If N or M has a nice approximation property with finite dimensional subalgebras, then the above commuting square for a single, sufficiently large k has complete information to recover the original $N \subset M$ (Popa). So classification of subfactors is reduced to the classification of certain commuting squares of finite dimensional C^*-algebras.

From a commuting square to a subfactor

Consider the following commuting square of finite dimensional C^*-algebras.
$$\begin{array}{ccc} A & \subset & B \\ \cap & & \cap \\ C & \subset & D. \end{array}$$
We can apply the Jones basic construction horizontally and vertically to get a double sequence $\{A_{k,l}\}_{k,l=0,1,2,\ldots}$ of finite dimensional C^*-algebras with $A_{k,l} \subset A_{k+1,l}$ and $A_{k,l} \subset A_{k,l+1}$. Then we get two subfactors $A_{0,\infty} \subset A_{1,\infty}$ and $A_{\infty,0} \subset A_{\infty,1}$ as certain limit algebras. If the original commuting square arises from $N \subset M$ as in the previous slide, then the former is anti-isomorphic to $N \subset M$ and the latter is isomorphic to $N \subset M$.

Bi-unitary connections

For an inclusion $A \subset B$ of finite dimensional C^*-algebras, we draw the Bratteli diagram as follows, where a dot represents a direct summand.



Look at the Bratteli diagrams of a commuting square. Choose one edge from each diagram corresponding to each of the four inclusions. We then get a complex number from the 4 edges. This is a notion of a bi-unitary connection considered by Ocneanu and Haagerup.

Bi-unitary connections and the basic construction

If one of the two subfactors $A_{0,\infty} \subset A_{1,\infty}$ and $A_{\infty,0} \subset A_{\infty,1}$ arising from a bi-unitary connection is of finite depth, so is the other. From now on, we assume this holds.

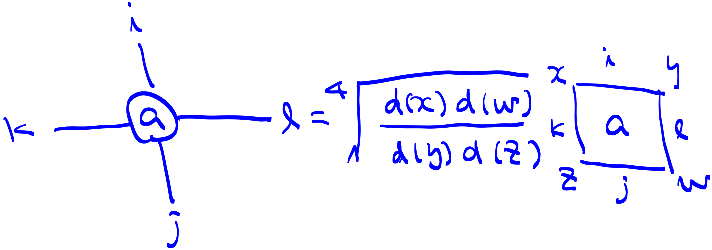
We would like to define a 4-tensor so that the labels for a wire are given by the edges of a Bratteli diagram and the value of a 4-tensor with four wires labeled is given by that of the square for a bi-unitary connection. However, we cannot concatenate such 4-tensors horizontally or vertically, because the index sets for labeling are different.

For resolving this issue, we use the horizontal and vertical basic constructions to get a new and larger commuting square involving $A_{k,l} \subset A_{k+2,l}$.

Bi-unitary connections, a fusion category and a 4-tensor

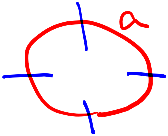
We have natural notions of a tensor product and irreducible decompositions for such bi-unitary connections corresponding to those for bimodules. In this way, we have a fusion category of bi-unitary connections equivalent to that of bimodules (Asaeda-Haagerup).

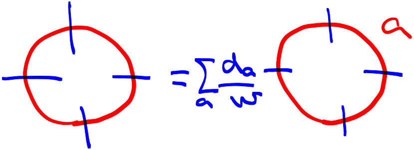
Label each irreducible bi-unitary connection with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \ldots$ and define 4-tensors as follows.



MPO and PMPO

Using such 4-tensors arising from a bi-unitary connection, we define a matrix product operator (MPO) and a projector matrix product operator (PMPO) as follows, where the length of a circle is 4.

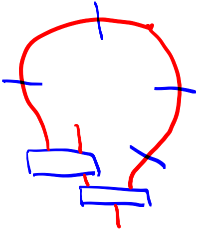


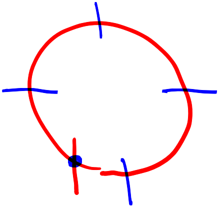


The product structure of these MPOs is isomorphic to that of relative tensor products of bimodules. We have a so-called zipper condition for these MPOs arising from this product structure.

An anyon algebra

Bultinck et al. introduced an anyon algebra for 4-tensors and obtained a modular tensor category describing topological order.

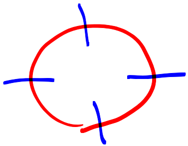


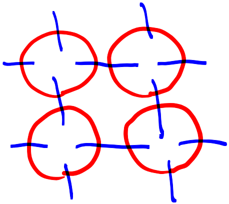


We can show that this anyon algebra is isomorphic to Ocneanu's tube algebra for the fusion category of bimodules arising from the subfactor. In particular, the resulting modular tensor categories are identical and the Verlinde formula holds for the setting of anyon algebra.

PEPS arising from a bi-unitary connection

For a PMPO of length 4, we have 4 exterior wires and 4 internal wires. By combining the internal 4 wires into one, we get a 5-tensor from this diagram.

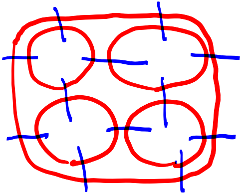


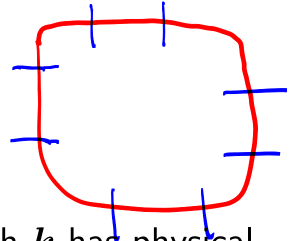


Bultinck et al. studied a PEPS on a square lattice as in the above picture, where the square lattice is of size $\mathbf{2} \times \mathbf{2}$ now. This gives a ground state of a local Hamiltonian.

PEPS and PMPO

For the PEPS in the right figure in the previous slide, a PMPO of length 8 acts from the exterior wires and PMPOs of length 4 acts from the internal wires. So this PEPS lives in the intersections of the ranges of these PMPOs.

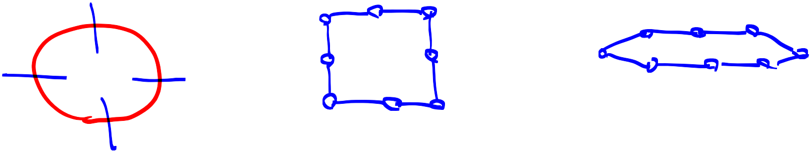




The range of a PMPO of length k has physical significance and we would like to identify this range with something important and well-studied in subfactor theory: higher relative commutants of a subfactor.

PMPO and the higher relative commutants

The range of a PMPO of length k is naturally identified with the higher relative commutants $A'_{\infty,0} \cap A_{\infty,k}$ of the subfactor $A_{\infty,0} \subset A_{\infty,1}$, since an element in this range is identified with a flat field of strings in subfactor theory.



Note that we have two subfactors $A_{0,\infty} \subset A_{1,\infty}$ $A_{\infty,0} \subset A_{\infty,1}$ in this study. The former appears in the anyon algebra and the latter appears here. They are known to be complex conjugate Morita equivalent.